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Edge-Disjoint Paths in Planar Graphs

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Given a planar graph $G=(V, E)$, find k edge-disjoint paths in G connecting k pairs of terminals specified on the outer face of G. Generalizing earlier results of Okamura and Seymour (*J. Combin. Theory Ser. B* **31** (1981), 75–81) and of the author (*Combinatorica* **2**, No. 4 (1982), 361–371), we solve this problem when each node of G not on the outer face has even degree. The solution involves a good characterization for the solvability and the proof gives rise to an algorithm of complexity $O(|V|^3 \log|V|)$. In particular, the integral multicommodity flow problem is proved to belong to the problem class P when the underlying graph is outerplanar. © 1985 Academic Press, Inc.

1. INTRODUCTION

The central topic of this paper is the following.

Edge-disjoint paths problem. Given an undirect graph $G=(V, E)$, find k edge-disjoint paths in G connecting k specified pairs of (not necessarily distinct) nodes of G.

The integral multicommodity flow problem is a capacitated version where each edge e has an integral capacity indicating how many paths are allowed to go through e. This problem is known to be NP-complete [3] while the edge-disjoint paths problem is trivial for $k=1$ and has a deep solution for $k=2$ [8, 10]. For bigger k the status is not known in general but there are important results for special classes of graphs. For a survey, see [7].

Here we solve the problem when G is planar, the terminals are positioned on the outer face, and each node not on the outer face has even degree. The solution involves a necessary and sufficient condition as well as a polynomial time algorithm. In particular, the results show that the

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integral multicommodity flow problem for outer-planar graphs belongs to the problem class P.

Sometimes it is convenient to specify the pairs to be conected by supplying graph H in which there is an edge uv for each terminal pair u, v. With such a graph the edge-disjoint paths problem is as follows: Given two edge-disjoint graphs $G=(V, E)$ and $H=(V, F)$ on the same node set, find $|F|$ edge-disjoint circuits in $G+H$ each of which uses exactly one edge of H. We call the elements of E and F *supply* and *demand* edges, respectively.

Throughout the paper we work with a connected graph $G=(V, E)$. For a subset $X \subset V$, $\delta_G(X)=\{uv: uv \in E,\ u \in X,\ v \in V-X\}$. A co-boundary $\delta_G(X)$ is called a *cut* if both X and $V-X$ induce connected subgraphs (in this case we call X a *cut-inducing* set). We use the notation $d_G(X)$ for $|\delta_G(X)|$. For $X, Y \subset V$ and capacity function $g \in R_+^E$ let $d_g(X, Y)$ denote $\Sigma(g(uv): u \in X-Y,\ v \in Y-X)$ and $d_g(X) =: d_g(X, V-X)$. We call the number $d_H(X)$ the *congestion* of co-boundary $\delta_G(X)$.

A graph is said to be *Eularian* if the degree of each node is even. We consider every planar graph G to be embedded into the plane. By the *outer face* of G we mean the boundary circuit of the infinite region. A node of G is called *inner* if it is not on the outer face. A planar graph is called *outer-planar* if no inner node exists. A planar graph is called *(s, t)-planar* if $s, t \in V$ and s, t are outer nodes.

2. Preliminaries

A simple necessary condition for the solvability of the edge-disjoint paths problem is the *cut criterion*:

$$d_H(X) \leqslant d_G(X) \qquad \text{for every } X \subset V.$$

It is easily seen that the cut criterion holds true if the inequality is required to be true only for cut-inducing sets X.

We call the number $s(X) := d_G(X) - d_H(X)$ the *surplus* of $\delta_G(X)$. The cut criterion says that the surplus is non-negative. A cut $\delta_G(X)$ (and sometimes the set X) is said to be *saturated* if $s(X)=0$. The cut criterion is not sufficient in general as Fig. 1 shows. However Okamura and Seymour proved

THEOREM 1. [6]. *If G is planar, $G+H$ is Eulerian, and every edge of H connects two nodes of the outer face of G, then the cut criterion is necessary and sufficient for the solvability of the edge-disjoint paths problem.*

Note that neither the assumption that $G+H$ is Eulerian nor the restriction on H can be removed as shown by examples in Fig. 1 and Fig. 2.

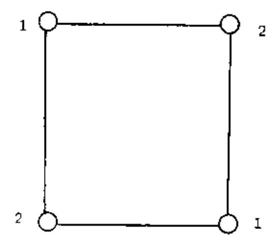

FIGURE 1

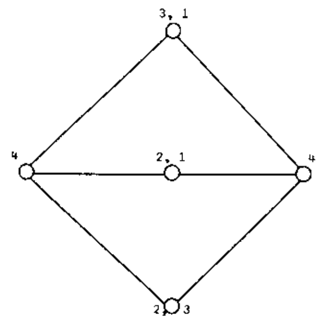

FIGURE 2

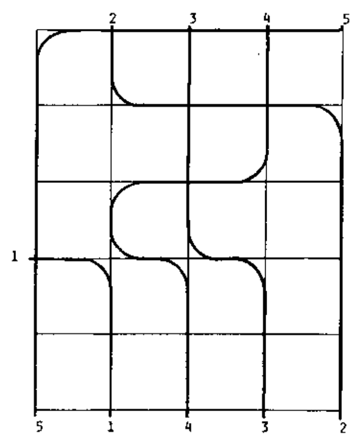

FIGURE 3

There are other special cases where the cut criterion is sufficient [7] but it is important to find further necessary criteria for situations when the cut criterion is not sufficient in general.

In [1] the edge-disjoint paths problem was solved in such a special case which we call the *grid model.* In a rectilinear grid (or plane lattice) we are given a closed rectangle T (bounded by lattice lines) and k pairs of distinct lattice points on the boundary of T. The rectangle T defines a finite subgraph G_T of the plane grid in the natural way (which has mn nodes when m horizontal and n vertical grid lines intersect T). A problem instance of the edge-disjoint paths problem in this grid model along with a solution is shown in Fig. 3.

In the grid model the cut criterion is not sufficient in general, as the example in Fig. 1 serves again as a counterexample. To formulate a more general necessary condition we need the concept of odd sets. A subset $X \subset V$ is called *odd* with respect to $G+H$ if $d_G(X)+d_H(X)$ is odd.

The crucial observation on odd sets is that for any solution to the edge-disjoint paths problem and for any odd set X at least one edge (actually an odd number of edges) in $\delta_G(X)$ cannot be used by the paths.

In [1] it turned out that it is enough to deal with only horizontal and vertical cuts. To be more precise, by a *column (row)* of G_T we mean a cut of G_T consisting only of horizontal (vertical) edges. Let $\{r_1, r_2,..., r_t\}$ be the set of satured rows ($t \geqslant 0$) and let c be any column. The removal of the edges in $r_1,...,r_t$ and c leaves $t+1$ components $T_1,..., T_{t+1}$ on the left-hand side of c (Fig. 4).

Denote by $q(c)$ the number of odd sets among $T_1,..., T_{t+1}$. For a row r of G_T the number $q(r)$ is defined analogously. Since every edge of a saturated cut must be used in a solution and at least one edge leaving an odd set cannot be used in a solution, for the solvability it is necessary that the number $q(c)$ cannot exceed the surplus $s(c)$ of column c. In [1] the following theorem was proved.

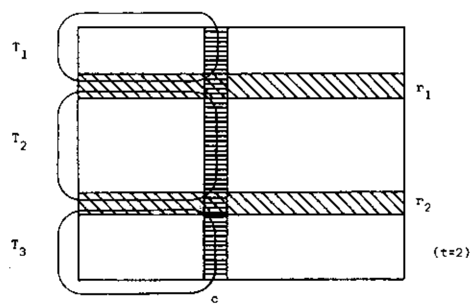

FIGURE 4

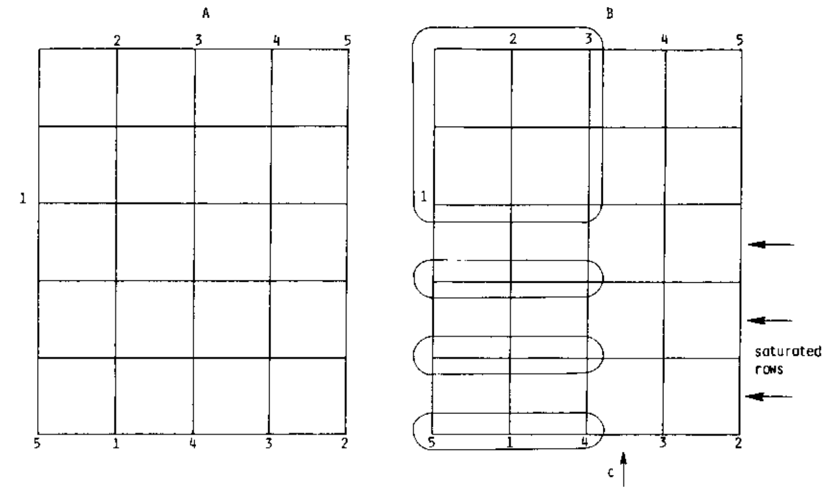

FIGURE 5

THEOREM 2 [1]. *We are given a rectangle T in a rectilinear grid and k pairs of distinct terminals on its boundary. There exist k edge-disjoint paths in G_T between the corresponding terminals if and only if $q(c) \leqslant s(c)$ for each column c and $q(r) \leqslant s(r)$ for each row r.*

The problem instance in Fig. 5A does not possess a solution since for column c in Fig. 5B we have $q(c)=4$ and $s(c)=2$. We mention two consequences of Theorem 2 which may be interesting for their own sake and may be useful in modelling layout problems of electric circuits.

COROLLARY 3 [1]. *In the grid model the cut criterion restricted to columns and rows is sufficient provided that no saturated columns or rows exist.*

COROLLARY 4 [1]. *If one member of each terminal pair is positioned on the upper boundary line of T while the other is on the lower boundary line of T and at least one corner point of T is not a terminal, then the edge-disjoint paths problem has a solution if and only if the cut criterion holds for every column.*

3. THE MAIN RESULT

The purpose of the present paper is to give a common generalization of Theorems 1 and 2. Before doing this let us observe some similarities and differences between the two models. Unlike the Okamura–Seymour case, in the grid model we have a special kind of planar graph, the terminals are distinct but it is not assumed that $G+H$ is Eulerian. On the other hand, in both models

(a) the underlying graph G is planar,

(b) the terminals are placed on the outer face of G,

(c) every inner node of G has even degree.

Our main result is

THEOREM 5. *Given a graph G and k pairs of (not necessarily distinct) terminals so that* (a), (b), (c) *are satisfied, there exist k edge-disjoint paths in G between the corresponding terminals if and only if*

$$\sum s(C_i) \geqslant \frac{q}{2} \tag{1}$$

for every family $\{C_1, C_2, ..., C_l\}$ of $l \leqslant |V|$ cuts of G, where q denotes the number of components in $G' = G - C_1 - \cdots - C_l$ which are odd with respect to $G+H$.

Remark. Observe that if no inner node exists, Theorem 5 provides a complete answer to the edge-disjoint paths problem in outer-planar graphs. Since the capacitated version can also be handled (Sect. 4) the integral multicommodity flow problem in outer-planar graphs can be considered solved.

Remark. Theorem 5 when specialized to the grid model provides a necessary and sufficient condition more complicated than that in Theorem 2. However, it is not difficult to derive (using the special structure of the grid model and the assumption made on the distinctness of the terminals) that in this case it suffices to restrict ourselves to column and row cuts. The (mostly technical) details are left to the reader.

Proof. Necessity. Suppose that there are k edge-disjoint paths between the terminals. For any odd set X at least one edge of G leaving X is not used. Thus at least $q/2$ edges from C_is cannot be used. On the other hand, in a cut C_i at most $s(C_i)$ edges may not be used which implies (1).

Sufficiency. Observe first that (1) implies the cut criterion by chosing l to be 1. If there are no odd sets, then by Theorem 1 we are done.

Obviously, a set X is odd precisely if X contains an odd number of odd nodes. By hypothesis each odd node is on the outer face of G. Let the set of odd nodes be $T = \{a_1, a_2, ..., a_{2n}\}$. (The subscripts reflect the order of the odd nodes on the outer face.)

The idea behind the proof is that we find an appropriate pairing of odd nodes and consider these pairs as new terminal pairs to be connected. In this extended problem there are no odd sets, so Theorem 1 can be applied provided that the cut criterion holds. Hence a pairing of odd nodes is "appropriate" if every cut C of G separates at most $s(C)$ new pairs. What we are going to prove is that such an appropriate pairing exists if (1) is satisfied.

For a subset $A_{i,j} = \{a_i, a_{i+1}, ..., a_j\}$, $1 < i \leqslant j \leqslant 2n$, set

$$p_1(A_{i,j}) = \min(s(C)\colon C \text{ a cut separating } A_{i,j} \text{ and } T - A_{i,j}). \tag{2}$$

Observe that $s(C) \equiv |A_{i,j}| \pmod 2$ for any cut C separating $A_{i,j}$ and $T - A_{i,j}$, so $p_1(A_{i,j}) \equiv |A_{i,j}| \pmod 2$. The next lemma is the crucial point in the proof. To formulate it, let D be a complete graph on nodes $a_1, a_2, ..., a_{2n}$. A set $A_{i,j} = \{a_i, a_{i+1}, ..., a_j\}$, $1 < i \leqslant j \leqslant 2n$, is called an *arc-set* and let p be a non-negative integer-valued function on the set of arc-sets such that $p(X) \equiv |X| \pmod 2$ for each arc-set X. The first element a_i of $A_{i,j}$ is denoted by $f(A_{i,j})$ the last element a_j is denoted by $\ell(A_{i,j})$.

PAIRING LEMMA. *Exactly one of the following two alternatives holds:*

(i) *There exists a perfect matching M of D such that*

$$d_M(X) \leqslant p(X) \text{ for every arc-set } X.$$

(ii) *There exists a family $\mathscr{F} = \{A_1, A_2, ..., A_l\}$ of arc-sets for which $f(A_i) \neq f(A_j)$, $\ell(A_i) \neq \ell(A_j)$ $(i \neq j)$ and*

$$\sum p(A_i) < \frac{q}{2},$$

where q denotes the number of components of odd cardinality in $D - \bigcup(\delta_D(A_i)\colon A_i \in \mathscr{F})$.

Proof. First, let M be a perfect matching satisfying (i) and $\mathscr{F} = \{A_1, A_2, ..., A_l\}$ satisfying (ii). Let $Q_1, ..., Q_q$ be the sets of odd components in $D - \bigcup \delta_D(A_i)$. Since each Q_j is left by an odd number of edges in M, the number z of edges in M which leave at least one Q_j is at least $q/2$. On the other hand z cannot exceed $\sum p(A_i)$. Thus (i) and (ii) cannot be true at the same time.

Second, let us assume (ii) does not hold.

Case 1. Suppose that $p(A) > 0$ for each arc-set A. We claim that the matching $M = \{a_1 a_2, a_3 a_4, ..., a_{2n-1} a_{2n}\}$ satisfies (i). Indeed, $d_M(A) \leqslant 2$ for each arc-set A. If $d_M(A) = 0$, then $d_M(A) \leqslant p(A)$. If $d_M(A) = 1$, then $|A|$ is odd and so is $p(A)$ whence $d_M(A) \leqslant p(A)$. Finally, if $d_M(A)=2$, then $|A|$ and $p(A)$ are even. But $p(A)>0$ from which $d_M(A) \leqslant p(A)$.

Case 2. Suppose that $p(A_0)=0$ for some arc-set A_0. Let A_0 be minimal. Now $|A_0|$ is even. If the last element c of A_0 is not a_{2n}, renumber the nodes (maintaining the same cyclic order) in such a way that the last element of A_0 should be a_{2n} and replace each arc-set containing c by its complement. This way we obtain an equivalent problem where $A_0 = \{a_{2k}, a_{2k+1}, ..., a_{2n}\}$ for some k, $1<k \leqslant n$.

We define a smaller problem on a complete graph D' on nodes $a_1, a_2, ..., a_{2k}$. Let $A_{i,j} = \{a_i, a_{i+1}, ..., a_j\}$ be an arc-set of D' $(1<i \leqslant j \leqslant 2k)$. Let $p'(A_{i,j}) = p(A_{i,j})$ if $j<2k$. Let $p'(A_{i,2k}) = \min(m_1, m_2)$, where $m_1 = \min(p(A_{i,2t-1})-1$: $t=k+1, ..., n)$ and $m_2 = \min(p(A_{i,2t})$: $t=k, ..., n)$. Obviously $p'(A) \equiv |A| \pmod 2$ and we claim that $p'(A) \geqslant 0$. Indeed, if we had $p'(A_{i,j})<0$ for some i, j then $j=2k$ and $p'(A_{i,2k}) = p(A_{i,2t-1})-1=-1$ for some t, $k+1 \leqslant t \leqslant n$. But then $|A_0 \cap A_{i,2t+1}|$ is odd and $\mathscr{F} = \{A_0, A_{i,2t+1}\}$ would violate (ii).

Claim. (ii) does not hold for $D'$and p'.

Proof. Suppose to the contrary that there exists a family $\mathscr{F}'$ satisfying (ii) with respect to D' and p'. We shall define a family $\mathscr{F}$ of arc-sets of D which satisfies (ii) with respect to D and p. This will contradict the assumption.

If there is no arc-set A in $\mathscr{F}'$ containing a_{2k}, then $\mathscr{F} = \mathscr{F}'$ satisfies (ii) with respect to D, p. If such an A occurs in $\mathscr{F}'$, then $A = A_{i,2k}$ for some i, $1<i \leqslant 2k$. (Note that at most one such an A exists.) If $p'(A_{i,2k}) = p(A_{i,2t})$ for some t, $k \leqslant t \leqslant n$, then $\mathscr{F} = \mathscr{F}' - \{A_{i,2k}\} \cup \{A_{i,2t}\}$ satisfies (ii) with respect to D, p. If $p'(A_{i,2k}) = p(A_{i,2t-1})-1$ for some t, $k+1 \leqslant t \leqslant n$, then $\mathscr{F} = \mathscr{F}' - \{A_{i,2k}\} \cup \{A_{i,2t-1}, A_0\}$ satisfies (ii) with respect to D, p and the claim is proved.

Applying the induction hypothesis to D', p', we get a perfect matching M' of D' satisfying (i). Let us define a perfect matching M of D, as follows. $M := M' \cup \{a_{2k}a_{2k+1}, a_{2k+2}a_{2k+3}, ..., a_{2n-1}a_{2n}\}$.

Claim. M satisfies (i) with respect to D, p.

Proof. By the construction of M and p' we have $d_M(A_{i,j}) \leqslant p(A_{i,j})$ for i, j, where $1<i \leqslant 2k$, $i \leqslant j \leqslant n$. For $A_{i,j}$ with $2k<i \leqslant j \leqslant 2n$ we have $p(A_{i,j})>0$ (by the minimality of A_0) from which $d_M(A_{i,j}) \leqslant p(A_{i,j})$ follows.

The second claim completes the proof of the pairing lemma.

Now apply the lemma to $p=p_1$ as defined in (2). If (i) holds, we have the appropriate pairing and we are done. If (ii) holds, let us consider the family $\mathscr{F} = \{A_1, A_2, ..., A_l\}$ in (ii). Let C_i $(i=1, 2, ..., l)$ be a cut of G separating A_i and $T-A_i$ for which $s(C_i)=p(A_i)$. We claim that

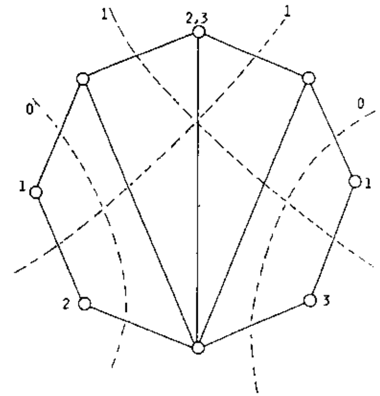

FIGURE 6

$C_1, C_2, ..., C_l$ violate (1). To see this, first observe that a component of $G - \bigcup(C_i: i=1, ..., l)$ cannot contain nodes from T belonging to distinct components of $D - \bigcup(\delta_D(A_i): i=1, ..., l)$. Let X be an odd component of this latter graph. The nodes of X belong to one or more componnents of $G - \bigcup(C_i: i=1, ..., l)$ but one of these components, denote it by Y, must contain an odd number of elements of X which is equivalent to saying that Y is odd with respect to $G+H$.

Consequently, there are at least q components of $G - \bigcup C_i$ which are odd with respect to $G+H$ and $\sum s(C_i) < q/2$, contradicting (1). This contradiction completes the proof of Theorem 5. ∎

In the example given in Fig. 6 the edge-disjoint paths problem does not have a solution since the four cuts indicated in the figure violate (1). The numbers on the cuts denote the surplus. Their sum is 2 while $q=8$ and so $\sum s(C_i) < q/2$.

By the proof of the pairing lemma, Theorem 5 implies the following result.

COROLLARY A. *Under the hypotheses of Theorem 5 the edge-disjoint paths problem has a solution whenever the surplus of every cut is positive.*

We can further specialize this result. Suppose we are given a triangle R in a triangular grid which is bounded by lattice lines. R defines a graph G_R in the natural way. Suppose that k pairs of distinct terminals are given on the boundary of R (see Fig. 7).

COROLLARY B. *There are always k edge-disjoint paths in G_R between the corresponding terminal pairs.*

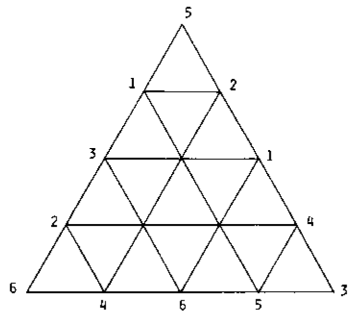

FIGURE 7

The proof is by showing that in this special case the surplus of each cut is positive, so Corollary A applies.

Remark. From the proof of the pairing lemma we see that if (i) does not hold, then there is a cut family violating (1) consisting of at most n members. It also follows that in Theorem 5 each component of G' contains a connected piece of the outer face of G.

Remark. If in Theorem 5 we drop the assumption that every inner node has even degree, then, as E. Tardos kindly pointed out, (1) is not sufficient in general; see the example in Fig. 8.

We close the section by presenting a conjecture stating that (1) might also be sufficient in another special case.

Conjecture. Suppose that $G+H$ is planar (but the terminals need not be on the outer face of G) then (1) is necessary and sufficient for the solvability of the edge-disjoint paths problem.

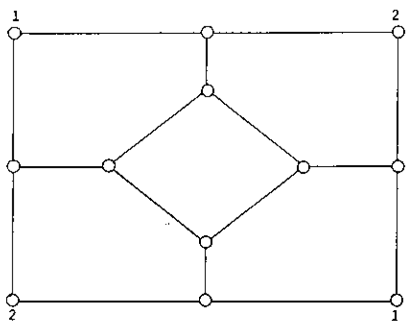

FIGURE 8

If $G+H$ is planar and Eulerian, then a theorem of Seymour [9] states that the cut criterion is already sufficient. If H contains at most 3 edges, the conjecture follows from a result of Korach [4].

4. ALGORITHMIC ASPECTS

The proof of Theorem 1 in [6] is constructive and actually yields a polynomial time solution algorithm of complexity $O(|V|^3 \log |V|)$. (We outline it below.) Since the proof of the pairing lemma also gives rise to a polynomial time algorithm to find the appropriate pairing, there is a polynomial time algorithm to find either the paths or the cuts in Theorem 5. In order to calculate the values $p_1(A_{i,j})$ we need $\binom{2n}{2}$ max-flow min-cut (MFMC) computations, where $2n$ is the number of odd nodes.

It is important to notice that each of the $p_1(A_{i,j})$ values can be calculatd by applying an MFMC computation to an (s, t)-planar graph. Namely, extend G by placing two new nodes s, t on the infinite region of G and adjoin edges connecting s and the nodes in $A_{i,j}$ and edges connecting t and the nodes in $T-A_{i,j}$. Let the capacities of each new edge be large. This way we get an (s, t)-planar graph where the value of the minimum cut separating s and t is $p_1(A_{i,j})$.

For (s, t)-planar graphs there is an $O(|V| \log |V|)$ MFMC algorithm [2, 6]. Hence the pairing algorithm needs $O(|V|^3 \log |V|)$ steps and the complexity of the overall algorithm is $O(|V)^3 \log |V|)$.

In [1] for the bipartite grid model (Corollary 4) an $O(N \log N)$ algorithm was developed, where N denotes the number of demand edges. Recently Mehlhorn and Preparata were able to generalize this method for the general grid model [5, Theorem 2]. The complexity of their algorithm is also $O(N \log N)$ (that does not depend on the size of the underlying grid).

There is a natural way to obtain a weighted version of Theorem 5. Suppose that G and H are given as in Theorem 5. With every edge e of G a positive integral capacity $g(e)$ is associated so that the sum of capacities at every inner node is even. Moreover, with every edge e of H a positive integral demand $h(e)$ is associated. The problem is to find a collection $\mathscr{F}$ of paths so that $\mathscr{F}$ contains $h(uv)$ paths connecting u and v for every $uv \in F$ and every edge e is used by at most $g(e)$ members of $\mathscr{F}$.

A possible special case of this problem which might have applications in circuit design is the weighted grid model where with each horizontal and vertical line a positive integer is associated which represents the capacity of every edge belonging to this line.

One can immediately observe that the weighted problem goes back to the problem in Theorem 5. Namely, replace every supply edge e by $g(e)$

parallel copies and every demand edge e by $h(e)$ parallel copies. The unweighted problem obtained this way satisfies the hypotheses of Theorem 5 so the weighted problem theoretically can be considered to be solved.

From an algorithmic point of view, however, this reduction is not at all satisfactory since the complexity of the resulting algorithm would include a factor proportional to the maximum capacity M. A proper polynomial algorithm is allowed only to involve a polynomial in $\log M$, the number of digits of M. We are going to show how this difficulty can be overcome.

Notice that there is no difficulty in finding the appropriate pairing since the proof of the pairing lemma provides a good algorithm if the values $p(A_{i,j})$ are available.

So the problem which remains to be overcome is finding an algorithm for the weighted version of the theorem of Okamura and Seymour [6]. To this end, suppose in addition that the sum of capacities plus the sum of demands at every outer node of G is even. The cut criterion in the weighted case is $d_g(X) \geq d_h(X)$ for every $X \subseteq V$. The surplus $d_g(X) - d_h(X)$ of a set X is denoted again by $s(X)$. By the assumptions on g and h $s(X)$ is always even.

We can suppose that G is 2-connected, for otherwise the problem can be decomposed at a cut-node into smaller problems. So we suppose that the boundary of G forms a simple circuit C with nodes $x_1, x_2, \ldots, x_n$ (in this order).

The idea behind the algorithm is a refinement of that of Okamura and Seymour, so we briefly summarize their method. Choose an edge on C, say $x_n x_1$, and a certain demand edge $x_k x_l$ $(k<l)$. Revise the sets of supply and demand edges as follows. Let $E' = E - x_n x_1$ and $F' = F - x_k x_l + x_1 x_k + x_l x_n$. One can immediately see that if the new problem with supply graph $G' = (V, E')$ and demand graph $H' = (V, F')$ has a solution, then so does the original one. Namely, paths P_{1k} and P_{ln} can be glued together via edge $x_n x_1$ to form a (possibly not simple) path between x_k and x_l. Since $G' + H'$ is Eulerian, G' is planar, and every edge of H' connects two outer nodes of G' the only question (in order to apply induction on $|E|$) is whether the cut criterion continues to hold. Okamura and Seymour proved that this is the case if x_k and x_l are chosen as follows. Let Y be a minimal saturated set for which $x_1 \in Y$, $x_n \notin Y$ (if no such set exists, let $Y = V - x_n$) and let $x_k x_l \in F$ be such that $x_k \in Y$ and l is as large as possible.

The method of Okamura and Seymour consists of applying iteratively the above reduction (including a decomposition into 2-connected parts when the reduced graph is not 2-connected). In each iterative step one edge of G is deleted, so the procedure stops after at most $|E|$ iterations. For the weighted case we need a lemma.

LEMMA. *If $d_h(X, Y) = 0$ for some $X, Y \subset V$, then*

$$s(X) + s(Y) = s(X \cap Y) + s(X \cup Y) + 2d_g(X, Y).$$

Proof. It is easily seen that the contribution of each edge is the same on the two sides of the equality. ∎

Denote $C_i = \{x_1, x_2, \ldots, x_i\}$ for $1 \leq i \leq n-1$. For each i calculate the minimum m_i of $d_g(X)$ over all subsets $X \subset V$ for which $C_i \subseteq X$ and $(C - C_i) \cap X = \emptyset$. This can simply be done by an MFMC computation with C_i as a sink set and $C - C_i$ as a source set. One can observe that the method for (s, t)-planar graphs can be used again [2, 6].

It is well known that there exists a unique minimal set X_i, where the minimum m_i is attained and the MFMC algorithm provides with this set as well. Let $s_i = s(X_i)$ and $m = \min(s_i: 1 \leq i \leq n-1)$.

Case 1. $m > 0$. Reduce $g(x_n x_1)$ by $z = \min(m, g(x_n x_1))$ and if z is odd, increase $h(x_n x_1)$ by one. The new problem satisfies the requirements (Eulerian property, cut criterion, etc.) so it suffices to deal with it. If $g(x_n x_1) \leq m$, then the new capacity of $x_n x_1$ is zero, therefore this edge can be left out and we obtain a smaller problem. If $m < g(x_n x_1)$, then in the resulting problem a saturated set arises which contains x_1 but does not contain x_n, that is, the new $m = 0$.

Case 2. $m = 0$. Let j denote the smallest subscript i for which $s_i = 0$. Choose a demand edge $x_k x_l$ $(k < 1)$ so that $x_k \in X_j$ and l is as large as possible. Let $\delta_1 = \min(s_i: i = 1, 2, \ldots, j-1)$ and $\delta = \min(g(x_n x_1), h(x_k x_l), \delta_1/2)$. Obviously $\delta > 0$.

Execute δ times the reduction procedure for x_1, x_n, x_k, x_l as it was described for the unweighted case. That is, decrease capacity $g(x_n x_l)$ and demand $h(x_k x_l)$ by δ and increase demands $h(x_1 x_k)$ and $h(x_l x_n)$ by δ. (It will not be disturbing that new demand edges may have arisen.) Like the unweighted case one can easily get a solution to the starting problem if a solution is available to the reduced one.

Claim. The cut criterion continues to hold.

Proof. Let $\mathscr{H} = \{X: \{x_1, x_n\} \cap X \neq \emptyset,\ x_k, x_l \notin X\}$. After the reduction the surplus of a cut-inducing set X is decreased if and only if X or $V - X \in \mathscr{H}$ and in this case the amount of the reduction is 2δ. So we have to show that $s(X) \geq 2\delta$ for $X \in \mathscr{H}$. By the maximal choice of l we have $d_h(X, X_j) = 0$ so the lemma applies:

$$s(X) = s(X) + s(X_j) = s(X \cap X_j) + s(X \cup X_j) + 2d_g(X, X_j).$$

If $x_1 \notin X$, then $x_n \in X$ and $x_n x_1$ connects X-X_j and X_j-X therefore

$d_g(X, X_j) \geqslant g(x_n x_1) \geqslant \delta$ from which $s(X) \geqslant 2\delta$ follows. If $x_1 \in X$, then $s(X \cap X_j) \geqslant \delta_1 \geqslant 2\delta$ from which $s(X) \geqslant 2\delta$ follows again and the claim is proved.

The algorithm consists of repeating the reduction procedure. We have three cases according to where the minimum is attained in the definition of δ.

Case 1. $\delta = g(x_n x_1)$. Then the new capacity of $x_n x_1$ is zero and $x_n x_1$ is deleted from G. Repeat the reduction procedure for the smaller problem.

Case 2. $\delta = h(x_k x_l)$. Repeat the reduction procedure with the same x_1, x_n, and X_j.

Case 3. $\delta = s_h/2$ for a certain h, $1 \leqslant h \leqslant j-1$. Repeat the reduction procedure with the same x_n, x_1.

A section of the algorithm between two occurances of Case 1 is called a *phase.* In order to prove that the procedure is a polynomial time algorithm, observe that $d_H(X_j)$ decreases by one if Case 2 occurs. Furthermore, during a reduction the demand of an edge occuring in a saturated cut cannot increase. Hence in the course of one whole algorithm Case 2 can occur at most $|V|^2$ times.

If Case 3 occurs, then in the reduced problem X_h is saturated and $h < j$. Thus in one phase Case 3 can occur at most $|V|$ times. In one phase we need $|V|$ MFMC computations. Since for (s, t)-planar there is an $O(|V| \log |V|)$ algorithms the complexity of the overall algorithm is $O(|V|^3 \log |V|)$.

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Note added in proof. Recently I learned that R. Hassin also developed an algorithm for the capacitated case of the Okamura–Seymour problem. See R. Hassin, On multicommodity flows in planar graphs, *Networks* **14** (1984), 225–235.

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