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Submodular functions in graph theory

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Abstract

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We describe various aspects of the use of submodular functions in graph theory. New proofs of theorems of Mader and of Tutte are provided as well as a new application on making a digraph k-edge-connected by adding a minimum number of edges.

1. Introduction

Edmonds [1] initiated systematic studies of submodular functions. Since then, it has turned out that submodular functions play an important role in combinatorial optimization and polyhedral combinatorics (for a survey, see [5, 9]). In this paper we outline the various applications of submodular functions in graph theory.

In Section 2, by providing proofs of classical theorems of Hall, Menger and Edmonds, we describe a basic technique based on submodular functions. Each of these theorems concerns cut-type conditions.

Section 3 is devoted to proving theorems involving partition-type necessary and sufficient conditions. Among others, a new proof is provided for Tutte's disjoint trees theorem. In Section 4 the splitting technique is introduced, while Section 5 is concerned with the uncrossing technique. As an application, we provide a simple proof of a difficult theorem of W. Mader on characterizing k-edge-connected directed graphs. In the last section we exhibit a recent application of submodular functions. It is a theorem about the minimum number of new edges to be added to a given digraph to make it k-edge-connected.

Let V be a finite ground set. Two subsets X, Y of V are called *intersecting* if none of $X \cap Y$, $X - Y$, $Y - X$ is empty. If, in addition, $V - (X \cup Y)$ is nonempty, X and Y are called *crossing*. For $s, t \in V$, we call a set X a $t\bar{s}$-set if $t \in X \subseteq V - s$.

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Let $\mathscr{F}$ be a family of subsets of V. $\mathscr{F}$ is called *cross-free* if there are no two crossing members of it. $\mathscr{F}$ is called *laminar* if it contains no two intersecting sets. $\mathscr{F}$ is called a *subpartition* of V if its members are pairwise disjoint nonempty subsets of V. If, in addition, every element of V belongs to a member of $\mathscr{F}$, $\mathscr{F}$ is called a *partition* of V.

Let $G=(V,E)$ be an undirected graph with node set V and edge set E. We denote an edge e connecting nodes u and v by uv or vu. This is not quite precise since there may be parallel edges between u and v. But this ambiguity will not cause any trouble.

For a directed graph $G=(V,E)$, a directed edge $e=uv$ is meant to be an edge from u to v. In this case vu means the oppositely directed edge. u is the *tail* of e, while v is the *head* of e.

Generally, by *graph* we mean an undirected graph and by *digraph* a directed graph. For a graph or digraph G and a subset X of nodes, $E_G(X)$ denotes the set of edges with both end-nodes in X and is called the set of edges induced by X. $S_G(X)$ denotes the set of edges with at least one end node in X. For $X, Y \subseteq V$, $d_G(X, Y)$ denotes the number of edges between $X-Y$ and $Y-X$ (in any direction). We define $d_G(X):=d_G(X, V-X)$. $\nabla G(X)$ denotes the set of edges between X and $V-X$. Such a set is called a *cut* with *sides* X and $V-X$. *Splitting off* a pair uv, vz of edges means that we replace the two edges uv, vz by a new edge uz. In a digraph G the *in-degree* $\rho_G(X)$ (*out-degree* $\delta_G(X)$) is the number of edges entering (leaving) X. When it causes no ambiguity, we will leave out the subscript G. A digraph $D=(V,A)$ is called an *arborescence* if D arises from a tree by orienting the edges in such a way that every node but one has one entering arc. The exceptional node, called the *root*, has no entering arc.

A digraph is called *k-edge-connected* if $\rho(X)\geqslant k$ for every $0 \subset X \subset V$. (For $k=1$ the term *strongly connected* is used.)

A set function $b: 2^V \to \mathbb{R}$ acting on the power set of a finite set V is called submodular if the inequality

$$b(X)+b(Y)\geqslant b(X\cap Y)+b(X\cup Y) \tag{1.1}$$

holds for every subset X and Y of V. In applications, often we encounter set functions satisfying the reverse inequality in (1.1) for every X, Y. Such a function is called a *supermodular function*. (In this note every occurring set function is meant to be 0 on the empty set.)

Let $G=(V,E)$ be a directed graph with node set V. It is not difficult to prove that the in-degree function ρ is submodular. Actually, one has the following identity:

$$\rho(X)+\rho(Y)=\rho(X\cup Y)+\rho(X\cap Y)+d(X, Y), \tag{1.2}$$

where $d(X, Y)$ denotes the number of edges between $X-Y$ and $Y-X$ (in any direction). To prove (1.2), one has to check that every edge of G has the same contribution to the two sides of (1.2).

Let $G=(V, W; E)$ be a bipartite graph. For $X \subseteq V$ let $\Gamma(X):=\{w\in W$: there is an edge $vw\in E$ with $v\in X\}$. Verbally, $\Gamma(X)$ is the set of neighbours of X. For $X, Y \subseteq V$ we have

$$\Gamma(X)\cup\Gamma(X)=\Gamma(X\cup Y) \quad \text{and} \quad \Gamma(X)\cap\Gamma(Y)\supseteq\Gamma(X\cap Y). \tag{1.3}$$

Condition (1.3) easily implies the submodularity of $|\Gamma(X)|$.

2. Three theorems from graph theory

We are going to prove three fundamental min–max theorems of graph theory.

Theorem 2.1 (Hall [7]). *In a bipartite graph $G=(V, W; E)$ there is a matching covering V if and only if*

$$|\Gamma(X)|\geqslant|X| \tag{2.1}$$

holds for every $X \subseteq V$.

Proof. The necessity of (2.1) is trivial. To see the sufficiency, we start with a definition and a lemma. A set $X \subseteq V$ is said to be *tight* if X satisfies (2.1) with equality.

Lemma 2.2. *The intersection and the union of two tight sets X and Y are tight.*

Proof of Lemma 2.2. By applying (2.1) to $X\cup Y$ and to $X\cap Y$ and using the submodularity of $|\Gamma|$, we have

$$\begin{aligned}|X|+|Y|&=|\Gamma(X)|+|\Gamma(Y)|\geqslant|\Gamma(X\cup Y)|+|\Gamma(X\cap Y)|\\&\geqslant|X\cup Y|+|X\cap Y|=|X|+|Y|.\end{aligned} \tag{2.2}$$

Hence equality must follow everywhere and, in particular, $|\Gamma(X\cup Y)|=|X\cup Y|$ and $|\Gamma(X\cap Y)|=|X\cap Y|$, that is, both $X\cup Y$ and $X\cap Y$ are tight. □

Proof of Theorem 2.1 (*conclusion*). Suppose that G is a minimal counterexample of Hall's theorem. It follows that deleting any edge of G would destroy (2.1). Thereby

(∗) for every edge sw ($s\in V$) of G there is a tight set X containing s so that s is the only neighbour of w in X.

There is a node $s\in V$ with $d(s)\geqslant 2$ since, otherwise, G itself would be a matching covering V, and then G would not be a counterexample. Let u and v be two neighbours of s and let P denote the intersection of tight sets P_u, P_v corresponding, respectively, to su and sv by (∗). By Lemma 2.2, P is tight.

At least one of u and v, say u, has a neighbour in $P-s$ since, otherwise, $P-s$ would violate (2.1). This contradicts (∗) since P_u and P_v include P. This contradiction shows that no counterexample may exist. □

Theorem 2.3 (Menger [12]) (directed, edge-version, in [3]). *In a directed graph $G=(V,E)$ there are k edge-disjoint paths from s to t if and only if the following cut condition*

$$\rho(X)\geqslant k \tag{2.3}$$

holds for every $t\bar{s}$-set $X\subseteq V$.

Proof. The necessity of the cut condition is obvious. To see its sufficiency, we use induction on the number of edges. Call a $t\bar{s}$-set T *tight* if $\rho(T)=k$.

Lemma 2.4. *The intersection and the union of any two tight sets X, Y are tight.*

Proof of Lemma 2.4. One has $k+k=\rho(X)+\rho(Y)\geqslant\rho(X\cap Y)+\rho(X\cup Y)\geqslant k+k$, from which equality must hold everywhere and the lemma follows. □

Proof of Theorem 2.3 (*conclusion*). We can assume that every edge e enters a tight set since, otherwise, e can be left out without violating (2.3). Let su be an edge of G with $u\neq t$. (If no such edge exists, then the theorem is trivial.) There is a tight set entered by su and, by Lemma 2.4, the intersection T of such sets is tight. There must be an edge uv with $v\in T$ for, otherwise, $\rho(T-u)<\rho(T)=k$, that is, $T-u$ would violate the cut condition.

Let G' denote the graph obtained from G by splitting off the edges su and uv. We claim that G' satisfies the cut criterion. Indeed, if a set X violates the cut criterion in G', then $u\in X$, $v\notin X$ and X is tight in G. But this contradicts the definition of T. By induction, there are k edge-disjoint paths in G' and, therefore, there are k edge-disjoint paths in G. □

Theorem 2.5 (Edmonds [2]). *Let $G=(V,E)$ be a digraph with a specified node s. There are k disjoint spanning arborescences of root s if and only if*

$$\rho(X)\geqslant k \tag{2.4}$$

for every set $X\subseteq V-s$.

Proof (Lovász [8]). The necessity is again clear. To prove the sufficiency, we proceed by induction on k. The case $k=0$ is trivial. Starting from s we are going to build up a subarborescence F of G rooted at s so that

(∗) $\rho_{E-F}(X)\geqslant k-1$ holds for every $X\subseteq V-s$.

If we can find such a spanning arborescence then, by applying the induction hypothesis to $G-F$ (with $k-1$), we are done.

In the general step let F be an arborescence satisfying (∗) and suppose that $V\neq V(F)$. We are going to find a one-edge-bigger arborescence F' satisfying (∗). Call a set $X\subseteq V-s$ *critical* if $\rho_{E-F}(X)=k-1$. Obviously, any critical set intersects $V(F)$.

Lemma 2.6. *The intersection and the union of two intersecting critical sets X and Y are critical.*

Proof of Lemma 2.6. One has $k-1+k-1=\rho_{E-F}(X)+\rho_{E-F}(Y)\geqslant\rho_{E-F}(X\cap Y)+\rho_{E-F}(X\cup Y)\geqslant k-1+k-1$, from which equality must hold everywhere and the lemma follows. □

Proof of Theorem 2.5 (*conclusion*). Let T be a minimal critical set not included in $V(F)$. (If no such set exists, let $T=V$.) There is an edge uv with $u\in V(F)\cap T$, $v\in T-V(F)$ for, otherwise, $\rho(T-V(F))=\rho_{E-F}(T-V(F))\leqslant k-1$, contradicting (2.4).

We claim that uv cannot enter any critical set. Indeed, if there were a critical set X entered by uv then, by Lemma 2.6, $X\cap T$ would be critical, contradicting the minimal choice of T.

Therefore, $F':=F+uv$ is an arborescence satisfying $(*)$ and F' is bigger than F. □

3. Partition condition

The three theorems proved in the preceding section have a feature in common. Each of them sounds like this: 'There exists something if and only if a certain inequality holds for every subset X'. Sometimes, more complicated conditions are required that include not only one set but also a subpartition of V. Here we provide two examples where this is the case. In Section 6 one more example will be shown.

Edmonds' theorem characterizes digraphs having k disjoint spanning arborescences rooted at a certain node s. But what if we are interested in finding k disjoint spanning arborescences with arbitrary roots? That is, there is no restriction on the k roots of the k arborescences to be found.

Theorem 3.1 (Frank [4]). *In a directed graph $G=(V,E)$ there are k disjoint arborescences if and only if*

$$\sum \rho(X_i)\geqslant k(t-1) \tag{3.1}$$

holds for every subpartition $\{X_1,X_2,\dots,X_t\}$ of V.

Proof. *Necessity.* Suppose $F_1,\dots,F_k$ are k disjoint spanning arborescences and $\mathcal{F}=\{X_1,X_2,\dots,X_t\}$ is a subpartition. Each F_i enters at least $t-1$ members of $\mathcal{F}$. Therefore, the contribution of one F_i to the sum $\sum\rho(X_i)$ is at least $t-1$. Since we have k disjoint arborescences, (3.1) follows.

Sufficiency. Assume that (3.1) holds. Add a new node s to G and also k parallel edges from s to every node of G. In this enlarged digraph, clearly,

$$\text{there are } k \text{ edge-disjoint paths from } s \text{ to every other node.} \tag{3.2}$$

Second, one by one, discard new edges as long as possible without violating (3.2). Let G' denote the final digraph and ρ' the in-degree function of G'. By Menger's theorem, (3.2) is equivalent to

$$\rho'(X) \geqslant k \quad \text{for every } X \subseteq V. \tag{3.3}$$

Call a subset $X \subseteq V$ *critical* if X satisfies (3.3) with equality and let $\mathscr{F} = \{X_1, \ldots, X_t\}$ denote the family of maximal critical subsets of V. We know from Lemma 2.6 that the intersection and the union of two intersecting critical sets are critical. This implies that the members of $\mathscr{F}$ are pairwise disjoint, that is, $\mathscr{F}$ is a subpartition of V.

Claim 3.2. *$\rho'(V) = k$, that is, V is critical.*

Proof of Claim 3.2. Indirectly, suppose there are $k+1$ edges $e_1, \ldots, e_{k+1}$ entering V. By the minimal property of G', discarding anyone of them destroys (3.3). Equivalently, each e_i enters a critical set and, hence, each e_i enters a member of $\mathscr{F}$. We have $kt = \sum \rho'(X_i) \geqslant k + 1 + \sum \rho(X_i)$, contradicting (3.1). □

Proof of Theorem 3.1 (*conclusion*). Since (3.3) holds true, Edmonds' theorem, when applied to G', shows that G' contains k disjoint spanning arborescences rooted at s. By Claim 3.2, each of these arborescences uses one single edge entering V. Hence, the restriction of these arborescences to V provides the desired k disjoint spanning arborescences of G. □

What about undirected graphs? What is a necessary and sufficient condition for the existence of k disjoint spanning trees of an undirected graph?

Theorem 3.3 (Tutte [13]). *A graph $G = (V, E)$ contains k disjoint spanning trees if and only if*

$$e_{\mathscr{F}} \geqslant k(t-1) \tag{3.4}$$

for every partition $\mathscr{F} = \{V_1, \ldots, V_t\}$ of V, where $e_{\mathscr{F}}$ denotes the number of edges connecting different V_i's. (That is $e_{\mathscr{F}} = \sum d(V_i)/2$.)

Proof. The necessity of (3.4) follows from the fact that, given a partition $\mathscr{F}$, any spanning tree must have at least $t-1$ edges connecting different members of $\mathscr{F}$. By Edmonds' theorem, the sufficiency of (3.4) follows immediately from the following orientation theorem.

Theorem 3.4. *Given a graph $G = (V, E)$ and a node $s \in V$, G has an orientation for which $\rho(X) \geqslant k$ for every $X \subseteq V - s$ if and only if (3.4) holds.*

Proof. If there is such an orientation, then $\rho(V_1) \geqslant k$ for each V_i not containing s and then $e_{\mathscr{F}} = \sum \rho(V_i) \geqslant k(t-1)$.

To see the sufficiency, extend G by a minimum number of edges sv ($v \in V$) so as to have a required orientation. If this minimum is zero, we are done; so, assume that it is positive. Let ρ denote the in-degree function of this orientation. We can assume that $\rho(s)=0$. Call a set $X \subseteq V-s$ *critical* if $\rho(X)=k$. Recall the following results.

Claim 3.5. *The intersection and the union of two critical sets with nonempty intersection are critical.*

Let $e=st$ be a new arc in the given orientation and let T be the set of nodes reachable from t along a path.

Claim 3.6. *If Z is critical and $T \cap Z \neq \emptyset$, then $Z \subseteq T$.*

Proof of Claim 3.6. Assume $Z \nsubseteq T$. For $Y := V-T$ we have $k=\rho(Y)+\rho(Z)=\rho(Y \cap Z)+\rho(Y \cup Z)+d(Y,Z) \geqslant k+0+d(Y,Z) \geqslant k$, where $d(Y,Z)$ denotes the number of arcs connecting $Y-Z$ and $Z-Y$ (in either direction). From this we get $\rho(Y \cup Z)=0$ and $d(Y,Z)=0$. The first equality implies that $t \in Z$ (by the definition of T and by the assumption that $T \cap Z \neq \emptyset$), while the second one implies that $t \notin Z$ (because of edge st); this contradiction proves the claim. □

Proof of Theorem 3.4 (*conclusion*). Consider the following cases.

Case 1: There is a node $v \in T$ which is not contained in any critical set. Let P be a directed path from t to v. Reorient the edges of P and discard e. The new orientation is still good, a contradiction to the minimality of the number of new su edges.

Case 2: Every node of T is in a critical set. Let $V_1, V_2, \ldots, V_{t-1}$ denote the maximal critical sets in T. By Claims 3.5 and 3.6, these are disjoint sets and form a partition of T. Let $V_t := V-T$ and $\mathscr{F} := \{V_1, \ldots, V_t\}$. Since $\rho(V_t)=0$, we have $k(t-1)=\sum(\rho(V_i)\colon i=1,\ldots,t-1)=\sum(\rho(V_i)\colon i=1,\ldots,t)=e'_{\mathscr{F}} > e_{\mathscr{F}}$, contradicting (3.4). (Here $e'_{\mathscr{F}}$ denotes the number of edges in the enlarged graph connecting different V_i's.) □

4. Splitting off

In Section 2, while proving Menger's theorem, we have already used the splitting-off technique. There is a great number of other applications of this technique and our purpose now is to show the one that will be an important ingredient for characterizing k-edge-connected digraphs.

Theorem 4.1. (Mader [11]). *Suppose that a node s of a digraph $G'=(V+s, E')$ satisfies $\delta'(s)=\rho'(s)$ and*

($*$) *for each pair of nodes x and y distinct from s, there are k edge-disjoint paths from x to y.*

Then, for every edge st, there is an edge vs such that vs and st can be split off without destroying ($*$).

Note that, by Menger's theorem, ($*$) is equivalent to

$$\rho'(X)\geqslant k, \tag{4.1a}$$

$$\delta'(X)\geqslant k \tag{4.1b}$$

for every proper subset $\emptyset\neq X\subset V$, where ρ' and δ' denote, respectively, the in-degree and out-degree function of G'.

Proof. In the proof we use the notation $V':=V+s$. The following identity is easy to prove. If $\delta(X\cap Y)=\rho(X\cap Y)$, then

$$\delta(X)+\delta(Y)=\delta(X-Y)+\delta(Y-X)+\bar{d}(X,Y), \tag{4.2}$$

where $\bar{d}(X, Y)$ denotes the number of edges between $X\cap Y$ and $V-(X\cup Y)$.

Lemma 4.2. *For G', if X, Y are intersecting subsets of nodes for which $\{s\}=X\cap Y$ and $\delta'(X)=\delta'(Y)=k$, then $\delta'(X-Y)=\delta'(Y-X)=k$ and $\bar{d}'(X,Y)=0$.*

Proof of Lemma 4.2. Applying (4.2), we obtain $k+k=\delta'(X)+\delta'(Y)=\delta'(X-Y)+\delta'(Y-X)+\bar{d}'(X,Y)\geqslant k+k+\bar{d}'(X,Y)$, from which $\delta'(X-Y)=\delta'(Y-X)=k$, and $\bar{d}'(X,Y)=0$ follows. □

Lemma 4.3. *Suppose for $A,B\subseteq V'$ that $\rho'(A)=\rho'(B)=k\leqslant\min(\rho'(A\cap B),\rho'(A\cup B))$. Then $\rho'(A\cap B)=\rho'(A\cup B)=k$ and $d'(A,B)=0$.*

Proof of Lemma 4.3. We have $k+k=\rho'(A)+\rho'(B)=\rho'(A\cap B)+\rho'(A\cup B)+d'(A,B)\geqslant k+k+d'(A,B)$, from which $k=\rho'(A\cap B)=\rho'(A\cup B)$, and $d'(A,B)=0$ follows. □

Call a subset $\emptyset\subset X\subset V$ *in-critical* if $\rho'(X)=k$ and *out-critical* if $\delta'(X)=k$. X is called *critical* if it is either out- or in-critical. (Note that V is never critical.)

Lemma 4.4. *Let A and B be two intersecting critical sets. Then either* (i) *$A\cup B$ is critical or* (ii) *$B-A$ is critical and $\bar{d}'(A,B)=0$.*

Proof of Lemma 4.4. If both A and B are in-critical and $A \cup B \subset V$ then, by Lemma 4.3, alternative (i) holds. If $A \cup B = V$, then Lemma 4.2, when applied to $X := V + s - A$, $Y := V + s - B$, implies (ii). The situation is analogous if both A and B are out-critical. Finally, let A be in-critical and B out-critical. Lemma 4.3, when applied to A and $V + s - B$, implies (ii). □

Proof of Theorem 4.1 (*conclusion*). A pair $\{vs, st\}$ of edges cannot be split off without violating (4.1) precisely if there is a critical set containing both v and t. Therefore, if there is no critical set containing t, then any pair vs, st can be split off.

For two intersecting critical sets A, B containing t, only alternative (i) may hold in Lemma 4.4 since $\bar{d}'(A, B) > 0$ in this case. Therefore, the union M of all critical sets containing t is critical again.

We claim that there is an edge vs with $v \in V - M$. Indirectly, suppose that no such edge exists. If M is in-critical, then $\delta'(V - M) < \rho'(M) = k$, contradicting (4.1b). If M is out-critical, then $\delta'(s) = \rho'(s)$ implies that $\rho'(V - M) = \delta'(M + s) < \delta'(M) = k$, contradicting (4.1a).

By the choice of M, no critical set contains both v and t; therefore, the pair $\{vs, st\}$ is splittable. □

5. Uncrossing

Another useful technique that finds many applications is the so-called uncrossing procedure. The power of this machinery is nicely shown by the following proof of another theorem of Mader [10]. The original proof was quite complicated.

Recall that a digraph $G = (V, E)$ is called *k-edge-connected* if $\rho(X) \geq k$ for every nonempty proper subset X of V. By Menger's theorem, this is equivalent to saying that, for any two nodes u and v, there are k edge-disjoint paths from u to v.

We say that G is *minimally k-edge-connected* if it is k-edge-connected, but deleting any edge destroys this property.

Theorem 5.1 (Mader [10]). *Every minimally k-edge-connected digraph with at least two nodes has two nodes with in- and out-degree k.*

Proof. Call a set *critical* if $\rho(X) = k$.

Lemma 5.2. *If X and Y are crossing critical sets, then both $X \cap Y$ and $X \cup Y$ are critical and $d(X, Y) = 0$.*

Proof of Lemma 5.2. We have $k + k = \rho(X) + \rho(Y) = \rho(X \cap Y) + \rho(X \cup Y) + d(X, Y) \geq k + k$. Whence, the lemma follows. □

Proof of Theorem 5.1 (*continued*). Choose a minimal family $\mathscr{R}$ of critical sets so that

$(*)$ every edge enters at least one member of $\mathscr{R}$.

By definition, such an $\mathscr{R}$ exists. If there are two crossing members X, Y of $\mathscr{R}$, replace X and Y by $X \cap Y$ and $X \cup Y$. By the first part of Lemma 5.2, the new family consists of critical sets and, since $d(X, Y)=0$, it satisfies $(*)$. Since $|X|^2+|Y|^2<|X \cap Y|^2+|X \cup Y|^2$, repeating this procedure we end up, in finitely many steps, with a cross-free family satisfying $(*)$. So, we assume that $\mathscr{R}$ is cross-free.

We are going to show that, for any given node s, there is a node t distinct from s such that $\rho(t)=\delta(t)=k$.

Let $\mathscr{F}:=\{X \in \mathscr{R}\colon s \notin X\}$, $\mathscr{H}:=\{V-X\colon s \in X \in \mathscr{R}\}$ and $\mathscr{L}:=\mathscr{H} \cup \mathscr{F}$. Suppose that $\sum(|X|\colon X \in \mathscr{L})$ is minimal. Note that $\mathscr{L}$ is laminar and $(*)$ transforms into

$(**)$ every edge either enters a member of $\mathscr{F}$ or leaves a member of $\mathscr{H}$ (or both).

Case 1: Every member of $\mathscr{L}$ is a singleton. Let $X:=\{x \in V-s\colon \{x\} \in \mathscr{F}\}$ and $Y:=\{x \in V-s\colon \{y\} \in \mathscr{H}\}$. We want to show that $X \cap Y \neq \emptyset$. Suppose that this is not the case. Then $(**)$ implies that $\delta(X)=0$, from which $X=\emptyset$ follows. But this is not possible since the head of any edge su must be in X.

Case 2: There is a member X of $\mathscr{L}$ with more than one element. Let X be minimal. By symmetry, we can assume that X is in $\mathscr{F}$.

Claim 5.3. *The digraph $(X, E(X))$ induced by X is strongly connected.*

Proof of Claim 5.3. Assume, indirectly, that there is a subset $\emptyset \neq Y \subset X$ for which no edge of G goes from $X-Y$ to Y. Since $\rho(Y) \geqslant k$ and $\rho(X)=k$, every edge entering X must enter Y and $\rho(Y)=k$. Therefore, in $\mathscr{F}$ we can replace X by Y, contradicting the minimal choice of $\mathscr{L}$. $\square$

Proof of Theorem 5.1 (*continued*). Let $A:=\{x \in X\colon \{x\} \in \mathscr{F}\}$ and $B:=\{x \in X\colon \{y\} \in \mathscr{H}\}$. If $A \cap B$ is nonempty, we are done. Suppose that $A \cap B=\emptyset$.

Claim 5.4. $A=\emptyset$.

Proof of Claim 5.4. $A \neq X$ for, otherwise, X can be left out from $\mathscr{F}$ without destroying $(**)$. If, indirectly, $A \neq \emptyset$ then, by Claim 5.3, there is an edge uv with $u \in A$, $v \in X-A$. However, such an edge would violate $(**)$. $\square$

Claim 5.5. $B=X$.

Proof of Claim 5.5. The tail of any edge induced by X must be in B; therefore, B is nonempty. If B, indirectly, is not X then, by Claim 5.3, there is an edge uv with $u \in X-B$, $v \in B$. However, such an edge would violate $(**)$. $\square$

Proof of Theorem 5.1 (*conclusion*). We have shown that $\rho(X)=k$ and $\delta(x)=k$ for every $x\in X$. Hence, $k|X|=\sum(\delta(x)\colon x\in X)=\delta(X)+|E(X)|\geqslant k+|E(X)|=k+\sum(\rho(x)\colon x\in X)-\rho(X)\geqslant k|X|$, from which equality follows everywhere. In particular, $\rho(x)=k$ for every $x\in X$. $\square$

By combining Theorems 4.1 and 5.1 we obtain the following theorem.

Theorem 5.6 (Mader [12]). *A digraph G is k-edge-connected if and only if G can be obtained starting from a single node by applying in any order the following two operations:*

Operation A: *Add a new edge connecting the existing nodes.*

Operation B: *Pick up k arbitrary (distinct) edges, subdivide each by a new node and then identify the k new nodes by shrinking them into one node.*

6. Augmenting digraphs

This section is devoted to demonstrating a recent application of the submodular technique. Let $G=(V,E)$ be a digraph which is not k-edge-connected. Our purpose is to make G k-edge-connected by adding new edges. What is the minimum number of new edges or, equivalently, when is it possible to make G k-edge-connected by adding at most γ new edges?

Theorem 6.1 (Frank [6]). *A digraph $G=(V,E)$ can be made k-edge-connected by adding at most γ new edges if and only if*

$$\sum(k-\rho(X_i))\leqslant\gamma \tag{6.1a}$$

and

$$\sum(k-\delta(X_i))\leqslant\gamma \tag{6.1b}$$

hold for every subpartition $\{X_1,X_2,\dots,X_t\}$ of V.

Proof. *Necessity.* Suppose $G'=(V,E\cup F)$ is a k-edge-connected supergraph of G, where F denotes the set of new edges. Then every subset X_i of V has at least $k-\rho(X_i)$ new entering edges. Therefore, the number of new edges in G' is at least $\sum(k-\rho(X_i))$ and (6.1a) follows. The proof of (6.1b) is analogous.

Let $G'=(V+s,E')$ be a digraph with in-degree and out-degree function ρ' and δ', respectively. The following lemma was proved in Section 4 (Lemma 4.3).

Lemma 6.2. *Suppose for $A,B\subseteq V$ that $\rho'(A)=\rho'(B)=k\leqslant\min(\rho'(A\cap B),\rho'(A\cup B))$. Then $\rho'(A\cap B)=\rho'(A\cup B)=k$ and $d'(A,B)=0$.*

Proof of Theorem 6.1 (*continued*). We prove the sufficiency in two steps. Let s be a node not in V and $V'\!:=V+s$.

Lemma 6.3. *G can be extended to a digraph $G'=(V+s, E')$ by adding a new node s, γ new edges entering s, and γ new edges leaving s in such a way that, for every subset $\emptyset \neq X \subset V$,*

$$\rho'(X) \geqslant k, \tag{6.2a}$$

$$\delta'(X) \geqslant k \tag{6.2b}$$

hold, where ρ' and δ' denote the in-degree and out-degree function of G', respectively.

Proof of Lemma 6.3. We prove that it is possible to add γ edges leaving s so that (6.2a) is satisfied. This will imply (by reorienting every edge of G) that it is possible to add γ edges entering s so that (6.2b) is satisfied. First we add a sufficiently large number of edges leaving s so as to satisfy (6.2a). (It certainly will do if we add k edges from s to v for every $v \in V$.) Second, discard new edges, one by one, as long as possible without violating (6.2a). Let G' denote the final extended digraph. The following claim implies Lemma 6.3. □

Claim 6.4. $\delta'(s) \leqslant \gamma$.

Proof of Claim 6.4. Call a subset $\emptyset \subset X \subset V$ *in-critical* if $\rho'(X)=k$. Let $S := \{v \in V, sv$ is an edge in $G'\}$. An edge sv cannot be left out from G' without violating (6.2a) precisely if sv enters an in-critical set. Therefore, by the minimality of G', there is a family $\mathcal{F} = \{X_1, X_2, \ldots, X_t\}$ of in-critical subsets of V covering S and we can assume that t is minimal.

Case 1: $\mathcal{F}$ consists of disjoint sets. Then we have $kt = \sum(\rho'(X_i)\colon i=1,\ldots,t) = \delta'(s) + \sum(\rho(X_i)\colon i=1,\ldots,t)$ and, hence, by (6.1a), $\delta'(s) = \sum(k-\rho(X_i)\colon i=1,\ldots,t) \leqslant \gamma$.

Case 2: There are two intersecting members A, B of $\mathcal{F}$. If $A \cup B \neq V$, then $A \cup B$ is in-critical by Lemma 6.2 and then, replacing A and B in $\mathcal{F}$ by $A \cup B$, we are in a contradiction with the minimal choice of t. Therefore, $A \cup B = V$.

Let $Y_1 := V - A$ and $Y_2 := V - B$. Then $\delta(Y_1) = \rho(A)$ and $\delta(Y_2) = \rho(B)$. By (6.2b), we have $\gamma \geqslant k - \delta(Y_1) + k - \delta(Y_2) = k - \rho(A) + k - \rho(B) \geqslant k - \rho'(A) + k - \rho'(B) + \delta'(s) = \delta'(s)$.

Therefore, the proof of the Claim 6.4 and Lemma 6.3 is complete. □

Proof of Theorem 6.1 (*conclusion*). The theorem immediately follows by γ repeated applications of Theorem 4.1. □

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