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AN ALGORITHM FOR THE UNBOUNDED MATROID INTERSECTION POLYHEDRON

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An algorithmic relation, between results of Edmonds, Cunningham, McDiarmid and Gröflin-Hoffman, is discussed.

INTRODUCTION

Throughout the paper we suppose two matroids M_1 and M_2 (without loops) on a finite groundset E with rank functions r_1, r_2 and a non-negative weight function w on E. Let us denote the maximum cardinality of a common independent set in A by r(A). It is known that $r(A) = \min(r_1(X) + r_2(A-X))$ [2]. In [2] Edmonds also proved the Matroid Polyhedron Intersection Theorem:

THEOREM 1. The linear system

$$x \geq 0, \quad x(A) \leq \min(r_1(A), r_2(A)) \quad \text{for} \quad A \subseteq E \tag{1}$$

defines the convex hull P of common independent sets of M_1 and M_2 and (1) is totally dual integral.

(A linear system $Ax \leq b$ is called totally dual integral or TDI if the linear programming dual $\min(yb: y \geq 0, yA = w)$ has an integral optimum for each integral w whenever the optimum exists. A basic feature of TDI systems is that they define a polyhedron whose facets contain integer points [4,8]).

Edmonds [3] also provided a good algorithm for optimizing a linear objective over P and for producing an optimal solution to the linear programming dual.

Fulkerson [6] proposed to investigate an unbounded polyhedron in connection with matroid intersections. Denoting by P_k the convex hull of k-element common independent sets of M_1 and M_2 Fulkerson conjectured and later Cunningham [1] and McDiarmid [9] independently proved that

$$P_k + R_+^E = \{x: x(A) \geq \max(0, k-r(E-A)) \quad \text{for} \quad A \subseteq E\}. \tag{2}$$

Finally, Gröflin and Hoffman [7] showed that the linear system in (2) is TDI.

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The original proof of Gröflin and Hoffman relies on the concept of lattice polyhedra and does not seem to provide an algorithm for finding the optimal solutions in the corresponding primal and dual linear programs. The purpose of this note is to present a constructive proof for the Gröflin-Hoffman theorem by describing such an algorithm.

For a subset $A \subseteq E$ and a weighting w on E, χ_A denotes the incidence vector of A and $w(A) = \Sigma(w(e): e \in A)$. For a number x let $x^+ = \max(0,x)$. Given a family $\mathcal{F}$ of subsets, $F \in \mathcal{F}$ is called *w-minimal* in $\mathcal{F}$ if $w(F) \leq w(X)$ for each $X \in \mathcal{F}$.

PROOF AND ALGORITHM

Without loss of generality we can suppose that $r_1(E) = r_2(E) = r(E) = k$. Then the theorem of Gröflin and Hoffman mentioned in the Introduction is as follows.

THEOREM 2. [7] For every integral weight function $w \geq 0$ the dual pair of linear programs

$$\min(wx: x(A) \geq k-r(E-A)) = \max(\Sigma_{A\subseteq E}\, y(A)(k-r(E-A)): \qquad (3)$$
$$y(A) \geq 0,\ \Sigma_{A\subseteq E}\, y(A)\chi_A = w)$$

have integral optimum solutions.

Proof and algorithm. Since an optimal integral vector in the left-hand side of (3) corresponds to a common base of M_1 and M_2, to prove Theorem 2 we have to find a common base B and an integral vector y which provide equality in (3). By complementary slackness, this is equivalent to showing that $y(A) > 0$ implies $(|B \cap A| =)\ x(A) = k-r(E-A)$ that is $B-A$ is a maximal cardinality common independent subset of $E-A$. Such a set A is called admissible (with respect to B). Thus our purpose is to find a common base B and a feasible vector y so that $y(A) > 0$ only if A is admissible.

In [5] we proved the following version of Theorem 1.

LEMMA 3. Given M_1, M_2, w, a common base B is w-minimal if and only if there are weights w_1, w_2 such that $w_1+w_2=w$ and B is a w_i-minimal base of M_i $i = 1,2$. Moreover, if w is integer-valued, w_i can be chosen integer-valued.

The proof of this lemma in [5] is by describing an algorithm which provides both the w-minimal B and the required weight splitting w_1, w_2. The present method starts with these data and constructs y from them.

Let $p_1 < \ldots < p_m$ and $q_1 < \ldots < q_n$ be the distinct values of the weights w_1, w_2, respectively and set $p_0 = q_0 = -\infty$. Arrange the elements of E into a two-dimensional array so that $x \in E$ is in entry (i,j) if $w_1(x) = p_i$ and $w_2(x) = q_j$. Note that there may be entries with more than one element in them. For an entry (i,j), set $A_{ij} = \{v \in E: w_1(v) \geq p_i,\ w_2(v) \geq q_j\}$. Call an entry (i,j) critical if $p_i + q_j > 0$ and $p_{i-1} + q_{j-1} < 0$. The key observation is the following

LEMMA 4. If (i,j) is critical, the set A_{ij} is admissible.

Proof. Let $A_1 = \{v \in E: w_1(v) < p_i\}$ and $A_2 = \{v \in E: w_2(v) < q_j\}$. Then $A_1 \cup A_2 = E-A_{ij}$ and since (i,j) is critical, $A_1 \cap A_2 = \emptyset$. What we show is that $B \cap A_h$ is a maximal cardinality independent subset of A_h in M_h $(h = 1,2)$. If, indirectly, there exists an element $v \in A_h-B$ such that $(B \cap A_h) + v$ is independent in M_h then there is an element $u \in B-A_h$ such that $B + v - u$ is a base of M_h. Since $w_h(v) < w_h(u)$ the w_h-weight of $B + v - u$ is strictly smaller than that of B, a contradiction. /

We are now in a position to define the dual solution y. Set

$$y(A) = \begin{cases} (p_i+q_j)^+ - (p_{i-1}+q_j)^+ - (p_i+q_{j-1})^+ + (p_{i-1}+q_{j-1})^+, & \text{if} \\ \qquad A = A_{ij} \text{ for some } 1 \leq i \leq m,\ 1 \leq j \leq n & \\ 0, & \text{otherwise} \end{cases}$$

Obviously $y(A) \geq 0$ and $y(A_{ij}) > 0$ implies that (i,j) is critical, and then, by Lemma 4, A_{ij} is admissible. The feasibility of y, that is $\Sigma_{A \ni e}\, y(A) = w(E)$ for $e \in E$, immediately follows by applying the next trivial lemma for $c_{ij} = (p_i+q_j)^+$.

LEMMA 5. Let $C = (c_{ij})$ be an m by n matrix. Then for $1 \leq s \leq m$, $1 \leq t \leq n$

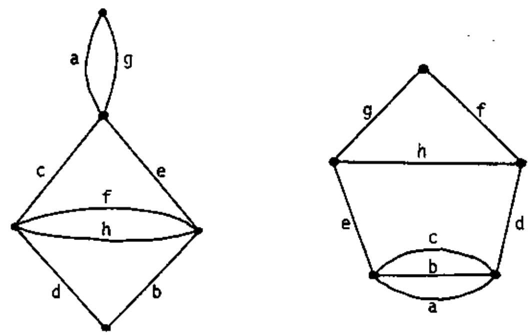
$$c_{st} = \Sigma_{i=1}^{s} \Sigma_{j=1}^{t} (c_{ij} + c_{i-1j-1} - c_{i-1j} - c_{ji-1})$$

where c_{oj} and c_{io} is meant to be 0.

By now the proof of Theorem 2 is complete. //

To illustrate the method consider the following example with two graphical matroids on eight elements

a	b	c	d	e	f	g	h	
1	3	1	5	5	0	8	6	: the weights



The common base and the weight splitting provided by the algorithm in [5]:
B = {a,e,d,f}

	a	b	c	d	e	f	g	h
w_1:	5	5	3	3	3	-2	5	3
w_2:	-4	-2	-2	2	2	2	3	3

The array:

5	a	b		g
3		c	d,e	h
-2			f	
	-4	-2	2	3

The optimal dual solution y:

$y(A_{23}) = 4 \quad A_{23} = \{d,e,g,h\}$

$y(A_{14}) = 1 \quad A_{14} = \{g,h\}$

$y(A_{31}) = 1 \quad A_{31} = \{a,b,g\}$

$y(A_{32}) = 1 \quad A_{32} = \{b,g\}$

$y(A_{22}) = 1 \quad A_{22} = \{c,d,e,g,h,b\}$.

Finally, we remark that Schrijver [10] proved a theorem for polymatroids analogous to the result of Gröflin and Hoffman. It can be shown that our method extends to polymatroids as well.

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