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On a theorem of Mader

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Abstract

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A relatively simple proof is given for (a slight strengthening of) a theorem of W. Mader on the existence of splittable pairs of edges in an undirected graph.

1. Introduction

In an undirected graph $G=(V+s, E)$ let $\lambda(u, v; G)$ (in short, $\lambda(u, v)$) denote the *local edge-connectivity* (or, simply, edge-connectivity) between u and v, that is, the maximum number of edge-disjoint paths connecting u and v. (By the undirected edge-version of Menger's theorem $\lambda(u, v)$ is the minimum cardinality of a cut separating u and v.)

Let $e=su$ and $f=sv$ be two distinct edges of G. *Splitting off* the pair $\{e, f\}$ means that we replace the two edges e, f by a new edge $h=uv$. (Note that if $u=v$, then h is a loop.) The resulting graph is denoted by G^{ef}. Clearly, $\lambda(x, y; G^{ef}) \leq \lambda(x, y; G)$. Call a pair $\{e, f\}$ of edges incident to s *splittable* if $\lambda(x, y; G^{ef}) = \lambda(x, y; G)$ holds for every $x, y \in V$, that is, after splitting $\{e, f\}$ off the edge-connectivity between every two nodes distinct from s remains the same.

Does every graph have a splittable pair? If G is a complete graph on four nodes, then G has no splittable pair of edges. If G is a tree on 5 nodes so that each edge is incident to s (that is G is the star $K_{4,1}$), then there is no splittable pair. These examples show that it is natural to assume that $d(s) \neq 3$ and that

there is no cut-edge incident to s. (*)

Mader [5], answering an earlier conjecture of L. Lovász, proved the following extremely powerful result.

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Theorem A (Mader, [5]). *Let $G = (V + s, E)$ be a connected undirected graph with $d(s) \neq 3$ for which $(*)$ holds, then there is a splittable pair $\{e, f\}$ of edges.*

(A recent application of Mader's theorem occurs in Frank [2] where it is a basic ingredient in a solution to the problem of augmenting graphs so as to satisfy local edge-connectivity prescriptions.)

Earlier Lovász [3–4] had proved that if $d(s)$ is even and $\lambda(u, v; G) \geq k \geq 2$ for every $u, v \in V$, then for a given edge $e = st$ there is an edge $f = su$ so that $\lambda(u, v; G^{ef}) \geq k$ for every $u, v \in V$. As a possible generalization he conjectured the following:

Theorem A′. *Let $G = (V + s, E)$ be a undirected graph for which $(*)$ holds and $d(s)$ is even. Then the set of edges incident to s can be partitioned into $d(s)/2$ disjoint splittable pairs.*

Thus following property will be useful.

Claim 1.1. *If $\{e, f\}$ is splittable in a graph G satisfying $(*)$, then G^{ef} also satisfies $(*)$.*

Proof. By $(*)$ it follows that $\lambda(u, v; G^{ef}) = \lambda(u, v; G) \geq 2$ holds for every pair $\{u, v\}$ of neighbours of s. Hence G^{ef} also satisfies $(*)$. □

Claim 1.2. *Theorems* A *and* A′ *are equivalent.*

Proof. Assume first the truth of Theorem A and let $\{e, f\}$ be a splittable pair. By Claim 1.1 Theorem A can be applied successively $d(s)/2$ times. Now Theorem A′ follows by observing that a pair splittable in G^{ef} is splittable in G, as well.

Conversely, assume that Theorem A′ is true. If $d(s)$ is even, there is nothing to prove so let $d(s)$ be odd. Then $d(s) \geq 5$. Let G' denote a graph arising from G by adding a new node x and three parallel edges connecting s and x. Property $(*)$ holds for G' and hence Theorem A′ applied to G'. Since $d(S) \geq 5$, among the $(d(s) + 3)/2$ splittable pairs provided by Theorem A′ at least one pair $\{e, f\}$ must consist of original edges. Clearly, $\{e, f\}$ is splittable in G, as well. □

If $d(s)$ is odd, then it is not necessarily true that for any given edge st there is an edge su such that $\{st, su\}$ is splittable, as is shown by Fig. 1.

However it immediately follows from Theorem A and Claim 1.1 that there are at most three such bad edges. In Section 5 we are going to show that actually there may be only one bad edge. More specifically, as a slight strengthening of Mader's theorem, the following will be shown.

Theorem B. *Suppose that in $G = (V + s, E)$ property $(*)$ holds and $d(s) \neq 3$. Then there are $\lfloor d(s)/2 \rfloor$ pairwise disjoint splittable pairs of edges incident to s.*

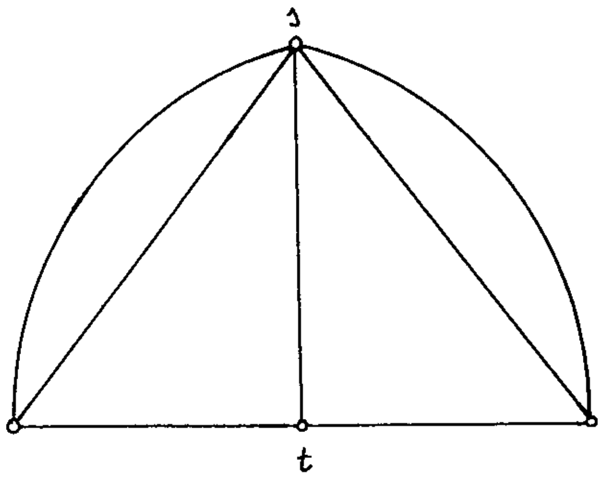

Fig. 1.

2. Notation, preliminaries

We will not distinguish between a one-element set $\{x\}$ and its element x. The union of a set X and an element y is denoted by $X+y$. For two sets X, Y, $X-Y$ denotes the set of elements in X but not in Y. $X \subset Y$ denotes that X is a subset of Y and $X \neq Y$. We will say that a subset $X \subseteq V$ *separates* two elements x and x' of V if $|X \cap \{x, x'\}| = 1$.

We denote an edge e connecting nodes u and v by uv or vu. This is not quite precise since there may be parallel edges between u and v. But this ambiguity will not cause any trouble. Both parallel edges and loops are allowed.

For a graph $G=(V, E)$ and for $X, Y \subseteq V$, $d(X, Y)$ denotes the number of edges between $X-Y$ and $Y-X$ and $\bar{d}(X, Y) := d(X \cap Y, V-(X \cup Y))$. Let $d(X) := d(X, V-X)$. The number $d(v)$ of edges incident to a node v is called the *degree* of v. Throughout the paper we will adopt the convention that for any function f concerning graph G the corresponding function concerning another graph G' is denoted by f'.

Deleting an edge e means that we leave out e from E while the node set V is unchanged. For the resulting graph we use the notation $G-e$. *Deleting* a subset C of nodes means that we leave out the elements of C and all the edges incident to some elements of C. The resulting graph is denoted by $G-C$. *Contracting* a subset C of nodes means a graph arising from G by adding a new node v_C to $G-C$ and $d(v, C)$ parallel edges between v and v_C for every $v \in V-C$. The resulting graph is denoted by G/C. We call an edge e of a graph $G=(V, E)$ a *cut edge* if $G-e$ has more components than G.

The following proposition is easy to prove if we observe that each edge has the same contribution to the two sides of the identities.

Proposition 2.1. *Let $H=(U, E)$ be an arbitrary graph and $X, Y \subseteq U$. Then*

$$d(X)+d(Y)=d(X \cap Y)+d(X \cup Y)+2d(X, Y), \tag{2.1}$$

$$d(X)+d(Y)=d(X-Y)+d(Y-X)+2\bar{d}(X, Y). \tag{2.2}$$

Let $G=(V+s, E)$ be a graph. Denote $R(X):=\max(\lambda(u, v)\colon u \in X, v \in V-X)$. Obviously $d(X) \geqslant R(X)=R(V-X)$. If equality holds, X is called *tight*. Let $s(X):=d(X)-R(X)$ denote the *surplus* of X. Clearly $s(X) \geqslant 0$. The following observation was already used in [2].

Proposition 2.2. *For arbitrary $X, Y \subseteq V$ at least one of the following inequalities holds*:

$$R(X)+R(Y) \leqslant R(X \cap Y)+R(X \cup Y), \tag{2.3α}$$

$$R(X)+R(Y) \leqslant R(X-Y)+R(Y-X). \tag{2.3β}$$

Proof. First observe that if Y is replaced by $V-Y$, then (2.3α) and (2.3β) transform into each other. Let (z, z') be a pair that maximizes $\lambda(z, z')$ over all pairs which are separated by at least one of X and Y. By symmetry we may assume that $z \in X$ and $z' \in V-X$. By replacing Y by $V-Y$ if necessary, we may also assume that $z \notin Y$.

If $z' \in Y$, then $\lambda(z, z')=R(X)=R(Y)=R(X-Y)=R(Y-X)$ and hence (2.3β) holds (actually with equality). If $z' \notin Y$, then $\lambda(z, z')=R(X)= R(X \cup Y)=R(X-Y)$. Clearly, $R(Y) \leqslant R(X \cap Y)$ or $R(Y) \leqslant R(Y-X)$. Accordingly, (2.3α) or (2.3β) holds. □

By combining the last two propositions we obtain the following.

Proposition 2.3. *For arbitrary $X, Y \subseteq V$ at least one of the following inequalities holds*:

$$s(X)+s(Y) \geqslant s(X \cap Y)+s(X \cup Y)+2d(X, Y), \tag{2.4α}$$

$$s(X)+s(Y) \geqslant s(X-Y)+s(Y-X)+2\bar{d}(X, Y). \tag{2.4β}$$

3. Properties of splitting

Let $G=(V+s, E)$ be an undirected graph satisfying (*). In this section $d(s)$ may be odd or even. We are going to exhibit some properties concerning the splitting off operation. Let $S:=\{v \in V\colon sv \in E\}$ denote the set of neighbours of s. Recall that a set X was called tight if $d(X)=R(X)$. We call a set $X \subseteq V$ *dangerous* if $d(X) \leqslant R(X)+1$, that is, $s(X) \leqslant 1$.

Claim 3.1. *A pair $\{su, sv\}$ is splittable if and only if there is no dangerous set X containing u and v.*

Proof. The existence of such an X clearly prevents $\{su, sv\}$ from being splittable. Conversely, suppose that $\{e=su, f=sv\}$ is not splittable. Let $G':=G^{ef}$. Then there is a pair $\{x, y\}$ of nodes for which $\lambda'(x, y)<\lambda(x, y)$ and there is a set $X \subseteq V$

separating x and y for which $d'(X) = \lambda'(x, y)$. Hence $d'(X) < d(X)$ and therefore $u, v \in X$. We have $d(X) - 2 = d'(X) = \lambda'(x, y) \leq \lambda(x, y) - 1 \leq R(X) - 1$, from which $d(X) \leq R(X) + 1$. That is, X is a dangerous set containing u and v. □

The following claim was already used by Mader in his proof.

Claim 3.2. *Let T be a tight set ($\emptyset \subset T \subseteq V$). A pair $\{e = su, f = sv\}$ of edges is splittable in G if the corresponding pair $\{e', f'\}$ is splittable in $G' := G/T$.*

Proof. For a subset Z of nodes of G for which either $Z \subseteq V - T$ or $T \subseteq Z \subseteq V$ let Z' denote the subset of nodes of G' corresponding to Z. For such a Z, clearly $R(Z') \geq R(Z)$ and $d(Z') = d(Z)$. Therefore if Z is dangerous in G, then Z' is dangerous in G'.

By Claim 3.1 if $\{e, f\}$ is not splittable in G, then there is a dangerous subset X for which $u, v \in X$. Clearly, $Z := X \cup T$ cannot be dangerous in G for otherwise Z' would be dangerous in G' and then $\{e', f'\}$ would not be splittable in G'. Hence $s(X \cup T) \geq 2$. Apply Proposition 2.3 to X and T. Alternative (2.4α) cannot hold since otherwise we would have

$$0 + 1 \geq s(T) + s(X) \geq s(X \cap T) + s(X \cup T) \geq 0 + 2.$$

Hence (2.4β) must hold. We have

$$\begin{aligned} 0 + 1 &\geq s(T) + s(X) \geq s(T - X) + s(X - T) + 2\bar{d}(X, T) \\ &\geq 0 + 0 + 2\bar{d}(X, T). \end{aligned}$$

Hence $2\bar{d}(X, T) = 0$ and $s(X - T) \leq 1$ follows. The equality shows that $u, v \in D := X - T$ while the inequality means that D is dangerous in G. Then D' is dangerous in G' showing that $\{e', f'\}$ is not splittable in G', a contradiction. □

Claim 3.3. *Suppose that*

$$\text{every tight set consists of one element.} \tag{3.1}$$

Then $\lambda(x, y) = \min(d(x), d(y))$ for every $x, y \in V$.

Proof. The claim immediately follows if we notice that a set $X \subseteq V$ is tight provided that X separates x and y and $\lambda(x, y) = d(X)$. □

4. Proof of Theorem A′

Recall that in Theorem A′ $d(s)$ is supposed to be even. By Claim 1.1 it suffices to prove that there is one splittable pair. Let $G = (V + s, E)$ be a counter-example with a minimum number of nodes. That is, we assume that there is no splittable pair of edges in G but the theorem holds for every smaller graph. From Claim 3.2 it follows that (3.1) holds for G. Let S denote the set of neighbours of s and let $t \in S$ be a node of minimum degree.

Claim 4.1. *$R(X-t) \geqslant R(X)$ holds for every set $X \subseteq V$ with $t \in X$, $|S \cap X| \geqslant 2$.*

Proof. Let $u \in S \cap (X-t)$. $d(u) \geqslant d(t)$ holds by the choice of t. $R(X)=\lambda(v, z)$ for some $v \in X$, $z \in V-X$. If $v \neq t$, then $R(X-t) \geqslant \lambda(v, z)=R(X)$, as required. If $v=t$, then by Claim 3.3 we have

$$R(X)=\lambda(t, z)=\min(d(t), d(z)) \leqslant \min(d(u), d(z))=\lambda(u, z) \leqslant R(X-t),$$

as required. □

Claim 4.2. *If X is dangerous, then $d(s, X) \leqslant d(s, V-X)$.*

Proof. Let $\alpha:=d(s, X)$ and $\beta:=d(s, V-X)$. We have

$$\begin{aligned} R(V-X)=R(X) \geqslant d(X)-1 &= d(V-X)-\beta+\alpha-1 \\ &\geqslant R(V-X)-\beta+\alpha-1 \end{aligned}$$

from which $\alpha \leqslant \beta+1$ follows. However, we cannot have equality for otherwise $d(s)=2\beta+1$ would follow but $d(s)$ is assumed to be even. □

Since no pair $\{st, su\}$ is splittable, Claim 3.1 implies that every element of S belongs to a dangerous set containing t. Let $\mathscr{L}$ be a minimal family of dangerous sets containing t so that $\bigcup(X: X \in \mathscr{L}) \supseteq S$.

Claim 4.3. $|\mathscr{L}| \geqslant 3$.

Proof. By Claim 4.2 $|\mathscr{L}| \geqslant 2$. Assume that $|\mathscr{L}|=2$, that is, $S \subseteq X \cup Y$ where $\mathscr{L}=\{X, Y\}$. By Claim 4.2

$$d(s, X) \leqslant d(s, V-X)<d(s, Y) \leqslant d(s, V-Y)<d(s, X),$$

a contradiction. Here the last inequality holds since $(S-X) \cup\{t\} \subseteq Y$. □

Let X_1, X_2, X_3 be three members of $\mathscr{L}$ and $\mathscr{F}:=\{X_1, X_2, X_3\}$. By the minimality of $\mathscr{L}$ each X_i contains an element x_i of S that does not belong to any other member of $\mathscr{F}$.

Claim 4.4. *For every two members X and Y of $\mathscr{F}$* (2.4β) *holds.*

Proof. Suppose, indirectly, that (2.4β) does not hold. Then by Proposition 2.3 (2.4α) holds. By the minimality of $\mathscr{L}, s(X \cup Y) \geqslant 2$. Therefore $1+1 \geqslant s(X)+s(Y) \geqslant s(X \cap Y)+s(X \cup Y) \geqslant 0+2$ and hence $s(X \cap Y)=0$ follows, that is, $X \cap Y$ is tight. Since (3.1) holds, $X \cap Y = \{t\}$. Then $X - Y = X - t$ and $Y - X = Y - t$ and by Claim 4.1 $R(X) \leq R(X - Y)$ and $R(Y) \leq R(Y - X)$. Therefore $s(X) + s(Y) \geq s(X - Y) + s(Y - X) + 2\bar{d}(X, Y)$, that is (2.4β) holds, a contradiction. □

Claim 4.5. *For every two members X and Y of $\mathcal{F}$, $|X - Y| = |Y - X| = 1$ and $\bar{d}(X, Y) = 1$.*

Proof. By Claim 4.4 we have

$$1 + 1 \geq s(X) + s(Y) \geq s(X - Y) + s(Y - X) + 2\bar{d}(X, Y) \geq 0 + 0 + 2.$$

Hence $\bar{d}(X, Y) = 1$ and both $X - Y$ and $Y - X$ are tight. Since (3.1) holds for G, the statement follows. □

Let $M := X_1 \cap X_2 \cap X_3$. From Claim 4.5 and from the minimality of $\mathcal{L}$ it follows that $X_i = M + x_i$ for $1 \leq i \leq 3$ and $\bar{d}(X_i, X_j) = 1 (1 \leq i < j \leq 3)$. Hence only one edge leaves M, the edge st. That is, st is a cut edge, contradicting (*) and this contradiction proves the theorem. □

5. Proof of Theorem B

By Theorem A′ we can assume that $d(s)$ is odd. Let us assume that

G is a minimal counter-example. (**)

Let $S := \{v \in V : sv \in E\}$ denote the set of neighbours of s. It is straightforward that $|S| \geq 2$. Claim 3.2 implies that (3.1) holds for G.

Claim 5.1. $d(s) = 5$.

Proof. Suppose that $d(s) \geq 6$. By Theorem A there is a splittable pair $\{e, f\}$. By Claim 1.1 (*) holds for $G' := G^{ef}$ and $d'(s) = d(s) - 2 \geq 4$. By the minimal choice of G Theorem B holds for G'. Thus there are $\lfloor d'(s)/2 \rfloor$ disjoint splittable pairs in G'. These pairs along with $\{e, f\}$ provide $\lfloor d(s)/2 \rfloor$ disjoint splittable pairs in G, contradicting (**). □

Claim 5.2. *If X is dangerous and $d(s, X) \geq 3$, then $d(s, X) = 3$ and $|V - X| = 1$.*

Proof. Since $d(s) = 5$ and $d(s, X) \geq 3$ we have

$$\begin{aligned} R(V - X) &\leq d(V - X) = d(X) - d(s, X) + d(s, V - X) \\ &\leq d(X) - 1 \leq R(X) = R(V - X). \end{aligned}$$

Hence $d(s, X) = 3$ and $d(s, V - X) = 2$. Moreover, $V - X$ is tight and therefore $V - X$ consists of one node. □

Claim 5.3. *There are no parallel edges incident to s.*

Proof. Let e_1 and e_2 be parallel edges connecting s and u. If the pair $\{e_1, g\}$ is splittable for every edge $g = sv$ not parallel to e_1, then let g_1 and g_2 be two edges incident to s that are not parallel to e_1. Now $\{e_i, g_i\}$ $(i = 1, 2)$ would be two splittable pairs despite of (**). So there is an edge $g = sv$ not parallel to e_1 for which $\{e_1, g\}$ is not splittable. Then there is a dangerous set X containing u and v.

By Claims 5.1 and 5.2 $d(s, V - X) = 2$ and $V - X$ consists of one node z. We obtained that $S = \{u, v, z\}$, that there are two parallel edges f_1, f_2 connecting s and z and just one edge (namely g) connecting s and v. However now $\{e_1, f_1\}$ is splittable since otherwise there is a dangerous set Y containing u and z and then $d(s, Y) \geqslant 4$ contradicting Claim 5.2. Therefore the pairs $\{e_i, f_i\}(i = 1, 2)$ are two disjoint splittable pairs, contradicting (**). □

Claim 5.4. *There is no dangerous set X with* $d(s, X) \geqslant 3$.

Proof. Let X be a dangerous set with $d(s, X) \geqslant 3$. By Claim 5.2 $V - X$ consists of one node z and $d(s, z) = 2$, contradicting Claim 5.3. □

Claim 5.5. *$G - s$ is connected.*

Proof. Let $G - s$ be disconnected. Since $d(s) = 5$ and (*) holds, $G - s$ has two components U and V. Let $e = su$, $f = sv$ be edges so that u and v belong to U and V, respectively. We claim that $\{e, f\}$ is splittable. For otherwise, by Claim 3.1, there is a dangerous set X containing u and v. Let $A := U \cap X$ and $B := V \cap X$. By symmetry we may assume that $R(A) \leqslant R(B)$. Then clearly $R(X) \leqslant R(B)$. We have $d(A) + d(B) - 1 = d(X) - 1 \leqslant R(X) \leqslant R(B) \leqslant d(B)$. It follows that $d(A) \leqslant 1$ and hence su is the only edge leaving A, that is, su is a cut-edge contradicting (*). Let e_1, e_2 be edges connecting s and U and f_1, f_2 edges connecting s and V. We have obtained that the pairs $\{e_i, f_i\}$ $(i = 1, 2)$ are splittable, contradicting (**). □

Let $t \in S$ be a node of minimum degree. Let G' denote the graph arising from G by deleting the edge st. Since $d(s) = 5$ and $G - s$ is connected, (*) holds for G'.

Claim 5.6. $\lambda'(x, y) = \lambda(x, y)$ *for every* $x, y \in V - t$.

Proof. Since (3.1) holds for G, the claim immediately follows. □

Claim 5.7. *If a pair* $\{e = su, f = sv\}$ *is splittable in* G', *then it is splittable in* G.

Proof. If the pair $\{e, f\}$ is not splittable in G, then there is a dangerous set X containing u and v. By Claim 5.4 $t \notin X$ and there is a node $z \in S-(X+t)$. By the choice of t, $d(t) \leqslant d(z)$.

$R(X)=\lambda(x, y)$ for some $x \in X, y \in V-X$. If $y \neq t$, then using Claim 5.6 we have

$$d(X)-1 \leqslant R(X)=\lambda(x, y)=\lambda'(x, y) \leqslant R'(X) \leqslant d'(X)-2=d(X)-2,$$

a contradiction. If $y=t$, then using Claims 3.3 and 5.6 we have

$$\begin{aligned} d(X)-1 \leqslant R(X)=\lambda(x, t)&=\min(d(x), d(t)) \leqslant \min(d(x), d(z))=\lambda(x, z) \\ &=\lambda'(x, z) \leqslant R'(X) \leqslant d'(X)-2=d(X)-2, \end{aligned}$$

a contradiction. □

Since $d'(s)=4$, Theorem A′ applies to G'. Hence there are two disjoint splittable pairs in G'. Claim 5.7 shows that these pairs are splittable in G, as well, contradicting (**) and thereby the proof is complete. □

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