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# Covering symmetric supermodular functions by graphs*

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**Abstract.** The minimum number of edges of an undirected graph covering a symmetric, supermodular set-function is determined. As a special case, we derive an extension of a theorem of J. Bang-Jensen and B. Jackson on hypergraph connectivity augmentation.

## 1. Introduction

T. Watanabe and A. Nakamura [WN1987] proved a min-max formula for the minimum number of new edges whose addition to a given undirected graph results in a $k$-edge-connected graph. E. Cheng [C1999] considered the problem of increasing the connectivity of hypergraphs by adding a minimum number of graph-edges, and provided a solution for the special case when the starting hypergraph is $(k-1)$-edge-connected. Extending further the results of Cheng, J. Bang-Jensen and B. Jackson [BJ1999] solved the general hypergraph connectivity augmentation problem. The purpose of the present paper is to derive a generalization of the theorem of Bang-Jensen and Jackson where, instead of a hypergraph whose connectivity is to be incereased, a symmetric supermodular function is specified to be 'covered' by undirected edges. The result, when specialized to hypergraphs, not only provides the theorem of Bang-Jensen and Jackson, but it actually gives rise to an extension when the connectivity of the hypergraph is to be increased inside a specified terminal set. This will be explained in Section 7.

Let us say some words about the proof methods. For proving their theorem, Watanabe and Nakamura used a sophisticated analysis of the structure of $k$-edge-connected graphs. A different proof, based on the splitting-off technique, was given by G.-R. Cai and Y.-G. Sun [CS1989]. A. Frank [F1992] used the the splitting-off technique in a different way and obtained a short proof of the theorem of Watanabe and Nakamura. This simplification enabled him to find several extensions such as local-edge-connectivity

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augmentation, minimum node-cost and degree-constrained augmentation. The same technique gave rise to a directed counterpart of the theorem of Watanabe and Nakamura.

The hypergraph connectivity augmentation problem is significantly more difficult than the graph augmentation problem since a new type of necessary conditions comes in. This was recognized by E. Cheng whose method for increasing optimally the edge-connectivity of a hypergraph by one is based on a decomposition technique of submodular functions due to W. Cunningham [C1983]. For solving the general hypergraph augmentation problem, Bang-Jensen and Jackson returned to the splitting technique but they also needed the inverse operation of splitting. The present method stems out of the work of Bang-Jensen and Jackson. Our original purpose was to show that their theorem can be extended to supermodular functions and we did this by invoking their 'back and forth' splitting method. The more abstract setting, however, enabled us to realize that the inverse splitting steps can be avoided. In this sense the present algorithm is conceptionally simpler than the one of Bang-Jensen and Jackson.

The structure of the paper is as follows. In the next section we introduce the basic notions, formulate the problems, and state the main results. Section 3 describes the properties of a reduction technique called projection. Section 4 is devoted to introduce and analyze the splitting-off operation in the supermodular setting. The proofs of the main theorems are included in Section 5 along with an extension of Theorem 2. We discuss algorithmic aspects in Section 6. The last section exhibits how to specialize the main results in order to obtain a generalization of the hypergraph connectivity augmentation theorem of Bang-Jensen and Jackson.

## 2. Preliminaries

Suppose we are given a finite ground-set $V$ and a non-negative, integer-valued function $p$ on the subsets of $V$. We say that $p$ is a **set-function on** $V$. We assume that $p(\emptyset) = 0$ and that $p$ is symmetric, that is, $p(X) = p(V - X)$ holds for every subset $X \subseteq V$. Furthermore, suppose that $p$ satisfies the supermodular inequality:

$$p(X) + p(Y) \leq p(X \cup Y) + p(X \cap Y) \tag{1a}$$

whenever $p(X) > 0$, $p(Y) > 0$ and $X, Y$ are crossing sets, that is, none of $V - (X \cup Y)$, $X \cap Y$, $X - Y$, $Y - X$ is empty. In this case we say that $p$ is **crossing supermodular.** It follows from the symmetry of $p$ that

$$p(X) + p(Y) \leq p(X - Y) + p(Y - X) \tag{1b}$$

holds for crossing sets $X, Y$. Note that (1a) automatically holds if $X \subseteq Y$ or $Y \subseteq X$. Likewise, (1b) holds if $X$ and $Y$ are disjoint or co-disjoint. (Two subsets of $V$ are called **co-disjoint** if their complements are disjoint).

For two subsets $X$ and $Y$, $X - Y$ denotes the set of elements of $X$ not belonging to $Y$. If $Y = \{y\}$, then set $Y$ is called a **singleton.** In this case $X - Y$ is denoted by $X - y$ and $X \cup Y$ is denoted by $X + y$. A subset containing an element $a$ and not containing an element $b$ is called an $a\bar{b}$**-set.** A set $X$ **separates** $u, v$ if $X$ contains exactly one of $u$ and $v$. Given a subset $T \subseteq V$, we say that $X$ is $T$**-separating** or that $X$ **separates** $T$



if $X \cap T \neq \emptyset$, $T - X \neq \emptyset$ (equivalently, $T$ has two elements separated by $X$). (Hence a subset $X$ separates $V$ if and only if $X$ is a nonempty, proper subset of $V$.) By a sub-partition $\mathcal{F} := \{V_1, \ldots, V_h\}$ of $V$ we mean a set of pairwise disjoint non-empty subsets of $V$. Here $h$ is the **size** of the sub-partition. We use the notation $\cup\mathcal{F} := \cup(X : X \in \mathcal{F})$. If $V = \cup\mathcal{F}$, then $\mathcal{F}$ is a **partition** of $V$. The sets $V_i$ are called the **members** of $\mathcal{F}$. We do not distinguish between a one-element set and its element. For example, if $p$ is a set-function on $V$ and $v \in V$, then we write $p(v)$ to denote $p(\{v\})$.

By a **hypergraph** we mean a pair $H = (V, A)$ where $V$ is the node-set of the hypergraph and $A$ is a collection of (possibly not distinct) subsets of $V$ with at least two elements. The members of $A$ are called **hyperedges.** A hyperedge with two elements is called a **graph-edge** or simply an **edge.** If every element of $A$ is an edge, we speak of an (undirected) graph. (We allow parallel edges but no loops.)

Let $d_H(X)$ denote the number of hyperedges intersecting both $X$ and $V - X$. For an element $v \in V$ we use the abbreviation $d_H(v)$ for $d_H(\{v\})$ and call it the **degree** of $v$. (Since we do not allow hyperedges of one element, $d_H(v)$ is the number of hyperedges containing $v$.)

Beside $p$ we are given a non-negative, integer-valued function $g : V \to \mathbf{Z}_+$. Throughout the paper we will use the notation $g(X) := \sum(g(v) : v \in X)$.

We say that $g$ **majorizes** a graph $G = (V, E)$ if

$$d_G(v) \leq g(v) \text{ for every node } v \in V. \tag{2}$$

$G$ is said to **cover** $p$ if

$$d_G(X) \geq p(X) \text{ for every subset } X \subseteq V. \tag{3}$$

One of our main concerns is to find a necessary and sufficient condition for the existence of a graph satisfying (2) and (3). (An analogous question concerning directed graphs was answered in [F1994].) The following 'cut-type' necessary condition expresses the natural requirement that the total upper bound on the degrees in $X$ should be at least the lower bound $p(X)$ for the number of edges entering $X$.

$$g(X) \geq p(X) \text{ for every subset } X \subseteq V, \tag{4}$$

For a formal proof of the necessity of (4), see (9) below. Unfortunately, (4) is not sufficient in general, as shown by an example consisting of a ground set $V$ with 4 elements and $p \equiv 1$, $g \equiv 1$. Here (4) is clearly satisfied but no graph $G$ may satisfy (3) since such a graph must be connected and hence it has at least three edges. On the other hand, $g \equiv 1$ implies that $G$ can have at most 2 edges.

This example motivates the following notions. A set $X$ for which $p(X) = g(X)$ is called **tight** (with respect to $p$). We call a partition $\mathcal{F} := \{V_1, \ldots, V_h\}$ of $V$ with $h \geq 4$ $p$**-full** if

$$p(\cup\mathcal{F}') \geq 1 \text{ for every sub-partition } \mathcal{F}' \text{ with } \emptyset \subset \mathcal{F}' \subset \mathcal{F} \text{ and} \tag{5}$$

$$\mathcal{F} \text{ has a member } V_i \text{ with } p(V_i) = 1. \tag{6}$$

We call the maximum size of a $p$-full partition the **dimension** of $p$ and denote it by $\dim(p)$. If there is no $p$-full partition, then $\dim(p)$ is defined to be 0.

**Theorem 1.** *Let $g : V \to \mathbf{Z}_+$ be a function and let $p : 2^V \to \mathbf{Z}_+$ be a symmetric, crossing supermodular set-function. There is a graph $G = (V, E)$ covering $p$ and majorized by $g$ if and only if (4) holds,*

$$g(V) \text{ is even} \tag{7a}$$

*whenever*

$$\text{every element of } V \text{ belongs to a tight set,} \tag{7b}$$

*and*

$$\dim(p) - 1 \le g(V)/2. \tag{8}$$

*Proof of the necessity.* Suppose that $G$ is a graph satisfying (2) and (3). Then

$$p(X) \le d_G(X) \le \sum (d_G(v) : v \in X) \le g(X) \tag{9}$$

and (4) follows. If $X$ is a tight set, then (9) implies that $d_G(v) = g(v)$ for every element $v \in X$. Therefore, if (7b) holds, then $g(V) = \sum(d_G(v) : v \in V)$. This sum is even and hence (7a) follows.

Let $\mathcal{F} := \{V_1, \dots, V_h\}$ be a $p$-full partition of size $h$. Then $G$ must have at least $h - 1$ edges. Indeed, by shrinking each $V_i$ into a node $v_i$ we obtain from $G$ a graph that must be connected by (5). Therefore $G$ must have at least $h - 1$ edges. On the other hand, since $G$ is majorized by $g$, $G$ has at most $g(V)/2$ edges. Therefore $h - 1 \le g(V)/2$ and (8) follows.

□

(Note that in this proof, among the two defining properties (5) and (6) of $p$-fullness, only (5) was used. The requirement in (6) has a special role that will be explained at the end of this section.)

In Section 5 (Lemma 6) we will prove that (7b) is equivalent to the existence of a partition of $V$ into tight sets. Here we show the following.

**Lemma 1.** *If Theorem 1 holds whenever $g(V)$ is even, then it holds when $g(V)$ is odd.*

*Proof.* Suppose that $g(V)$ is odd. Then, by the hypothesis (7), there is an element $s \in V$ which does not belong to any tight set. This means, in particular, that $g(s) > 0$. Revise $g$ by reducing $g(s)$ by 1 and let $g'$ denote the resulting function. Clearly, $g'(V)$ is even. Since $s$ does not belong to any tight set, (4) holds for $(g', p)$. Clearly, $\dim(p) - 1 \le \lfloor g(V)/2 \rfloor = g'(V)/2$. That is, both (4) and (8) hold for $(g', p)$. By hypothesis, the theorem holds for $(g', p)$ and therefore there is a graph $G$ covering $p$ and majorized by $g'$. Then $G$ is clearly majorized by $g$, as well.

□

By this lemma, we will henceforth assume that $g(V)$ is even. Our second result is about the smallest graph covering $p$. In Section 6 a generalization of Theorem 2 will be formulated and proved.



**Theorem 2.** *Let $p : 2^V \to \mathbf{Z}_+$ be a symmetric, crossing supermodular function and $\gamma$ a positive integer. There exists a graph $G = (V, E)$ with at most $\gamma$ edges covering $p$ if and only if*

$$\sum(p(X) : X \in \mathcal{P}) \le 2\gamma \tag{10}$$

*holds for every partition $\mathcal{P}$ of $V$ and*

$$\dim(p) - 1 \le \gamma. \tag{11}$$

*Proof of the necessity.* We have already seen the necessity of (11). To prove the necessity of (10), suppose that there is a graph $G = (V, E)$ covering $p$ and let $\mathcal{P}$ be a partition of $V$. Then $2\gamma \ge \sum(d_G(X) : X \in \mathcal{P}) \ge \sum(p(X) : X \in \mathcal{P})$, that is, (10) indeed holds.

□

In the sequel we need the following notions. A set $X$ was called tight if $p(X) = g(X)$. If, in addition, $1 = p(X) = g(X)$, we say that $X$ is a **1-tight** set. Note that a set $X$ with $g(X) = 0$ is always tight since $p$ was assumed to be non-negative. We call a set $X$ **near-tight** if $p(X) = g(X) - 1$. A tight or a near-tight set is called **dangerous.** A dangerous set is **maximal** if it is not a proper subset of any dangerous set.

Let $Z \subset V$ be a subset. Define $V' = V - Z + z$ where $z$ is a new element. For $X \subseteq V'$ let $p'(X) := p(X)$ if $z \notin X$ and let $p'(X) := p(V' - X)$ if $z \in X$. Furthermore let $g'(z) := g(Z)$ and $g'(v) := g(v)$ for every $v \in V - z$. We say that $(g', p')$ arises from $(g, p)$ by **contracting** $Z$. We can extend this notion. Let $\mathcal{F} := \{V_1, V_2, \dots, V_h\}$ be a sub-partition of $V$ and let $(g', p')$ denote the pair arising from $(g, p)$ by contracting all the members of $\mathcal{F}$. (Clearly $(g', p')$ does not depend on the order in which the separate set-contractions are carried out). We say that $(g', p')$ arises from $(g, p)$ by **contracting** $\mathcal{F}$.

We conclude the section by investigating the question whether the necessary conditions above are indeed good characterizations. Suppose that $p$ is given by an evaluation oracle. Condition (4) is an appropriate certificate in the sense that if someone claims that a certain set $X$ violates (4), that is, $g(X) > p(X)$, then with one oracle call we can immediately check the truth of this statement. Likewise, for a given partition $\mathcal{P}$, we can check (10) by calling the evaluation oracle $|\mathcal{P}| \le |V|$ times.

The situation with conditions (8) and (11) is slightly more complicated. How can we check whether a given partition $\mathcal{F} := \{V_1, \dots, V_h\}$ is $p$-full? (6) can clearly be tested by $h$ oracle calls. The problematical case is to test $\mathcal{F}$ for (5) since this requires to evaluate $p(\cup_{i \in I} V_i)$ for all possible index subsets $\emptyset \subset I \subset \{1, \dots, h\}$, an exponential number of oracle calls. To overcome this difficulty, we prove the following claims which will also be useful later.

*Claim 1.* Let $p'$ be a symmetric crossing supermodular set-function on a ground-set $V'$ and $Z$ a subset of $V'$ with $|Z| \ge 2$. Suppose that there is an element $s$ of $Z$ for which $p'(s) = 1$ and $p'(\{s, z\}) \ge 1$ for every $z \in Z - s$. Then

$$p'(X) \ge 1 \text{ for every subset } X \text{ such that } s \in X \subset Z, \text{ and} \tag{12a}$$

$$p'(Z) \ge 1 \text{ if } Z \subset V'. \tag{12b}$$

*Proof.* What we have to show is that $p'(X) > 0$ for every set $X$ with $s \in X \subset Z$ and that $p'(Z) > 0$ in case $Z \neq V'$. Suppose indirectly that this is not true and let $X$ be a smallest subset of $Z$ containing $s$ for which $p'(X) = 0$. Then $X \subset V'$ and, by hypothesis, $|X| \geq 3$. Let $u \in X - s$ and $X' := X - u$. By the minimality of $X$, we have $p'(X') \geq 1$. Now sets $X'$ and $Y := \{s, u\}$ are crossing and by (1a) we have $1 + 1 \leq p'(X') + p'(Y) \leq p'(X' \cap Y) + p'(X' \cup Y) = p'(s) + p'(X) = 1 + 0$, a contradiction.

□

*Claim 2.* Let $p$ be a symmetric crossing supermodular set-function on a ground-set $V$ and $\mathcal{F}$ a sub-partition of $V$ with $|\mathcal{F}| \geq 2$. Suppose that there is a member $S$ of $\mathcal{F}$ for which $p(S) = 1$ and $p(S \cup F) \geq 1$ for every $F \in \mathcal{F}$. Then

$$p(\cup\mathcal{F}') \geq 1 \text{ for every sub-partition } \mathcal{F}', S \in \mathcal{F}' \subset \mathcal{F}, \text{ and} \tag{13a}$$

$$p(\cup\mathcal{F}) \geq 1 \text{ if } \cup\mathcal{F} \subset V. \tag{13b}$$

If $\mathcal{F}$ is a partition of $V$, then (5) holds.

*Proof.* Contract $\mathcal{F}$ and let $V'$ and $p'$ denote the arising ground-set and set-function, respectively. Let $Z$ denote the subset of $V'$ arising from the contraction of $\mathcal{F}$. The first part of the claim follows by applying Claim 1. When $\mathcal{F}$ is a partition of $V$, the symmetry of $p$ implies (5).

□

By Claim 2 and (6) we immediately have:

**Corollary 1.** *A partition $\mathcal{F} := \{V_1, \ldots, V_h\}$ with $h \geq 4$ is $p$-full if and only if there is a member $Z \in \mathcal{F}$ for which $p(Z) = 1$ and $p(Z \cup V_i) \geq 1$ for every $i = 1, \ldots, h$.*

□

By Corollary 1 we can easily test a partition $\mathcal{F}$ for $p$-fullness by at most $2h \leq 2|V|$ oracle calls. First compute $p(V_i)$ for each $i = 1, \ldots, h$. If none of these values is 1, then (6) is violated and $\mathcal{F}$ is not $p$-full. If at least one of these values is one, we may assume that $p(V_1) = 1$. Compute $p(V_1 \cup V_i)$ for each $i = 2, 3, \ldots, h$. If at least one of these values is zero, then $\mathcal{F}$ is not $p$-full. Otherwise $\mathcal{F}$ is $p$-full.

(As we have already noted earlier, while proving the necessity of (8) and (9), assumption (6) has not been required. The only role of (6) is to make possible to test a given partition $\mathcal{F}$ for $p$-fullness. Without assuming (6), the problem of deciding whether a partition $\mathcal{F}$ satisfies (5) includes an NP-complete problem. Indeed, let $G = (V, E)$ be an arbitrary graph in which we want to decide whether there is a cut with at least $k$ edges where $k$ is a given integer. [This is the max-cut problem, a well-known NP-complete problem]. Let $\mathcal{F}$ be the partition of $V$ in which every member consists of one element of $V$. Let $p(X) = \max(0, k - d_G(X))$ for $\emptyset \subset X \subset V$ and $p(\emptyset) = p(V) = 0$. Then $p$ is crossing supermodular and symmetric. Now $\mathcal{F}$ satisfies (5) if and only if $k > d_G(X)$ for every subset $\emptyset \subset X \subset V$. In other words, $\mathcal{F}$ satisfies (5) if and only if the maximum cut is smaller than $k$.)



## 3. Projection

In this section we introduce a reduction technique called projection. In the sequel, projection will be used in two separate levels. First, it will have a simplifying role in the proofs of Theorems 1 and 5 by enabling us to get rid of elements $v$ for which $g(v) = 0$. A second application of projection occurs in Section 5 where it is used to derive an extension of Theorem 2.

Let $p$ be a symmetric, crossing supermodular set-function on $V$ and $r$ an element of $V$. Define a set-function $p'$ on $V' := V - r$, as follows. $p'(\emptyset) := p'(V') := 0$ and $p'(X) := \max(p(X), p(X + r))$ for $\emptyset \subset X \subset V'$. It is easy to check that $p'$ is symmetric and crossing supermodular. $p'$ is called the **simple projection** of $p$ **along** $r$ or the projection of $p$ **to** $V'$. For any subset $\emptyset \subset X \subset V'$, let $\hat{X} := X$ if $p'(X) = p(X)$ (i.e., $p(X) \geq p(X + r)$) and let $\hat{X} := X + r$ if $p'(X) > p(X)$ (i.e., $p(X) < p(X + r)$). Clearly, $p'(X) = p(\hat{X})$. For a sub-partition $\mathcal{P}'$ of $V'$, let $\hat{\mathcal{P}}' := \{\hat{X} : X \in \mathcal{P}'\}$.

The usability of this operation depends on whether the necessary conditions in Theorems 1 and 2 can be preserved under projection. The next claim will imply that this is the case with respect to (10).

*Claim 3.* If two non-empty subsets $X, Y \subset V'$ are disjoint, then so are $\hat{X}$ and $\hat{Y}$. If $\mathcal{P}'$ is a sub-partition of $V'$, then $\hat{\mathcal{P}}'$ is a sub-partition of $V$ for which

$$\sum(p(\hat{X}) : \hat{X} \in \hat{\mathcal{P}}') = \sum(p'(X) : X \in \mathcal{P}'). \tag{14}$$

*Proof.* If indirectly both $\hat{X}$ and $\hat{Y}$ contain $r$, then $p(\hat{X}) \geq p(X)+1$ and $p(\hat{Y}) \geq p(Y)+1$ and by (1b) we obtain $p(\hat{X}) + p(\hat{Y}) \leq p(\hat{X} - \hat{Y}) + p(\hat{Y} - \hat{X}) = p(X) + p(Y) < p(\hat{X}) + p(\hat{Y})$, a contradiction. The second part of the claim follows from the first.

□

In order to preserve (8) or (11) under projection, it would be satisfactory to assert that a $p'$-full partition of $V'$ can always be 'extended' to a $p$-full partition of $V$. However, we can prove this only under an additional assumption (see Lemma 3 below). Fortunately, in the application this technical requirement will be satisfied. We need a preparatory lemma.

**Lemma 2.** *Suppose that*

$$p'(X) \geq 1 \textit{ for every subset } \emptyset \subset X \subset V' \tag{15}$$

*and that there is a subset $S := \{s_1, s_2, s_3, s_4\}$ of $V'$ so that $p'(s_i) = 1$ $(1 \leq i \leq 4)$. Let $\mathcal{N} := \{X : \emptyset \subset X \subset V',\ p(X) = 0\}$. Then $V'$ has an element $z$ covering all the members of $\mathcal{N}$.*

*Proof.* There is nothing to prove if $\mathcal{N} = \emptyset$ so we assume that $\mathcal{N} \neq \emptyset$.

*Claim A.* There are no two disjoint members of $\mathcal{N}$.

*Proof.* Suppose that $X, Y \in \mathcal{N}$ are disjoint. By Claim 3 $\hat{X}$ and $\hat{Y}$ are also disjoint and hence one of them, say $\hat{X}$, does not contain $r$. But then, by (15), $1 \leq p'(X) = p(X)$, contradicting the assumption that $X \in \mathcal{N}$.

□

If there is an element $z \in V'$ with $p(z) = 0$, then $\{z\} \in \mathcal{N}$ and, by Claim A, $z$ satisfies the requirement of Lemma 2. So we may assume that

$$p(v) \geq 1 \text{ for every } v \in V'. \tag{16}$$

*Claim B.* If an element $s \in S$ belongs to a member $N$ of $\mathcal{N}$, then there is an element $x \in N - s$ so that $\{s, x\} \in \mathcal{N}$.

*Proof.* Since $p(s) = 1$ and $p(N) = 0$, $N$ has at least two elements. By Claim 1, $N - s$ has an element $x$ for which $p(\{s, x\}) = 0$, that is, $\{s, x\} \in \mathcal{N}$.

□

*Claim C.* Every element $s \in S$ belongs to a member of $\mathcal{N}$.

*Proof.* Let $N$ be a minimal member of $\mathcal{N}$. By (16), $|N| \geq 2$. We are done when $s \in N$, so assume that $s \notin N$. If $S - s \subseteq N$, then $|N| \geq 3$ on one hand, but Claim B and the minimality of $N$ imply that $|N| = 2$, on the other. Therefore $S - s \not\subseteq N$ and there is an element $s' \in S - s$ with $s' \notin N$.
Now one of the sets $X := N + s$ and $Y := \{s, s'\}$ belongs to $\mathcal{N}$ for otherwise we would have $1 + 1 \leq p(X) + p(Y) \leq p(X - Y) + p(Y - X) = p(N) + p(s) = 0 + 1$, a contradiction.

□

Let $H = (V', A)$ denote the graph for which $A := \{xy : \{x, y\} \in \mathcal{N}\}$. By Claim A, $H$ does not have two disjoint edges. Therefore either there is a node $z \in V'$ covering all the edges in $A$ or else $A$ consists of three elements forming a triangle. This latter case, however, cannot occur since, by Claims B and C, each of the four elements of $S$ is covered by a member of $A$.

By changing indices if necessary, we may suppose that $z \notin S - s_4$. ($z$ may or may not be equal to $s_4$.) Then $zs_i \in A$ (that is, $\{z, s_i\} \in \mathcal{N}$) $(1 \leq i \leq 3)$. To conlude the proof of the lemma, we show that $z$ covers every member of $\mathcal{N}$. If this is not true, then let $X$ be a minimal member of $\mathcal{N}$ not covered by $z$. By Claim A, $X$ intersects $\{z, s_1\}$ therefore $s_1 \in X$. Claim B and the minimality of $X$ imply that $|X| = 2$. Therefore $X$ is disjoint from at least one of the sets $\{z, s_2\}$ and $\{z, s_3\}$, contradicting Claim A. This contradiction completes the proof of Lemma 2.

□□

The next lemma shows how to transform a certain $p'$-full partition of $V'$ into a $p$-full partition of $V$.

**Lemma 3.** *Let $\mathcal{F}_1$ be a $p'$-full partition of $V'$ for which*

$$|\mathcal{F}_1| - 2 \geq \sum(p'(X) : X \in \mathcal{F}_1)/2. \tag{17}$$

*Then there is a member $Z$ of $\mathcal{F}_1$ for which $\mathcal{F} := \mathcal{F}_1 - \{Z\} \cup \{Z + r\}$ is a $p$-full partition of $V$.*



*Proof.* By contracting the members of $\mathcal{F}_1$ we may assume that each member of $\mathcal{F}_1$ is a singleton. Then the $p'$-fullness of $\mathcal{F}_1$ implies (15) and inequality (17) transforms into

$$|V'| - 2 \geq \sum(p'(v) : v \in V')/2. \tag{18}$$

We claim that the number $\alpha$ of elements $v$ of $V'$ for which $p'(v) = 1$ is at least 4. Indeed, from (18) we have $2|V'| - 4 \geq \sum(p'(v) : v \in V') \geq \alpha + 2(|V'| - \alpha) = 2|V'| - \alpha$ from which $\alpha \geq 4$. Let $S := \{s_1, s_2, s_3, s_4\} \subseteq V'$ be a subset of $V'$ so that $p'(s_i) = 1$ $(1 \leq i \leq 4)$ and apply Lemma 2.

Suppose that the elements of $V'$ are $v_1, \ldots, v_h$ and that $v_1$ is an element satisfying the property in Lemma 2. Then

$$p(X) \geq 1 \text{ for every subset } \emptyset \subset X \subseteq V' - v_1. \tag{19}$$

Now the family $\mathcal{F} := \{\{v_1, r\}, \{v_2\}, \ldots, \{v_h\}\}$ is a partition of $V$ and we claim that $\mathcal{F}$ is $p$-full. Indeed, by the symmetry of $p$, (19) implies (5). To see (6), recall that there is an element (actually at least three) $x$ of $V'$ different from $v_1$ for which $p'(x) = 1$. Then $1 \leq p(x) \leq p'(x) = 1$, hence $p(x) = 1$ and (6) holds.

□

We extend the notion of elementary projection. For a given subset $\emptyset \subset T \subset V$, define the **projection** $p_T$ **of** $p$ to $T$, as follows. $p_T(\emptyset) := p_T(T) := 0$ and $p_T(X) := \max(p(X') : X' \subset V, X' \cap T = X)$ for $\emptyset \subset X \subset V$. It can be seen that $p_T$ arises from $p$ by a sequence of simple projections along the elements of $V - T$ and therefore $p_T$ is symmetric and crossing supermodular if $p$ is. We say that a sub-partition $\mathcal{P}$ of $V$ is $T$**-separating,** if every member of $\mathcal{P}$ is $T$-separating. Let $\dim_T(p)$ denote the maximum size of a $T$-separating $p$-full partition of $V$.

**Theorem 3.** *Let $\gamma$ be an integer. Suppose that*

$$\sum(p(X) : X \in \mathcal{P}) \leq 2\gamma \text{ for every } T\text{-separating sub-partition } \mathcal{P} \text{ of } V \tag{20}$$

*and*

$$\dim_T(p) - 1 \leq \gamma. \tag{21}$$

*Then*

$$\sum(p_T(X) : X \in \mathcal{P}) \leq 2\gamma \text{ for every sub-partition } \mathcal{P} \text{ of } T \tag{22}$$

*and*

$$\dim(p_T) - 1 \leq \gamma. \tag{23}$$

*Proof.* We use induction on $|V - T|$. There is nothing to prove if this number is zero since if $T = V$, then (20) is equivalent to (22) and (21) is equivalent to (23). So suppose that there is an element $r \in V - T$. Let $V' = V - r$ and let $p'$ be the simple projection of $p$ to $V'$. We are going to prove that (20) and (21) hold for $p'$, that is,

$$\sum(p'(X) : X \in \mathcal{P}) \leq 2\gamma \text{ for every } T\text{-separating sub-partition } \mathcal{P} \text{ of } V' \tag{20'}$$

and

$$\dim_T(p') - 1 \le \gamma. \tag{21'}$$

From this the theorem will follow by induction.

If $\mathcal{P}$ is a $T$-separating sub-partition of $V'$, then, by Claim 3, $\hat{\mathcal{P}}$ is a $T$-separating sub-partition of $V$. Hence (14) and (20) imply (20'). To see (21'), suppose indirectly that there is a $p'$-full $T$-separating partition $\mathcal{F}_1$ of $V'$ for which $|\mathcal{F}_1| - 2 \ge \gamma$. By (20'), $\gamma \ge \sum(p'(X) : X \in \mathcal{F}_1)/2$ and hence (17) is satisfied. Lemma 3 provides with a $p$-full partition $\mathcal{F}$ of $V$ for which $|\mathcal{F}| = |\mathcal{F}_1|$. $\mathcal{F}$ is also $T$-separating and hence (21) fails to hold. This contradiction proves the theorem.

□

Theorem 3 will be used in Section 5 to prove an extension of Theorem 2. Here we derive the following corollary that will be used in the proofs of Theorem 1 and 5 to get rid of elements $v$ for which $g(v) = 0$.

**Theorem 4.** *Let $T \subset V$ be a set for which $g(V - T) = 0$. If $g$ and $p$ satisfy* (4), *then $g$ and $p_T$ satisfy* (4), *that is,*

$$p_T(X) \le g(X) \text{ for every subset } X \subset T. \tag{24}$$

*If $g$ and $p$ satisfy* (4) *and* (8), *then $g$ and $p_T$ satisfy* (8), *that is,*

$$\dim(p_T) - 1 \le g(T)/2 \ \ (= g(V)/2). \tag{25}$$

*Proof.* By the definition of $p_T$, there is a set $\bar{X}$ so that $p_T(X) = p(\bar{X})$ and $X = T \cap \bar{X}$. It follows from $g(V - T) = 0$ that $g(X) = g(\bar{X})$. By applying (4) to $\bar{X}$, we obtain $g(X) = g(\bar{X}) \ge p(\bar{X}) = p_T(X)$, and (24) follows.

Let $\gamma := g(V)/2$. We claim that (20) holds. Let $\mathcal{P}$ be a $T$-separating sub-partition of $V$. By using (4), (24), and the definition of $p_T$, we obtain $\sum(p(X) : X \in \mathcal{P}) \le \sum(p_T(X \cap T) : X \in \mathcal{P}) \le \sum(g(X \cap T) : X \in \mathcal{P}) \le g(T) = g(V) = 2\gamma$, that is, (20) holds. Since $\dim_T(p) \le \dim(p)$, (21) is a consequence of (8). Therefore Theorem 3 applies and (23) is the same as (25).

□

## 4. Splitting off

The technique of splitting off a pair of edges of a graph so as to maintain certain connectivity properties has been a major tool in the solution of several connectivity augmentation problems. Here we introduce an analogous reduction method concerning $g$ and $p$. We assume that (4) holds but (8) is not required until we mention it explicitly.

Suppose that

$$g(V) \text{ is even and } g \text{ is positive on at least two elements of } V. \tag{26}$$

Let $u$ and $t$ be two elements of $V$ with $g(u) \ge 1$, $g(t) \ge 1$. Let $g^{tu}(u) := g(u) - 1$, $g^{tu}(t) := g(t) - 1$ and $g^{tu}(x) := g(x)$ if $x \in V - \{t, u\}$. Furthermore, define $p^{tu}(X) := \max(0, p(X) - 1)$ if $X$ separates $u$ and $t$ and $p^{tu}(X) := p(X)$ otherwise. It is easy to see that $p^{tu}$ is symmetric and crossing supermodular. We say that the pair $(g^{tu}, p^{tu})$ arises from the pair $(g, p)$ by **splitting off** $(g, p)$ at $\{t, u\}$.



We are interested in finding a pair $\{t, u\}$ whose splitting preserves (4), that is,

$$g^{tu}(X) \ge p^{tu}(X) \text{ for every subset } X \subseteq V. \tag{27}$$

Such a pair is called **splittable** (with respect to $p$ and $g$). We call a $p$-full partition $\mathcal{F} := \{V_1, \dots, V_h\}$ of $V$ ($h \ge 4$) **1-tight** if each member of $\mathcal{F}$ is 1-tight. We claim that if $t, u$ satisfy (27), then

$$\text{there is no 1-tight } p\text{-full partition.} \tag{28}$$

Indeed, assume that $u \in V_1, t \in V_2$. For $X := V_1 \cup V_2$ and $Y := V_2 \cup V_3$ we have $1 + 1 \le p(X) + p(Y) \le p(X - Y) + p(Y - X) = p(V_1) + p(V_3) = 1 + 1$ from which $p(X) = 1$ follows. Thus $p^{tu}(X) = 1$ and $g^{tu}(X) = 0$, contradicting (27). The following theorem states the converse.

**Theorem 5.** *Assume that $g$ and $p$ satisfy* (4), (26) *and* (28). *For an arbitrary element $t$ of $V$ with $g(t) > 0$ there is an element $u$ with $g(u) > 0$ such that* (27) *holds.*

*Proof.* It follows from the definitions that (27) holds if and only if no dangerous set includes both $u$ and $t$, or equivalently, no maximal dangerous set includes both $u$ and $t$.

We may assume that the theorem is true whenever the ground-set has fewer elements than $|V|$. First suppose that there is an element $r$ for which $g(r) = 0$. Let $V' := V - r$ and let $p'$ denote the projection of $p$ along $r$. By Theorem 2.5, (4) holds for $p'$. Condition (26) trivially holds for $V'$. Finally, we claim that (28) holds for $p'$. For if there is a 1-tight $p'$-full partition $\mathcal{F}_1$ of $V'$, then (17) is satisfied since $|\mathcal{F}_1| \ge 4$ implies that $|\mathcal{F}_1| - 2 \ge |\mathcal{F}_1|/2 = \sum(p'(X) : X \in \mathcal{F}_1)/2$. Let $\mathcal{F}$ denote tha $p$-full partition of $V$ provided by Lemma 3. Since $\mathcal{F}_1$ is 1-tight and $g(r) = 0$, it follows that $g(X) = 1$ for each member of $\mathcal{F}$, that is, $\mathcal{F}$ is a 1-tight $p$-full partition of $V$, violating (28).

Since we assumed that the theorem holds for $V'$, there is a pair $\{t, u\}$ that is splittable with respect to $p'$. Since a $p' \ge p$, this pair is splittable with respect to $p$, as well.

Therefore we can assume that $g$ is positive everywhere. In this case every 1-tight set is a singleton.

*Claim A.* If $X$ and $Y$ are maximal dangerous sets containing $t$, then they are crossing.

*Proof.* Since $X \cap Y \ne \emptyset$, $X - Y \ne \emptyset$, $Y - X \ne \emptyset$, we have to show that $X \cup Y \ne V$. Suppose indirectly that $X \cup Y = V$. Then the symmetry of $p$ and (4) imply that $(g(X) - 1) + (g(Y) - 1) \le p(X) + p(Y) = p(Y - X) + p(X - Y) \le g(Y - X) + g(X - Y) = g(X) + g(Y) - 2g(X \cap Y) \le g(X) + g(Y) - 2$. Hence we have equality throughout and, in particular, $g(X \cap Y) = 1$. We also have $g(X - Y) = p(X - Y) = p(Y) = g(Y) - 1 = g(Y - X)$ from which $g(V) = g(X - Y) + g(Y - X) + g(X \cap Y) = 2g(X - Y) + 1$, that is, $g(V)$ is odd, a contradiction.

□

*Claim B.* Let $X$ and $Y$ be crossing dangerous sets for which $p(X) \geq 1$, $p(Y) \geq 1$. Then $g(X \cap Y) = 1$, sets $X - Y$, $Y - X$ are tight and sets $X$, $Y$ are near-tight.

*Proof.* Since $g$ is positive everywhere, $g(X \cap Y) \geq 1$. By (1b) and (4) we have $(g(X) - 1) + (g(Y) - 1) \leq p(X) + p(Y) \leq p(X - Y) + p(Y - X) \leq g(X - Y) + g(Y - X) = g(X) + g(Y) - 2g(X \cap Y) \leq g(X) + g(Y) - 2$ from which we have $g(X \cap Y) = 1$, $g(X) - 1 = p(X)$, $g(Y) - 1 = p(Y)$, $p(X - Y) = g(X - Y)$, $p(Y - X) = g(Y - X)$.

□

*Claim C.* Let $X$ and $Y$ be crossing dangerous sets for which $p(X) \geq 1$, $p(Y) \geq 1$ and suppose that $X$ is maximal dangerous. Then $X \cap Y$ is 1-tight and $p(X \cup Y) = g(X \cup Y) - 2 > 0$.

*Proof.* By the maximality of $X$, the set $X \cup Y$ is not dangerous, that is, $p(X \cup Y) \leq g(X \cup Y) - 2$. By this, (1a) and (4), we have $(g(X)-1)+(g(Y)-1) \leq p(X)+p(Y) \leq p(X \cap Y)+p(X \cup Y) \leq g(X \cap Y)+g(X \cup Y)-2 = g(X)+g(Y)-2$. Hence we must have $p(X \cap Y) = g(X \cap Y)$, $p(X \cup Y) = g(X \cup Y) - 2$. By Claim B, $g(X \cap Y) = 1$, that is, $X \cap Y$ is 1-tight. By Claim B, $X$ and $Y$ are near-tight and hence $g(X \cup Y) = g(X)+g(Y)-g(X \cap Y) = g(X)+g(Y)-1 = (p(X)+1)+(p(Y)+1)-1 \geq 2+2-1$ from which $g(X \cup Y) - 2 > 0$.

□

*Claim D.* Let $X_1, X_2, X_3$ be three maximal dangerous sets for which $t \in Z_0 := X_1 \cap X_2 \cap X_3$. Let $Z_i := X_i - t$ let $\mathcal{Z} := \{Z_i : i = 0, \dots, 3\}$. Then each member of $\mathcal{Z}$ is 1-tight.

*Proof.* By Claim C $X_j \cap X_k$ is 1-tight and hence $\{t\} = X_j \cap X_k$ from which $Z_0 = \{t\}$ is 1-tight. By Claim B, $X_j$ is near-tight and $Z_j = X_j - X_k (= X_j - t)$ is tight.

For $1 \leq i < j \leq 3$, let $X_{ij} := X_i \cup X_j$ and $Z_{ij} := Z_i \cup Z_j (= X_{ij} - t)$. By Claim B, $X_k$ is near-tight and $p(X_{ij}) = g(X_{ij}) - 2 \geq 1$. We claim that $Z_{ij}$ is dangerous. Suppose indirectly that $p(Z_{ij}) \leq g(Z_{ij}) - 2$. Since $p(X_k) \geq 1$, $p(X_{ij}) \geq 1$, $X_k - X_{ij} \neq \emptyset$, $X_{ij} - X_k \neq \emptyset$, (1b) applies and we have $[g(X_{ij})-2]+[g(X_k)-1] = p(X_{ij})+p(X_k) \leq p(X_{ij}-X_k)+p(X_k-X_{ij}) = p(Z_{ij})+p(Z_k) \leq g(Z_{ij})-2+g(Z_k) = [g(X_{ij})-1]-2+[g(X_k)-1]$, and this contradiction shows that $Z_{ij}$ is indeed dangerous.

By applying Claim C to $X := X_i$ and $Y := Z_{ij}$, we obtain that $X \cap Y = Z_i$ is 1-tight.

□

Returning to the proof of the theorem, suppose indirectly that for every element $u$ there is a dangerous set containing $t$ and $u$. That is, there is a family $\mathcal{D} := \{X_1, X_2, \dots, X_r\}$ of maximal dangerous sets containing $t$ so that $V = \cup(X_i : i = 1, \dots, r)$. By the maximality of $X_i$, each $X_i$ has at least two elements and hence $g(X_i) \geq 2$ and $p(X_i) \geq g(X_i)-1 \geq 1$.

By (26), $r$ is at least 2. By Claim A, we cannot have $r = 2$, that is, $r \geq 3$. It follows from Claim D that the set $Z_0 := \{t\}$ and the sets $Z_i := X_i - t$ are pairwise disjoint 1-tight sets. Since $p(Z_0 \cup Z_i) = p(X_i) \geq 1$, Claim 2 implies that $\mathcal{Z} := \{Z_0, Z_1, \dots, Z_r\}$ is a 1-tight $p$-full partition of $V$, contradicting (28). This contradiction proves the theorem.

□□



## 5. Proofs and an extension of Theorem 2

Let $p$ and $g$ be the same as in Theorem 1 and suppose that, in addition, (8) holds. In the preceding section we showed for every element $t$ with $g(t) \geq 1$ that there is an element $u$ with $g(u) \geq 1$ so that splitting off the pair $\{t, u\}$ does not destroy (4). Our present purpose is to show that, with an appropriate choice of $t$, such a splitting off operation automatically preserves (8), as well. This will immediately imply Theorem 1.

We call an element $z \in V$ **critical,** if $g(z) = 1$ and $z$ belongs to a 1-tight set. Let $K$ denote the set of critical elements and $P$ the set of elements where $g$ is positive. We define an auxiliary graph $D = (K, A)$ so that an edge $uv$ belongs to $A$ if

$$p_P(\{u, v\}) \geq 1 \tag{29}$$

where $p_P$ denotes the projection of $p$ to $P$. (Equivalently, $uv$ belongs to $A$ if there is a set $X$ with $p(X) \geq 1$ so that $u, v \in X$ and $g(X - \{u, v\}) = 0$.) We choose a node $t$ by the following rule.

**Selecting node $t$.** *If $A$ is empty, then let $t$ be any element of $P$. If the maximum degree of $D$ is two and one of the components of $D$ is a tirangle, then let $t$ be an element of this triangle. In the remaining case let $t$ be a node of $D$ with maximum degree.*

**Lemma 4.** *Let $t$ be the element of $V$ defined above and let $\mathcal{F} := \{V_1, \dots, V_h\}$ be a $p$-full partition of $V$ for which $h - 1 = g(V)/2$. If $V_i$ is the member of $\mathcal{F}$ containing $t$, then*

$$g(v) \leq 1 \text{ for every } v \in V_i. \tag{30}$$

*Proof.* Suppose first that the lemma holds for the special case when $g$ is positive everywhere (that is $P = V$). This implies the general result. Indeed, suppose that $V - P \neq \emptyset$ and consider the projecion $p_P$ of $p$ on $P$. The auxiliary graph belonging to the projection is the same as the one belonging to $p$, that is, $D$. (24) and (25) hold by Theorem 4 which is equivalent to saying that, (4) and (8) hold for $p_P$. Furthermore, $g(X) \geq p(X) \geq 1$ for each member $X$ of $\mathcal{F}$. Since $g(V - T) = 0$, each member of $\mathcal{F}$ intersects $T$ and hence $\mathcal{F}_P = \{X \cap P : X \in \mathcal{F}\}$ is a $p_P$-full partition of $P$ for which $h - 1 = g(P)/2$. Since $g$ is positive on every element of $P$, (30) follows.

Hence it suffices to prove the theorem for the special case when $P = V$. In this case $p_P = p$, every 1-tigth set consists of one element and an element $v$ is critical if and only $g(v) = p(v) = 1$ (that is, $\{v\}$ is 1-tight).

*Claim A.* Every component of $D$ is a clique, or a path, or a circuit.

*Proof.* It suffices to show that the neighbours of a node of degree at least three form a clique of $D$. Suppose that a node $a$ of $D$ is adjacent in $D$ to nodes $b, c, d$. We show that $bc \in A$. Indeed, by Claim 1 implies $p(X) \geq 1$ for $X := \{a, b, c\}$. For $Y := \{a, d\}$ we have $1 + 1 \leq p(X) + p(Y) \leq p(X - Y) + p(Y - X) = p(\{b, c\}) + p(d) = p(\{b, c\}) + 1$ from which $p(\{b, c\}) \geq 1$, that is, $bc \in A$.

□

*Claim B.* Let $ab$ and $bc$ be edges of $D$ and let $U \subseteq V - \{a, c\}$ be a set for which $b \in U$, $p(U) \geq 1$, $|U| \geq 2$. Then $p(U + b) \geq 1$, $p(U + c) \geq 1$, $p(U + a - b) \geq 1$, $p(U + c - b) \geq 1$.

*Proof.* By the symmetrical role of $a$ and $c$, it suffices to show the first inequality.

By (1a) we have $1 + 1 \leq p(U) + p(\{b, c\}) \leq p(U + c) + p(b) = p(U + c) + 1$ from which $p(U + c) \geq 1$. Analogously, $p(U + a) \geq 1$. Furthermore, by (1b), $1 + 1 \leq p(U + a) + p(\{bc\}) \leq p(U + a - b) + p(c) = p(U + a - b) + 1$ from which $p(U + a - b) \geq 1$. The $p(U + c - b) \geq 1$ follows analogously.

□

*Claim C.* $\mathcal{F}$ contains at least two 1-tight members. If it contains exactly two, then $g(V_i) = 2$ for all other members of $\mathcal{F}$.

*Proof.* Let $\alpha$ denote the number of 1-tight members of $\mathcal{F}$. If a member of $\mathcal{F}$ is not 1-tight, then $g(V_i) \geq 2$. Therefore $2h - 2 = g(V) = \sum_i g(V_i) \geq \alpha + 2(h - \alpha) = 2h - \alpha$ and the claim follows.

□

Suppose that the 1-tight members of $\mathcal{F}$ are $V_1, \ldots, V_\alpha$. Since we have assumed that $P = V$, these sets are one-element sets. Let $V_i = \{v_i\}$ $(1 \leq i \leq \alpha)$ and $V_0 := \{v_1, \ldots, v_\alpha\}$. Since $\mathcal{F}$ is $p$-full, $V_0$ is a clique of $D$. Therefore the maximum degree of a node of $D$ is at least $\alpha - 1$.

*Case 1.* $\alpha = 2$. (30) trivially holds if $V_i$ is 1-tight. If $V_i$ is not 1-tight, then, by Claim C, $g(V_i) = 2$. Since $g$ is positive everywhere, (30) follows.

*Case 2.* $\alpha \geq 3$. By the selection rule, $t$ has at least two neighbours and the neighbours form a clique.

*Claim D.* Let $U$ be a member of $\mathcal{F}$ with $|U| \geq 2$. Then there are no two adjacent edges $ab$ and $bc$ of $D$ so that $b \in U \subseteq V - \{a, c\}$.

*Proof.* Suppose indirectly that these edges exist. We show that

$$p(\{v_1, b\}) \geq 1. \qquad (31)$$

This is true if $v_1 = a$. If $v_1 \neq a$, then by Claim B and (1b) we have $1 + 1 \leq p(U + v_1) + p(U + a - b) \leq p(\{v_1, b\}) + p(a) = p(\{v_1, b\}) + 1$ from which (31) follows.

By applying Claim B to $\{a, b, v_1\}$ we obtain that $p(U + v_1) \geq 1$ and hence $1 + 1 \leq p(U + v_1) + p(\{a, b\}) \leq p(U + v_1 - b) + p(a) = p(U + v_1 - b) + 1$ from which

$$p(U + v_1 - b) \geq 1. \qquad (32)$$



Let $\mathcal{F}'$ denote the partition of $V$ arising from $\mathcal{F}$ by replacing $U$ by $U' := \{b\}$ and $U'' := U - b$. By (31), $p(V_1 \cup U') = p(\{v_1, b\} \geq 1$. By (32), $p(V_1 \cup U'') = p(U + v_1 - b) \geq 1$. Therefore Claim 1 implies that $\mathcal{F}'$ is $p$-full. From this, in turn, we get $\dim(p) - 1 \geq (h + 1) - 1 = g(V)/2 + 1$, contradicting (8).

□

Let us consider the member $V_i$ of $\mathcal{F}$ containing $t$ and let $C$ be the component of $D$ containing $t$. Since $V_0$ is a clique of at least three elements, the choice of $t$ implies that $C$ is also a clique of at least three elements. By Claim D, $C$ may have at most one element not in $V_i$. Hence $C$ and $V_0$ are disjoint.

We show that $C \subseteq V_i$. Suppose indirectly that there is an element $b \in C - V_i$ (that is, $C - V_i = \{b\}$) and $b$ belongs to $V_j$. Now $|V_j| \geq 2$ for otherwise $V_j = \{b\}$ and $1 \leq p(V_j) \leq g(b) = 1$ from which $V_j$ is 1-tight, that is, $j \leq \alpha$, contradicting that $V_0 \cap C = \emptyset$. Now $|C| \geq 3$ and $|C - V_i| = 1$ imply $|C \cap V_i| \geq 2$, that is, the set $U := V_j$ has an element $b$ having two neighbours outside $V_j$. This contradicts Claim D.

Since $g(V_i) \geq g(C) \geq |C| \geq \alpha$ we obtain that $2h - 2 = g(V)/2 = \sum(g(X) : X \in \mathcal{F})/2 \geq \alpha + g(V_i) + 2(h - \alpha - 1) \geq \alpha + \alpha + 2h - 2\alpha - 2 = 2h - 2$ from which equality follows throughout and, in particular, $g(C) = |C|$. That is, $V_i = C$ and (30) follows.

□ □

By Theorem 5 there is an element $u$ for which (27) holds.

**Lemma 5.** *For $u$ and $t$*

$$\dim(p^{tu}) - 1 \leq g^{tu}(V)/2 (= g(V)/2 - 1). \qquad (33)$$

*Proof.* Suppose indirectly that (33) fails to hold and let $\mathcal{F} := \{V_1, \ldots, V_h\}$ be a $p^{tu}$-full partition of $V$ for which $h - 1 > g^{tu}(V)/2$, that is, $h \geq g(V)/2 + 1$. $\mathcal{F}$ contains at least one 1-tight member (1-tight with respect to $p$ for otherwise we would have $h \geq g(V)/2 + 1 = \sum_i g(V_i)/2 + 1 \geq 2h/2 + 1$. Therefore there is a critical element of $V$ and then, by the selection rule, $t$ is critical. Since $p \geq p^{uv}$, $\mathcal{F}$ satisfies (5), and hence $\mathcal{F}$ is $p$-full and it must have exactly $\dim(p)$ members.

Let $V_i$ denote the member of $\mathcal{F}$ containing $t$. If $u$ also belongs to $V_i$, then by Lemma 4, $g(u) = 1$. Since $p(\{t, u\}) \geq 1$ and $g(\{t, u\}) = 2$, the set $\{t, u\}$ is dangerous, contradicting the assumption that (27) holds. Therefore $u$ does not belong to $V_i$. Let $u$ belong to $V_j$.

None of $V_i$ and $V_j$ is 1-tight (with respect to $p$) since if $V_i$, say, is 1-tight, then $p^{tu}(V_i) = 0$, contradicting the assumption that $\mathcal{F}$ is $p^{tu}$-full. Since $h - 1 = g(V)/2$, $\mathcal{F}$ includes at least two 1-tight members: $V_1$ and $V_2$. For $X := V_1 \cup V_i$ and $Y := V_2 \cup V_i$ we obtain $1 + 1 \leq p(X) + p(Y) \leq p(X - Y) + p(Y - X) \leq p(V_1) + p(V_2) = 1 + 1$ from which $p(V_1 \cup V_i) = p(X) = 1$. Since $u \notin V_1 \cup V_i$, $p^{tu}(V_1 \cup V_i) = 0$, contradicting the assumption that $\mathcal{F}$ is $p^{tu}$-full.

□ □

*Proof of Theorem 1.* As we have proved already the necessity of the conditions, we turn to the sufficiency and assume (4) and (8) to hold. It was also shown that $g(V)$ can be assumed to be even. We use induction on $g(V)$. The theorem is trivial if there is at most one element $t$ for which $g(t) > 0$. In this case (4) and the symmetry of $p$ imply that $p$ is identically zero and therefore the empty graph $G = (V, \emptyset)$ satisfies the requirements. Thus we may assume (26). Condition (28) holds since if there is a 1-tight $p$-full partition $\mathcal{F} := \{V_1, \ldots, V_h\}$ $(h \geq 4)$, then $h = \sum g(V_i) = g(V)$ and, by (8), $h - 1 \leq g(V)/2$ from which $h \leq 2$.

By Theorem 5 and Lemma 5 there is a pair $\{t, u\}$ of elements of $V$ with $g(t) > 0$, $g(u) > 0$ for which (27) and (33) hold. (27) means that $(g^{tu}, p^{tu})$ satisfies (4). (33) means that $(g^{tu}, p^{tu})$ satisfies (8). By induction, there is a graph $G'$ covering $p^{tu}$ and majorized by $g^{tu}$. Let $G$ be a graph arising from $G'$ by adding the edge $tu$. It follows from the definition of $g^{tu}$ and $p^{tu}$ that $G$ covers $p$ and is majorized by $g$.

□

We conclude the section by proving Theorem 2. We start by showing that (7b) can be formulated in a more aesthetic form.

**Lemma 6.** (7b) *is equivalent to the existence of a partition of V into tight sets.*

*Proof.* If such a partition exists, then (7b) is satisfied. Conversely, assume that (7b) holds and let $\mathcal{F}$ be a family of tight sets covering $V$ so that $|\mathcal{F}|$ is minimum. We are done if $\mathcal{F}$ is a partition of $V$ so suppose that $X \cap Y \neq \emptyset$ for two members of $\mathcal{F}$. Then $X \cup Y = V$, for otherwise $X, Y$ are crossing and the union of two crossing tight sets is tight, that is, $X$ and $Y$ could be replaced by $X \cup Y$, contradicting the minimality of $|\mathcal{F}|$.

Now $X \cup Y = V$ implies that $\mathcal{F} = \{X, Y\}$. We have $g(X) = p(X) = p(V - X) \leq g(V - X) \leq g(Y) = p(Y) = p(V - Y) \leq g(V - Y) \leq g(X)$ from which equality follows everywhere. In particular, $X - Y$ and $Y - X$ are tight and $g(V - X) = g(Y)$, that is, $g(X \cap Y) = 0$, and hence $X \cap Y$ is also tight. Therefore $\{X - Y, X \cap Y, Y - X\}$ is a partition of $V$ into tight sets.

□

*Proof of Theorem 2.* We have proved already the necessity of conditions (10) and (11). To derive their sufficiency, let $g' : V \to \mathbf{Z}_+$ be a function for which (4) is satisfied and suppose that $g'$ is minimal with respect to this property. Then every element of $V$ belongs to a tight set. By Lemma 6 there is a partition $\mathcal{P}$ of $V$ into tight sets. From (10) $g'(V) = \sum(g'(X) : X \in \mathcal{P}) = \sum(p(X) : X \in \mathcal{P}) \leq 2\gamma$. Increase $g'$ on an arbitrary element by $2\gamma - g'(V)$. The resulting function $g$ satisfies (4) and $g(V) = 2\gamma$. Hence (11) implies (8). Therefore the conditions of Theorem 1 are met and hence there is a graph $G$ covering $p$ and majorized by $g$. Clearly, $G$ has at most $g(V)/2 = \gamma$ edges.

□

Finally, we show the following extension of Theorem 2.

**Theorem 6.** *Let $p : 2^V \to \mathbf{Z}_+$ be a symmetric, crossing supermodular function, T a subset of V, and $\gamma$ a positive integer. There exists an undirected graph $G = (V, E)$ with at most $\gamma$ edges so that*

$$d_G(X) \geq p(X) \textit{ for every subset } X \subset V \textit{ separating } T \tag{34}$$



*if and only if*

$$\sum(p(X) : X \in \mathcal{P}) \leq 2\gamma \textit{ for every T-separating sub-partition } \mathcal{P} \textit{ of } V \tag{35}$$

*and*

$$\dim_T(p) - 1 \leq \gamma. \tag{36}$$

*If* (35) *and* (36) *hold, then graph G may be chosen so that $d_G(v) = 0$ for every node $v \in V - T$ (that is, all edges of G have both end-nodes in T).*

*Proof.* The proof of the necessity of conditions is analogous to that in Theorem 2. For the sufficiency, we assume (35) and (36) to hold. Since these coincide, respectively, with (20) and (21), Theorem 2.1 applies and it implies (22) and (23). Now Theorem 2, when applied to $p_T$, implies the theorem.

□

## 6. Algorithmic aspects

The proof of Theorem 1 gives rise to a polynomial time algorithm, provided that an oracle to

$$\text{minimize } (m(X) + d_z(X) - p(X) : X \subset V) \tag{37}$$

is available where $m$ is a function on $V$ and $z$ is a non-negative functions on the (unordered) pairs of the elements of $V$, and $d_z(X) := \sum(z(uv) : u \in X, v \notin X)$. Minimization means that a set $X$ attaining the minimum is computed. [Grötschel, Lovász and Schrijver, invented a polynomial time algorithm (that relies on the ellipsoid method) to compute the minimum of a submodular function using only an evaluation oracle. Because $p$ is supermodular, $g + d_z - p$ is submodular and therefore the required minimization oracle (37) is, in principle, always available. In the applications on hypergraph connectivity to be described in the next section, (37) can be realized via network flow techniques.]

The algorithm consists of a sequence of splitting operations which are carried out as long as there are two elements with positive $g$. We describe only the first step when a certain pair $\{t, u\}$ is split off $z(\{t, u\})$ times. The same rule applies to the pair $(g', p')$ at any intermediate step where $g'(v) := d_z(v) (v \in V)$ and $p'(X) := p(X) - d_z(X) (X \subset V)$ and $z(\{u, v\})$ denotes the number of how many times the pair $\{u, v\}$ have been perviously split off.

We say that a pair $\{t, u\}$ is **simple,** if $\min(g(u), g(t)) = 1$, and **multiple** if $\min(g(u), g(t)) \geq 2$. With the help of (37), compute $\alpha_1 := \min(g(X) - p(X) : t \in X, u \in X)$. (This can be done by applying (37) to $m$ where $m(t)$ and $m(u)$ are the negative of a suitably big number and $m(v) := g(v)$ for $v \in V - t - u$. This way the minimizing set $X$ is forced to contain $t$ and $u$).

If $\{t, u\}$ is simple, we carry out $\alpha := \min(\lfloor \alpha/2 \rfloor, 1)$ (which is 0 or 1) splitting at $\{t, u\}$ and say that the splitting is **simple.** If $\{t, u\}$ is multiple, we carry out $\alpha := \min(\lfloor \alpha/2 \rfloor, g(t) - 1, g(u) - 1)$ splitting operations at $\{t, u\}$ in one step. and call this step a **multiple splitting.** (In this case $\alpha$ can be 0, 1 or bigger.) A splitting (simple or multiple) is called **non-trivial** if $\alpha > 0$.

Let $P$ the set of elements where $p$ is positive. For a given subset $Z \subset T$ we can compute $p_P(Z)$ with one application of (37). (Define $m(v) = 0$ for $v \in V - P$, $m(v) = M$ for $v \in P - Z$, and $m(v) = -M$ for $v \in Z$, where $M$ is a big number. Then the minimum is attained on a set $X$ for which $Z = X \cap T$ and $p_P(Z) = p(X)$.)

Making use of Claim A of Section 5, the auxiliary graph $D = (K, A)$ belonging to $(g, p)$ can be computed by at most $2|V|$ oracle calls. Hence we can determine the element $t \in P$ as described before Lemma 4. If $g(t) > 1$, (which can happen if $A = \emptyset$) then choose any multiple pair $\{t, u\}$ that has not yet been considered for splitting and perform a multiple splitting at $\{t, u\}$. If $g(t) = 1$, Choose an element $u \in P$ so that the pair $\{t, u\}$ has not been considered for simple splitting and perform a simple splitting at $\{t, u\}$.

By this rule, every pair appears in at most one simple and in at most one multiple splitting. Note that performing a multiple splitting does not affect the auxiliary digraph $D$. Therefore the rules above comply with the choice described in the proof of Theorem 1 in Sections 4 and 5. The proof ensured that, as long as there are two elements with positive $g$, there is a non-trivial splitting. We can conclude that the algorithm terminates after at most $2|V|^2$ splitting operations. (This can be reduced to $O(|V|)$ by using a technique described in [F1992, pp. 51].)

As far as the algorithmic side of Theorem 2 is concerned, the proof at the and of Section 4 required the computation of a minimal function $g$ satisfying (4). This can be done by calling (37) at most $|V|$ times. (Indeed, start with a large $g$ satisfying (4), consider the elements of $V$ in any order $v_1, \dots, v_n$ and iterate the following step. Compute $\mu_i := \min(g(X) - p(X) : v_i \in X \subset V)$ and reduce $g(v_i)$ by $\min(\mu_i, g(v_i))$.)

Finally, we show how Theorem 6 can be handled algorithmically. By Theorem 3 all we need to show is that the minimization oracle (37) is available with respect to $p_T$. Let $m_T$ be a function on $T$ and $z_T$ a non-negative function on the pair of elements of $T$. Define $m : V \to \mathbf{Z}$ by $m(v) := m_T(v)$ if $v \in T$ and $m(v) = 0$ if $v \in V - T$. For $u, v \in V$ define $z(\{u, v\}) := z_T(\{u, v\})$ if $u, v \in T$ and zero otherwise. It is easy to see that if $X$ is a minimizer for (37), then $X' := X \cap T$ minimizes $m_T(X') + d_{z_T}(X') - p_T(X')$ over subsets $X'$ of $T$.

## 7. Increasing the connectivity of hypergraphs

Given a hypergraph $H' = (V, A')$, a subset $\emptyset \subset C \subset V$ is called a **component** of $H'$, if $d_{H'}(C) = 0$ and $d_{H'}(X) > 0$ for every $\emptyset \subset X \subset C$. Given a subset $T \subset V$, we let $c_T(H')$ denote the number of components of $H'$ having a non-empty intersection with $T$. Given a positive integer $k$, $H'$ is said to be ***k*-edge-connected in *T*,** if

$$d_{H'}(X) \geq k \text{ for every subset } \emptyset \subset X \subset V \text{ separating } T. \tag{38}$$

When $T = V$ in (38) we say that $H'$ is ***k*-edge-connected.**

Suppose we are given a hypergraph $H = (V, A)$, a specified subset $T$ of $V$, and a positive integer $\gamma$. When is it possible to add at most $\gamma$ new graph-edges to $H$ so that the resulting hypergraph $H^+ = (V, A^+)$ is $k$-edge-connected in $T$? When the starting hypergraph $H$ itself is a graph, this problem was solved for $T = V$ by Watanabe and Nakamura [WN1987] and for arbitrary $T$ by Frank [F1992]. In the case $T = V$, the problem was solved by E. Cheng [C1999] when $H$ is a $(k-1)$-edge-connected hypergraph and by J. Bang-Jensen and B. Jackson [BJ1999] when $H$ is arbitrary. Here we prove the following generalization of the theorem of Bang-Jensen and Jackson.



**Theorem 7.** *A hypergraph $H = (V, A)$ can be made $k$-edge-connected in $T$ by adding at most $\gamma$ new graph-edges if and only if*

$$\sum(k - d_H(X) : X \in \mathcal{P}) \leq 2\gamma \text{ for every sub-partition } \mathcal{P} \text{ of } V \text{ separating } T \tag{39}$$

*and*

$$c_T(H') - 1 \leq \gamma \text{ for every hypergraph } H' = (V, A') \text{ arising from } H \text{ by leaving out } k-1 \text{ hyperedges.} \tag{40}$$

*If* (39) *and* (40) *hold, the new edges can be chosen so as to connect elements of $T$.*

Note that condition (40) is void if $H$ has less than $k-1$ hyperedges. When $H = (V, A)$ is a hypergraph and $A_0 \subseteq A$ is a subset of hyperedges, the hypergraph $H' = (V, A - A_0)$ is said to arise from $H$ by **leaving out** $A_0$. For a subset $\emptyset \subset Z \subset V$, we say that $H' := (V', A')$ arises from $H$ **by contracting** $Z$, if $V' := V - Z + z$, where $z$ is a new node, and $A' = \{X' \subseteq V' : X' = X - Z + z \text{ for } X \in A, X \cap Z \neq \emptyset, X - Z \neq \emptyset\} \cup \{X' \subseteq V' : X' = X \text{ for } X \in A, X \cap Z \neq \emptyset\}$. In a hypergraph $H$ the **contraction of a sub-partition** $\mathcal{P}$ of $V$ means that we contract separately the members of $\mathcal{P}$.

*Proof.* To see the necessity of the conditions, suppose that there is a graph $G = (V, E)$ with $|E| \leq \gamma$ for which $H^+ := (V, A + E)$ is $k$-edge-connected in $T$. Then $d_H(X) + d_G(X) \geq k$ for every set $X$ separating $T$. Hence $\sum(k - d_H(X) : X \in \mathcal{P}) \leq \sum(d_G(X) : X \in \mathcal{P}) \leq 2\gamma$ for every sub-partition $\mathcal{P}$ of $V$ separating $T$, that is, (39) holds.

Let $H'$ be the hypergraph occurring in (40). Let $G' = (V', E')$ denote the graph arising from $G$ by contracting the components of $H'$ and let $T'$ denote the subset of elements of $V'$ which arise by contracting a $T$-separating component of $H'$. Because $H'$ arises from $H$ by leaving out exactly $k-1$ hyperedges, $d_H(X) < k$ for every subset $X$ being the union of some components of $H'$. Since such a set $X$ is $T$-separating and $H + G$ is $k$-edge-connected, at least one edge of $G$ must connect $X$ and $V - X$. Therefore $T'$ belongs to one component of $G'$ and hence $c_T(H') - 1 = |T'| - 1 \leq |E'| \leq |E| \leq \gamma$, proving (40).

To prove the sufficiency, let us assume that conditions (39) and (40) hold. The following claim is well-known. Its proof is simply by checking the possible contributions of one hyperedge to the two sides of (41).

*Claim 4.* For arbitrary subsets $X, Y \subseteq V$,

$$d_H(X) + d_H(Y) \geq d_h(X \cap Y) + d_H(X \cup Y). \tag{41}$$

Moreover, if equality occurs, then there is no hyperedge $Z$ of $H$ for which

$$Z \subseteq X \cup Y, Z \cap (X - Y) \neq \emptyset, Z \cap (Y - X) \neq \emptyset. \tag{42}$$

□

Define a set-function $p$ on $V$ by $p(\emptyset) = p(V) = 0$ and

$$p(X) := \max(0, k - d_H(X)) \text{ for } \emptyset \subset X \subset V. \tag{43}$$

Then $p$ is symmetric and, by (41), crossing supermodular. Clearly, a set $E$ of at most $\gamma$ edges satisfies the requirement of the theorem precisely if $G = (V, E)$ satisfies (34). By Theorem 6, all we have to show is that (20) and (21) hold. Since (20) is exactly the same as (39), our goal is to derive (21).

Suppose indirectly that (21) does not hold, that is, there is a $p$-full partition $\mathcal{F} := \{V_1, \ldots, V_h\}$ of $V$ so that

$$h - 2 \geq \gamma. \tag{44}$$

Let $H_0 = (V, A_0)$ be the hypergraph consisting of those hyperedges of $H$ intersecting at least two members of $\mathcal{F}$.

**Lemma 7.** *$H_0$ has exactly $k - 1$ hyperedges.*

*Proof.* For the lemma we may assume that each member of $\mathcal{F}$ consists of one element for otherwise we can contract $\mathcal{F}$. We claim that the number $\alpha$ of those members of $\mathcal{F}$ for which $p(X) = 1$ is at least four. Indeed, $h - 2 \geq \gamma \geq \sum(p(X) : X \in \mathcal{F})/2$. Hence $2h - 4 \geq \sum(p(X) : X \in \mathcal{F}) \geq \alpha + 2(h - \alpha) = 2h - \alpha$, from which $\alpha \geq 4$. That is, $V$ has a four-element subset $S = \{s_i : 1 \leq i \leq 4\}$ for which $p(s_i) = 1$ $(1 \leq i \leq 4)$. Since $\mathcal{F}$ is $p$-full, $p(X) \geq 1$ for every subset $\emptyset \subset X \subset V$.

*Claim A.* $p(X) = 1$ holds for every $s_i\bar{s}_j$-set $X$.

*Proof.* Induction on $|X|$. If $|X| = 1$, then $p(X) = p(s_i) = 1$. Suppose now that $|X| \geq 2$ and that the claim is true for any proper subset of $X$ containing $s_i$. For any element $z \in X - s_i$ we have $1 + 1 \leq p(X) + p(\{z, s_j\}) \leq p(X - z) + p(s_j) = 1 + 1$ from which $p(X) = 1$.

□

By the definition of $p$, Claim 4 implies:

*Claim B.* If $X$ and $Y$ are two crossing subsets of $V$ for which

$$p(X) = p(Y) = p(X \cap Y) = p(X \cup Y) = 1, \tag{45}$$

then there is no hyperedge $Z$ in $H_0$ satisfying (42).

□

*Claim C.* Every hyperedge $Z$ of $H_0$ contains $s_j$.

*Proof.* Suppose indirectly that $s_j \notin Z$. Let $z$ be an arbitrary element of $Z$. If $|Z| = 2$, then, by $|S| = 4$, there is an element $s_i \in S - Z - s_j$. If $|Z| \geq 3$, then let $s_i$ be an arbitrary element of $S - z - s_j$. ($s_i$ may or may not belong to $Z$). Let $X := \{z, s_i\}$ and $Y := Z - z + s_i$. By this definition, $X, Y$ are crossing and $Z$ satisfies (42). On the other hand, (6.8) holds by Claim A and hence $Z$ does not satisfy (42) by Claim $B$, a contradiction.

□



Since $1 = p(s_j) = k - d_{H_0}(s_1)$, the element $s_1$ is contained in exactly $k - 1$ hyperedges of $H_0$. By Claim $C$ every hyperedge of $H_0$ contains $s_1$ and, the lemma follows.

□□

Returning to the proof of the theorem (and regarding that $\mathcal{F}$ was assumed to consist of one-element members only for the proof of Lemma 7), we can conclude that, by leaving out the $k - 1$ hyperedges of $H_0$ from $H$, we obtain a hypergraph $H'$ in which every hyperedge is a subset of a member of $\mathcal{F}$. Since each member $V_i$ of $\mathcal{F}$ is $T$-separating, $V_i$ includes a $T$-separating component of $H'$, therefore $c_T(H') \geq h$. By combining this with (44), we obtain that (40) fails to hold, contradicting the assumption.

□□□

**Corollary 2.** *Let $H' = (V, A')$ be a hypergraph which is $k$-edge-connected in $T$. Suppose that $H'$ contains a graph-edge $u = uv$ for which $v \notin T$. Then there are two elements $x, y$ of $T$ so that replacing edge $e$ by a new edge $f := xy$, the resulting hypergraph $H^+ = (V, A^+)$ is also $k$-edge-connected in $T$.*

*Proof.* Let $H = (V, A)$ be the hypergraph arising from $H'$ by deleting $e$. $H$ can be made $k$-edge-connected in $T$ by adding one edge. Then, by Theorem 7, $H$ can be made $k$-edge-connected by adding one edge with both end-nodes in $T$.

□

We conclude the paper by mentioning that the algorithms outlined in the preceding section can be specialized to the hypergraph connectivity augmentation problem. It can be shown by using standard reduction techniques that the required oracle (37) is available via max-flow min-cut computations.

The algorithm of Bang-Jensen and Jackson also uses the splitting operation but it requires a kind of inverse-splitting operation, as well. Our algorithm is simpler in that it does not need inverse-splitting.

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