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PRESERVING AND INCREASING LOCAL EDGE-CONNECTIVITY IN MIXED GRAPHS *

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Abstract. Generalizing and unifying earlier results of W. Mader, and A. Frank and B. Jackson, we prove two splitting theorems concerning mixed graphs. By invoking these theorems we obtain min-max formulae for the minimum number of new edges to be added to a mixed graph so that the resulting graph satisfies local edge-connectivity prescriptions. An extension of Edmonds's theorem on disjoint arborescences is also deduced along with a new sufficient condition for the solvability of the edge-disjoint paths problem in digraphs. The approach gives rise to strongly polynomial algorithms for the corresponding optimization problems.

Key words. mixed graphs, splitting theorems, edge-connectivity, augmentation, branchings, network synthesis

AMS subject classifications. 05C40, 68R10, 90B

1. Introduction and preliminaries. Our main concern, the edge-connectivity augmentation problem, is as follows: given a mixed graph M, what is the minimum number (or, more generally, the minimum cost) γ of new edges to be added to M so that in the resulting graph M', the local edge-connectivity $\lambda(x, y; M')$ between every pair of nodes x, y is at least a prescribed value $r(x, y)$?

Several special cases were solved earlier for directed and undirected graphs. First, let M be undirected. When $r \equiv 1$, the minimum cost augmentation problem reduces to a minimum cost tree problem. For $r \equiv 2$, the problem was solved independently by Eswaran and Tarjan [4] and Plesnik [22]. For this case, the minimum cost augmentation problem is already NP-complete.

The uniform case $r \equiv k$ for an arbitrary integer $k \geq 2$ was first solved by Watanabe and Nakamura [24], who developed a polynomial time algorithm as well as a min-max relationship. Slightly later, Cai and Sun [1] also solved this special case. The algorithm of Watanabe and Nakamura has been improved by Naor, Gusfield, and Martel [21]. Neither of these algorithms gives rise to a strongly polynomial time algorithm in the capacitated case. The first such approach was given by Frank [6]. The same paper includes a complete solution of the generalization to arbitrary (symmetric) demand functions $r(u, v)$.

For directed augmentation, the case $r \equiv 1$ was solved by Eswaran and Tarjan [4] while the general uniform case $r \equiv k (\geq 1)$ was solved by Frank [6]. Another interesting approach is by Gabow [9]. A related problem on augmentation was solved by Gusfield

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[12], who described a way of adding a minimum number of directed or undirected edges to a mixed graph so that each edge belongs to a (possibly mixed) circuit with no backward directed edge. (Note, however, that our general mixed augmentation problem is not a generalization of Gusfield's.) Finally, several degree-constrained and node-cost variants were also solved by Frank [6].

On the negative side, for directed graphs the nonuniform demand problem was shown to be NP-complete by Frank [6] even if $r(u,v) \equiv 1$ for every pair of nodes u,v of a specified subset $T \subset V$ and $r(u,v) \equiv 0$ otherwise. In this light, relatively little space is left for possible generalizations admitting good characterizations and/or polynomial time algorithms. (This sentence may serve as an excuse in case the reader feels that the hypothesis of the generalizations we discuss below is more technical than necessary.)

In the present paper we show how the augmentation problem for mixed graphs can be solved for certain demand functions that are more general than the uniform one. (By a *mixed graph* M we mean a graph that may have both directed and undirected edges.) When the starting graph is mixed, one may wish to add both directed and undirected edges. Unfortunately, we do not have anything to say about this general case. Our results concern only the two extremes, when either only directed edges or undirected edges are allowed to be added to the given mixed graph M.

Splitting off a pair of edges $e = us, f = st$ means that we replace e and f by a new edge ut. The resulting mixed graph will be denoted by M^{ef}. This operation is defined only if both e and f are undirected (respectively, directed) and then the newly added edge ut is considered undirected (directed). Accordingly, we speak of undirected or directed splittings.

Two theorems of W. Mader concerning directed and undirected splittings are important tools in the proofs by Frank in [6]. Here we follow an analogous line, and the basis for the present generalization is an extension of the existing splitting theorems. When a splitting-off operation is performed, the local edge-connectivity never increases. The content of the splitting-off theorems is that under certain conditions there is an appropriate pair $\{e = us, f = st\}$ of edges whose splitting preserves all local or global edge-connectivity between nodes distinct from s.

An interesting by-product of our investigations is an extension of Edmonds's theorem on the existence of k disjoint arborescences [2]. A new sufficient condition will also be deduced for the existence of k edge-disjoint paths in a directed graph connecting specified pairs of nodes.

Given two elements s,t and a subset X of a ground-set U, we say that X is an *$s\bar{t}$-set* if $s \in X, t \notin X$. X *separates s from t* (or s and t) if $|X \cap \{s,t\}| = 1$. A family $\{X_1,\ldots,X_t\}$ of pairwise disjoint, nonempty subsets of U is called a *subpartition.*

Let $G = (U,E)$ be an undirected graph. $d_G(X,Y)$ denotes the number of undirected edges between $X-Y$ and $Y-X$. $\bar{d}_G(X,Y) := d_G(X,U-Y)(= d_G(U-X,Y))$. $d_G(X)$ stands for $d_G(X,U-X)$. Observe that $\bar{d}_G(X,Y) = \bar{d}_G(U-X,U-Y)$. When it will not cause ambiguity we shall leave out the subscript.

PROPOSITION 1.1. *For $X,Y \subseteq U$,*

$$d_G(X) + d_G(Y) = d_G(X \cap Y) + d_G(X \cup Y) + 2d_G(X,Y), \tag{1.1a}$$

$$d_G(X) + d_G(Y) = d_G(X - Y) + d_G(Y - X) + 2\bar{d}_G(X,Y). \tag{1.1b}$$

For a directed graph $D = (U,A), \rho_D(X)$ denotes the number of edges entering $X, \delta_D(X) := \rho_D(U-X)$, and $\beta_D(X) := \min(\rho_D(X),\delta_D(X))$. Note that $\beta_D(X) = \beta_D(U-X)$. $d_D(X,Y)$ denotes the number of edges with one end in $X-Y$ and one end in $Y - X$. $\bar{d}_D(X,Y) := d_D(X, U-Y) (= d_D(U-X, Y)$. An *out-arborescence* F is a directed tree in which every node but one has in-degree 1 and the exceptional node, called the *root*, is of in-degree 0. (Equivalently, there is a directed path from the root to every other node of F.)

PROPOSITION 1.2. *For* $X, Y \subseteq U$,

$$\rho_D(X) + \rho_D(Y) = \rho_D(X \cap Y) + \rho_D(X \cup Y) + d_D(X,Y). \tag{1.2a}$$

If $\delta_D(X \cap Y) = \rho_D(X \cap Y)$, *then*

$$\rho_D(X) + \rho_D(Y) = \rho_D(X - Y) + \rho_D(Y - X) + \bar{d}_D(X,Y). \tag{1.2b}$$

If $\delta_D(X \cup Y) = \rho_D(X \cup Y)$, *then*

$$\rho_D(X) + \rho_D(Y) = \delta_D(X - Y) + \delta_D(Y - X) + \bar{d}_D(X,Y). \tag{1.2c}$$

If $\delta_D(X \cap Y) = \rho_D(X \cap Y)$ *or* $\delta_D(X \cup Y) = \rho_D(X \cup Y)$, *then*

$$\beta_D(X) + \beta_D(Y) \geq \beta_D(X - Y) + \beta_D(Y - X) + \bar{d}_D(X,Y). \tag{1.2d}$$

Proof. Equation (1.2a) follows by showing that each edge has the same contribution to the two sides. A similar argument shows that $\rho_D(X) + \rho_D(Y) = \rho_D(X - Y) + \rho_D(Y - X) + \bar{d}_D(X,Y) + (\rho_D(X \cap Y) - \delta_D(X \cap Y))$ holds for any digraph D from which (1.2b) follows. The derivation of (1.2c) is analogous.

Let us prove (1.2d). The two cases are clearly equivalent: Substitute $U - X$ for X and $U - Y$ for Y. So assume that $\delta_D(X \cap Y) = \rho_D(X \cap Y)$. If $\beta_D(X) = \rho_D(X)$ and $\beta_D(Y) = \rho_D(Y)$ then by (1.2b), $\beta_D(X) + \beta_D(Y) = \rho_D(X) + \rho_D(Y) = \rho_D(X - Y) + \rho_D(Y - X) + \bar{d}_D(X,Y) \geq \beta_D(X - Y) + \beta_D(Y - X) + \bar{d}_D(X,Y)$. The case when $\beta_D(X) = \delta_D(X)$ and $\beta_D(Y) = \delta_D(X)$ is analogous. Finally, suppose that $\beta_D(X) = \rho_D(X)$ and $\beta_D(Y) = \delta_D(Y)$. Let $Y' := U - Y$. Then, applying (1.2a) to X and Y' we get $\beta_D(X) + \beta_D(Y) = \rho_D(X) + \rho_D(Y') = \rho_D(X \cap Y') + \rho_D(X \cup Y') + d_D(X, Y') = \rho_D(X - Y) + \delta_D(Y - X) + \bar{d}_D(X,Y) \geq \beta_D(X - Y) + \beta_D(Y - X) + \bar{d}_D(X,Y)$. □

Let $M = (U, A \cup E)$ be a mixed graph composed as the union of a directed graph $D = (U, A)$ and an undirected graph $G = (U, E)$. Let $\rho_M(X) := \rho_D(X) + d_G(X), \delta_M(X) := \delta_D(X) + d_G(X)$, and $\beta_M(X) := \min(\rho_M(X), \delta_M(X))$. We say that a node v of a M is *di-Eulerian* if $\rho_D(v) = \delta_D(v)$. M is called di-Eulerian if every node of M is di-Eulerian.

By combining Propositions 1.1 and 1.2, we obtain the following proposition.

PROPOSITION 1.3. *For* $X, Y \subseteq U$,

$$\rho_M(X) + \rho_M(Y) = \rho_M(X \cap Y) + \rho_M(X \cup Y) + d_D(X,Y) + 2d_G(X,Y). \tag{1.3a}$$

If $\delta_D(X \cap Y) = \rho_D(X \cap Y)$, *then*

$$\rho_M(X) + \rho_M(Y) = \rho_M(X - Y) + \rho_M(Y - X) + \bar{d}_D(X,Y) + 2\bar{d}_G(X,Y). \tag{1.3b}$$

If $\delta_D(X \cup Y) = \rho_D(X \cup Y)$, *then*

$$\rho_M(X) + \rho_M(Y) = \delta_M(X - Y) + \delta_M(Y - X) + \bar{d}_D(X,Y) + 2\bar{d}_G(X,Y). \tag{1.3c}$$

If $\delta_D(X \cap Y) = \rho_D(X \cap Y)$ *or* $\delta_D(X \cup Y) = \rho_D(X \cup Y)$, *then*

$$\beta_M(X) + \beta_M(Y) \geq \beta_M(X - Y) + \beta_M(Y - X) + \bar{d}_D(X,Y) + 2\bar{d}_G(X,Y). \tag{1.3d}$$

By a *feasible path* (or simply *path*) of a mixed graph M we mean a sequence $\{v_0, v_0v_1, v_1, v_1v_2, v_2, \ldots, v_{n-1}, v_{n-1}v_n, v_n\}$, where each v_iv_{i+1} is a directed or undirected edge of M. The *local edge-connectivity* $\lambda(s,t;M) = \lambda(s,t)$ from s to t is in the maximum number of edge-disjoint paths from s to t. By a version of Menger's theorem, this is equal to the minimum of $\delta_D(S) + d_G(S)$ over all $s\bar{t}$-sets S. Note that $\lambda(s,t)$ can be computed by a max-flow min-cut (MFMC) computation.

Let U be a set and $r(x,y)(x,y \in U)$, an arbitrary symmetric, nonnegative function. Define a set function R as follows. Let $R(\emptyset) = R(U) = 0$, and for $X \subset U$ let

$$R(X) := \max(r(x,y) : x,y \in U, X \text{ separates } x \text{ and } y). \tag{1.4}$$

Clearly, $R(X) = R(U - X)$.

LEMMA 1.1. *For arbitrary $X, Y \subseteq U$, at least one of the following two inequalities holds:*

$$R(X) + R(Y) \le R(X \cap Y) + R(X \cup Y), \tag{1.5a}$$

$$R(X) + R(Y) \le R(X - Y) + R(Y - X). \tag{1.5b}$$

Proof. First observe that if Y is replaced by $U - Y$, then (1.5a) and (1.5b) transform into each other. Let (z, z') be a pair that maximizes $r(z,z')$ over all pairs which are separated by at least one of the sets X and Y. By symmetry we may assume that $z \in X$ and $z' \in U - X$. By replacing Y by $U - Y$, if necessary, we may also assume that $z \notin Y$.

If $z' \in Y$, then $r(z,z') = R(X) = R(Y) = R(X-Y) = R(Y-X)$, and hence (1.5b) holds (actually with equality). If $z' \notin Y$, then $r(z,z') = R(X) = R(X \cup Y) = R(X - Y)$. Clearly, $R(Y) \le R(X \cap Y)$ or $R(Y) \le R(Y - X)$. Accordingly, (1.5a) or (1.5b) holds. □

Let $M = (U, A \cup E)$ be a mixed graph with a specified node s satisfying $\rho_M(s) = \delta_M(s)$. Throughout this paper we will use the notation $V := U - s$. Let

$$T(M) := \{x \in V : \rho_M(x) \neq \delta_M(x)\} \tag{1.6}$$

be the set of non-di-Eulerian nodes. Observe that $\rho_M(T(M)) = \delta_M(T(M))$ and hence $T(M)$ never consists of one element. Let k be a positive integer and assume that

$$\lambda(x,y;M) \ge k \quad \text{for every } x,y \in T(M). \tag{1.7}$$

Suppose that $r(x,y)$ satisfies

$$r(x,y) \le k \quad \text{for every } x,y \in U, \quad \text{and} \tag{1.8a}$$

$$r(x,y) = k \quad \text{for every } x,y \in T(M). \tag{1.8b}$$

For $X \subseteq U$ define $q(X) := R(X) - \beta_M(X)$.

LEMMA 1.2. *For $X, Y \subseteq U$, at least one of the following two inequalities holds:*

$$q(X) + q(Y) \le q(X \cap Y) + q(X \cup Y) - (2d_G(X,Y) + d_D(X,Y)), \tag{1.9a}$$

$$q(X) + q(Y) \le q(X - Y) + q(Y - X) - (2\bar{d}_G(X,Y) + \bar{d}_D(X,Y)). \tag{1.9b}$$

Proof. Since $q(X) = q(U - X)$, the inequalities in (1.9) transform into each other when X is replaced by its complement $U - X$. Therefore, we can assume that $\beta_M(X) = \rho_M(X) \le \delta_M(X)$ and $\beta_M(Y) = \rho_M(Y) \le \delta_M(Y)$.

If R satisfies (1.5a), then by subtracting (1.3a) from (1.5a) we obtain (1.9a). Now assume that R does not satisfy (1.5a). Then at least one of $T(M) \cap X \cap Y$ and $T(M) - (X \cup Y)$ is empty, otherwise (1.4) and (1.8) would imply that $R(X) = R(Y) = R(X \cap Y) = R(X \cup Y) = k$, and hence (1.5a) would hold. Therefore (1.3d) holds. Furthermore, by Lemma 1.1, R satisfies (1.5b). Subtracting (1.3d) from (1.5b) we obtain (1.9b). □

For $x, y \in U$, let us define

(1.10a) $$r_M(x,y) := \min(k, \lambda(x,y;M)) \qquad \text{if } x, y \in V,$$

(1.10b) $$r_M(x,y) := 0 \quad \text{if } s \in \{x,y\}.$$

Note that r_M depends on M, s, and k and satisfies (1.8).

LEMMA 1.3. $r_M(x,y) = r_M(y,x)$.

Proof. This clearly holds if $\lambda(x,y;M) = \lambda(y,x;M)$ and, by (1.10b), $s \in \{x,y\}$. Assume that $x, y \in V$ and $\lambda(x,y;M) < \lambda(y,x;M)$. There is an $x\bar{y}$-set X for which $\delta_M(X) = \lambda(x,y;M)$. We cannot have $\lambda(x,y;M) < k$ since at least one of the sets X and $U - X$, say X, is then disjoint from $T(M)$. But then $\rho_M(X) = \delta_M(X)$ and hence $\lambda(x,y;M) = \delta_M(X) = \rho_M(X) \geq \lambda(y,x;M)$, a contradiction. Therefore, $k \leq \lambda(x,y;M) < \lambda(y,x;M)$, that is, $r_M(x,y) = k = r_M(y,x)$ as required. □

Define

(1.11) $$R_M(X) := \max(r_M(x,y) : X \text{ separates } x \text{ and } y).$$

Note that by (1.10) $R_M(X) = R_M(V - X)$ for every $X \subseteq V$.

LEMMA 1.4. *For any subset $X \subseteq V$ separating nodes $x, y \in V$,*

(1.12a) $$\beta_M(X) \geq R_M(X) \geq r_M(x,y).$$

Moreover, if $\lambda(x,y;M) \leq k$, then there is a subset X_0 of V separating x and y for which

(1.12b) $$\beta_M(X_0) = r_M(x,y).$$

Proof. By symmetry we may assume that $x \in X$ and $y \in V - X$. Then $r_M(x,y) \leq \lambda(x,y;M) \leq \delta_M(X)$ and $r_M(y,x) \leq \lambda(y,x;M) \leq \rho_M(X)$. From this and Lemma 1.3 we get $\beta_M(X) \geq r_M(x,y)$. This, in turn, along with the definition of $R_M(X)$, implies (1.12a).

If $\lambda(x,y;M) \leq k$, then $\lambda(x,y;M) = r_M(x,y)$ and, by Menger's theorem, there is a subset $X_0 \subset V$ separating x and y for which $\beta_M(X_0) = \lambda(x,y;M)$ and (1.12b) follows. □

Let $s_M(X) := \beta_M(X) - R_M(X)$. By Lemma 1.4, $s_M(X) \geq 0$ for every $X \subseteq V$. We call a nonempty set $X \subseteq V$ *tight* (*dangerous*) if $s_M(X) = 0\,(s_M(X) \leq 1)$. We may distinguish between the two possible types of tight (dangerous) sets by use of the prefix "in" or "out." Note that V is not tight if $\rho_M(s), \delta_M(s) > 0$. Lemma 1.2 immediately provides the following lemma.

LEMMA 1.5. *For $X, Y \subseteq V$, one of the following inequalities holds:*

(1.13a) $$s_M(X) + s_M(Y) \geq s_M(X \cap Y) + s_M(X \cup Y) + 2d_G(X,Y) + d_D(X,Y),$$

(1.13b) $$s_M(X) + s_M(Y) \geq s_M(X - Y) + s_M(Y - X) + 2\bar{d}_G(X,Y) + \bar{d}_D(X,Y).$$

We are going to prove two splitting theorems for M. In §2 each edge incident to s is directed, while in §3 the edges incident to s are all undirected.

2. Directed splitting. When a splitting operation is carried out, the local edge-connectivity may drop. There are theorems for directed graphs stating that global or local edge-connectivities may be preserved by an appropriate choice of edges to be split off. One is from Mader [20].

THEOREM 2.1. *Let $D = (V + s, A)$ be a directed graph for which $\lambda(x, y; D) \geq k$ for every $x, y \in V$ and $\rho(s) = \delta(s)$. Then, for every edge $f = st$ there is an edge $e = us$ such that $\lambda(x, y; D^{ef}) \geq k$ for every $x, y \in V$.*

The next theorem was proven by Frank [5] and Jackson [13].

THEOREM 2.2. *Let $D = (V + s, A)$ be a directed Eulerian graph, that is, $\rho(x) = \delta(x)$ for every node x of D. Then, for every edge $f = st$ there is an edge $e = us$ such that $\lambda(x, y; D^{ef}) = \lambda(x, y; D)$ for every $x, y \in V$.*

Our first result is a common generalization of these two theorems. (Recall the definition of function $r_M(x, y)$ in (1.10).)

THEOREM 2.3. *Let $M = (V + s, A \cup E)$ be a mixed graph, satisfying* (1.7). *Assume that s is incident only with directed edges and $\rho_M(s) = \delta_M(s) > 0$. Then, for every edge $f = st$, there is an edge $e = us$ such that*

$$(2.1) \qquad \lambda(x, y; M^{ef}) \geq r_M(x, y) \quad \textit{for every } x, y \in V.$$

If $M = D$ is a directed graph and $\lambda(x, y; D) \geq k$ for every $x, y \in V$, then $r_M(x, y) = k$ and we are back at Theorem 2.1. If $M = D$ is directed Eulerian graph and $k := \max(\lambda(x, y; D) : x, y \in V)$, then we are back at Theorem 2.2.

We call a pair $\{e, f\}$ satisfying (2.1) *splittable.* This is equivalent to requiring that

$$(2.2) \qquad r_{M^{ef}}(x, y) \geq r_M(x, y) \quad \text{for every } x, y \in V.$$

Repeatedly applying Theorem 2.3 $\rho_M(s)$ times, one obtains the following theorem.

THEOREM 2.4. *Let $M = (V + s, A \cup E)$ be a mixed graph satisfying* (1.7). *Assume that s is incident only with directed edges and $\rho_M(s) = \delta_M(s)$. Then the edges entering and leaving s can be matched into $\rho_M(s)$ disjoint pairs so that $\lambda(x, y; M^+) \geq r_M(x, y)$ for every $x, y \in V$, where M^+ denotes the mixed graph arising from M by splitting off all these pairs.*

Proof of Theorem 2.3. We may assume that every edge of M is directed since replacing each undirected edge with a pair of oppositely directed edges does not affect the local edge-connectivities. (Incidentally, this means that having a mixed graph in Theorem 2.3 rather than a directed one is not a big thing; the point is that Theorems 2.1 and 2.2 can be combined into one.) Note that for edges $e = us, f = st$, one has $\beta_{M^{ef}}(X) = \beta_M(X) - 1$ if $u, t \in X$ and $\beta_{M^{ef}}(X) = \beta_M(X)$ otherwise.

CLAIM 2.1. *A pair $\{e = us, f = st\}$ is splittable in M if and only if there is no tight set X containing u and t.*

Proof. First suppose that X is a tight set containing u and t. Then $\beta_{M^{ef}}(X) + 1 = \beta_M(X) = R_M(X)$. There are nodes $x \in X, y \in V - X$ such that $R_M(X) = r_M(x, y)$. By applying (1.12) to M^{ef} we obtain $r_{M^{ef}}(x, y) \leq \beta_{M^{ef}}(X) < \beta_M(X) = R_M(X) = r_M(x, y)$, that is, $\{e, f\}$ is not splittable.

Conversely, suppose that $\{e, f\}$ is not splittable. Then there are nodes $x, y \in V$ such that $\lambda(x, y; M^{ef}) < r_M(x, y) \leq k$. Then $r_{M^{ef}}(x, y) = \lambda(x, y; M^{ef})$, and by applying Lemma 1.4 to M^{ef} we see that there is a set $X \subset V$ separating x and y for which

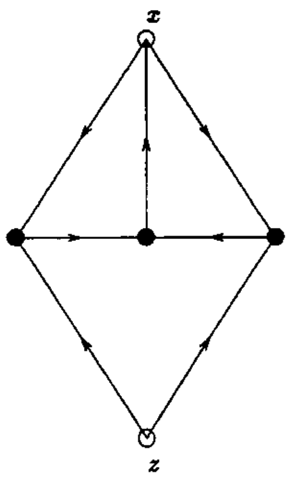

FIG. 1. *Here $k = 2, T$ consists of three nodes, and there are no two edge-disjoint out-arborescences rooted at z with both containing every element of T. Observe that $\rho(x) = 1 < \delta(x) = 2$ holds for the only node x not in T.*

$\beta_{M^{ef}}(X) = r_{M^{ef}}(x,y)$. Using (1.12a) we have $\beta_M(X) - 1 \leq \beta_{M^{ef}}(X) = r_{M^{ef}}(x,y) \leq r_M(x,y) - 1 \leq R_M(X) - 1 \leq \beta_M(X) - 1$. Hence equality follows everywhere. In particular, $\beta_M(X) = R_M(X)$, that is, X is tight. Also, $\beta_M(X) - 1 = \beta_{M^{ef}}(X)$, that is, $u, t \in X$. □

CLAIM 2.2. *There are no two maximal tight $t\bar{s}$-sets.*

Proof. Assume, indirectly, that X and Y are such sets. Apply Lemma 1.5. If (1.13a) holds, then we have $0 + 0 = s_M(X) + s_M(Y) \geq s_M(X \cap Y) + s_M(X \cup Y) + d_D(X,Y) \geq 0$, from which $s_M(X \cup Y) = 0$, contradicting the maximality of X and Y.

If (1.13b) holds, then we have $0 + 0 = s_M(X) + s_M(Y) \geq s_M(X - Y) + s_M(Y - X) + \bar{d}_D(X,Y) \geq 0 + 0 + \bar{d}_D(X,Y)$. Hence $\bar{d}_D(X,Y) = 0$, which contradicts the existence of the edge st. □

If there is no tight $t\bar{s}$-set, then choose an arbitrary edge $e = us$ of M. If there are tight $t\bar{s}$-sets, then by Claim 2.2 there is a unique maximal one denoted by X. We claim that there is an edge $e = us$ with $u \notin X$. Assume this is not the case. Then the existence of the edge st and the fact that $\rho_M(s) = \delta_M(s)$ imply that $\delta_M(V-X) < \rho(X)$ and $\rho_M(V - X) < \delta(X)$, that is, $\beta_M(V - X) < \beta_M(X)$. This is impossible, however, since $\beta_M(V - X) \geq R_M(V - X) = R_M(X) = \beta_M(X)$. By Claim 2.1 the pair $\{us, st\}$ is splittable. □

Mader [20] showed how his Theorem 2.1 implies the following basic result of Edmonds on edge-disjoint arborescences.

THEOREM 2.5 [2], [3]. *In a digraph $D = (U, A)$ with a special node z there are k edge-disjoint spanning out-arborescences of root z if and only if $\rho(X) \geq k$ holds for each subset $X \subseteq U - z$ of nodes (or, equivalently, there are k edge-disjoint paths from z to every other node of D.)*

The following possible generalization naturally emerges. In addition to z, we are given a subset $T \subseteq U - z$ so that $\rho(X) \geq k$ for every subset $X \subseteq U - z, X \cap T \neq \emptyset$. Is it true that there are k edge-disjoint out-arborescences rooted at z so that each contains every element of T? The answer is yes if $T = U - z$ (by Edmonds's theorem) or if $|T| = 1$ (by Menger's theorem). But Lovász [15] found the example in Fig. 1 which shows that such a statement is not true in general. In this light, the following result might have some value.

THEOREM 2.6. *Let $D = (U, A)$ be a digraph with a special node z called a root,* *and let* $T' := \{x \in U - z : \rho(x) < \delta(x)\}$. *Assume that* $\lambda(z, x; D) \geq k (\geq 1)$ *for every* $x \in T'$. *Then there is a family* $\mathcal{F}$ *of* k *edge-disjoint out-arborescences rooted at* z *so that every node* $x \in U$ *belongs to at least* $r(x) := \min(k, \lambda(z, x; D))$ *members of* $\mathcal{F}$.

Proof. The theorem is trivial if $|U| = 2$, so suppose that $|U| \geq 3$. We may assume that there is no edge in D entering z. Now $U - T' - z$ is nonempty; otherwise $\rho(x) < \delta(x)$ would hold for every node of D, which is not possible since $\sum \rho(x) = |A| = \sum \delta(x)$.

Let $s \in U - T' - z$ be a node for which $r(s)$ is minimum. By the hypothesis made on $T', r(s) \leq r(x)$ for every $x \in U$. Extend D by adding $\rho(x) - \delta(x)$ parallel edges from x to z for each $x \in U - T' - z$, and k parallel edges from x to z for each $x \in T'$. Let D' denote the resulting digraph.

Clearly, $T(D') \subseteq T' + z$. We claim that (1.7) holds for D'. This is equivalent to saying that $\rho_{D'}(X) \geq k$, and $\delta_{D'}(X) \geq k$ holds for every subset $X \subseteq V - z$ for which $X \cap T'$ is nonempty. The first inequality follows from the hypothesis. The second one follows from the fact that $\delta_{D'}(X) = \rho_{D'}(X) - \sum(\rho_{D'}(x) - \delta_{D'}(x) : x \in X) \geq \rho_{D'}(X) \geq k$.

We can apply Theorem 2.3, which implies that there are edges $e = us, f = st$ such that $\lambda(z, x; D_1) \geq r(x)$ holds for every $x \in U - s$, where D_1 denotes the digraph arising from D' by splitting off e and f. It is also clear that $\lambda(z, s; D_1) \geq r(s) - 1$.

By induction there is a family $\mathcal{F} = \{F_1, \ldots, F_k\}$ of k edge-disjoint out-arborescences in D_1 rooted at z such that each node x belongs to at least $r(x)$ members of $\mathcal{F}$ for $x \in U - s$, and s belongs to at least $r(s) - 1$ members of $\mathcal{F}$. Let $a = ut$ denote the edge of D_1 which results from the splitting of $f = st$ and $e = us$.

Suppose first that one member of $\mathcal{F}$, say F_1, contains a. If (i) s is not contained in F_1, define $\bar{F}_1 := F_1 - a + e + f$. If (ii) s is contained in F_1, let P denote the unique subpath of F_1 from z to s with its last edge $h = ws$. If P does not use a, define $\bar{F}_1 := F_1 - a + f$. If P uses a, define $\bar{F}_1 := F_1 - a - h + e + f$. Finally, if no member of $\mathcal{F}$ contains a, define $\bar{F}_1 := F_1$.

By these constructions $\bar{F}_1$ is an out-arborescence of D containing each node belonging to F_1 plus, possibly, node s. Hence we have a family $\bar{\mathcal{F}} = \{\bar{F}_1, F_2, \ldots, F_k\}$ of k out-arborescences of D so that each node x other than s belongs to at least $r(x)$ of them, and s belongs to at least $r(s) - 1$ of them. If s belongs to at least $r(s)$ members of $\bar{\mathcal{F}}$, then this family satisfies the requirements of the theorem. If (i) occurred, then we are surely in this case.

Suppose s is contained in precisely $r(s) - 1$ members of $\bar{\mathcal{F}}$. Then (ii) occurs and by the choice of $s, r(x) \geq r(s)$ for every $x \in U$. Hence every node x is in strictly more members of $\bar{\mathcal{F}}$ than s is. Therefore, there is a member F of $\mathcal{F}$ containing x but not s. By the construction of $\bar{\mathcal{F}}$, at least one of the edges $e = us$ and $h = ws$ is not used by the out-arborescences of $\bar{\mathcal{F}}$. Accordingly, choose x to be u or w. We conclude that by replacing the out-arborescence F by $F + xs$ in $\bar{\mathcal{F}}$, we obtain a family of k out-arborescences satisfying the requirements. □

Clearly, if in Theorem 2.6, $\lambda(z, x; D) \geq k$ holds for every $x \in U$, then we are back at Edmonds's theorem. Another special case may also be worth mentioning. Call a digraph $D = (U, A)$ with root z a *preflow digraph* if $\rho(x) \geq \delta(x)$ holds for every $x \in U - z$. (The name arises from an MFMC algorithm of Karzanov [14] and Goldberg and Tarjan [11], where a preflow was defined as a function on the edge-set of a digraph so that the in-sum is at least the out-sum at every node except the root.) An easy, well-known fact from network flow theory is that any flow from the source to the terminal may be decomposed into path-flows. The following corollary may be considered as a generalization.

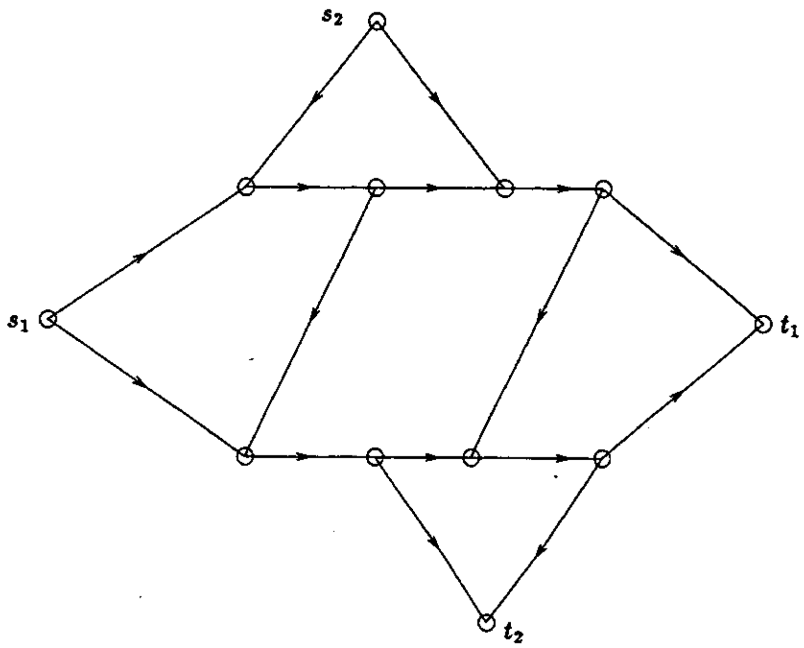

FIG. 2. *Here $k = 2$ but the example can easily be generalized to arbitrary k. In fact, we can have arbitrarily many paths between s_i and t_j $(i, j = 1, 2, \ldots, k)$ and still not have the edge-disjoint paths.*

COROLLARY 2.1. *In a preflow digraph $D = (U, A)$, for any integer $k(\geq 1)$ there is a family $\mathcal{F}$ of k edge-disjoint out-arborescences of root z such that every node x belongs to $\min(k, \lambda(z, x; D))$ members of $\mathcal{F}$. In particular, if $k := \max(\lambda(z, x; D) : x \in U - z)$, then every x belongs to $\lambda(z, x; D)$ members of $\mathcal{F}$.*

Y. Shiloach [23] pointed out that Edmonds's theorem immediately implies the following pretty result. Given k pairs $(s_1, t_1), \ldots, (s_k, t_k)$ of nodes in a k edge-connected digraph D, there are edge-disjoint paths from s_i to t_i $(i = 1, \ldots, k)$.

Using Theorem 2.6 we have the following generalization.

COROLLARY 2.2. *Let $(s_1, t_1), \ldots, (s_k, t_k)$ be k pairs of nodes in a digraph $D = (U, A)$ such that for every node x with $\rho(x) < \delta(x)$ or $x = t_j$ there are edge-disjoint paths from s_i to x $(i = 1, \ldots, k)$. Then there are edge-disjoint paths from s_i to t_i $(i = 1, \ldots, k)$.*

Proof. Extend the digraph by a new node z and an edge zs_i for each $i = 1, \ldots, k$. By Theorem 2.6 there are k edge-disjoint out-arborescences rooted at z such that each contains every t_i. Since there are k edges leaving z, each edge zs_i belongs to one of these out-arborescences denoted by F_i. Now F_i includes a path P_i from s_i to t_i $(i = 1, \ldots, k)$, and these paths satisfy the requirements. □

Note that if we only impose the condition in Corollary 2.2 on the vertices $t_i, i = 1, 2, \ldots, k$, then D may not have edge-disjoint paths from s_i to t_i $(i = 1, 2, \ldots, k)$. This can be seen from the example in Fig. 2.

3. Undirected splitting. Generalizing earlier results of Lovász [16], [17] (see also [18]), Mader proved the following powerful theorem on undirected splitting. For a short proof, see Frank [7]. In what follows $U = V + s$ will denote the node set of the graph in question. We will use the terms $R_M, r_M, \delta_M, \rho_M$, and β_M introduced in §1.

THEOREM 3.1 [19]. *Let $G = (V + s, E)$ be a (connected) undirected graph in which $0 < d_G(s) \neq 3$ and there is no cut-edge incident to s. Then there exists a pair of edges $e = su, f = st$ such that $\lambda(x, y; G) = \lambda(x, y; G^{ef})$ holds for every $x, y \in V$.*

The main result of this section is an extension of Mader's theorem to mixed graphs. Let $M = (V + s, A \cup E)$ be a mixed graph composed from a digraph $D = (V + s, A)$ and an undirected graph $G = (V + s, E)$ so that s is incident only with undirected edges. By a *cut-edge* of a mixed graph M we mean an edge $e \in E$ such that $M - e$ is disconnected (in the undirected sense).

THEOREM 3.2. *Suppose that in $M = (V + s, A \cup E)$, node s is incident only with undirected edges, $0 < d_M(s) \neq 3$, and*

$$\text{(3.1)} \qquad \textit{there is no cut-edge incident to } s.$$

Let $k \geq 2$ be an integer satisfying (1.7). *Then there is a pair of edges $e = su, f = st$ such that*

$$\text{(3.2a)} \qquad \lambda(x, y; M^{ef}) \geq r_M(x, y) \quad \textit{for every } x, y \in V.$$

We call a pair $\{e, f\}$ satisfying (3.2a) *splittable*. Inequality (3.2a) is equivalent to the following:

$$\text{(3.2b)} \qquad r_{M^{ef}}(x, y) \geq r_M(x, y) \quad \text{for every } x, y \in V.$$

In order to make repeated splittings, the following lemma is useful.

LEMMA 3.1. *If $\{e, f\}$ is splittable in a mixed graph M satisfying the hypothesis of Theorem* 3.2, *then M^{ef} satisfies* (3.1).

Proof. Let x and y be two neighbours of s. We claim that $\lambda(x, y; M) \geq 2$. Indeed, $\lambda(x, y; M) \geq 1$ since $xs, ys \in E$ by the assumption. If $\lambda(x, y; M) = 1$, then there is an $x\bar{y}$-set X with $\delta_D(X) = 0$ and $d_G(X) = 1$. Let h denote the unique edge of G between X and $V + s - X$. Then h is either xs or ys. Since $k \geq 2$, one of the sets X and $V + s - X$ is disjoint from $T(M)$. By (1.7), $\rho_D(X) = \delta_D(X)(= 0)$, showing that h is a cut-edge incident to s. Thus $\lambda(x, y; M) \geq 2$. By (3.2a), $\lambda(x, y; M^{ef}) \geq 2$ for every two neighbours x, y of s. This implies the claim. □

A closely related form of Theorem 3.2 is as follows.

THEOREM 3.3. *Let $M = (V + s, A \cup E)$ be a mixed graph with a node s such that s is incident only with undirected edges, $d(s)$ is even, and* (3.1) *holds. Let $k \geq 2$ be an integer satisfying* (1.7). *Then the set of edges incident to s can be matched into $d(s)/2$ disjoint pairs so that $\lambda(x, y; M^+) \geq r_M(x, y)$ for every $x, y \in V$, where M^+ denotes the mixed graph arising from M by splitting off all these pairs.*

This theorem is analogous to Theorem 2.4 except that Theorem 3.3 does not hold for $k = 1$ (see Fig. 3).

CLAIM 3.1. *Theorems* 3.2 *and* 3.3 *are equivalent.*

Proof. First assume the truth of Theorem 3.2 and let $\{e, f\}$ be a splittable pair. By Lemma 3.1, Theorem 3.2 can be applied $d_G(s)/2$ times. Theorem 3.3 follows by observing that a pair splittable in M^{ef} is splittable in M as well.

Conversely, assume that Theorem 3.3 is true. If $d_G(s)$ is even, then there is nothing to prove, so assume that $d_G(s)$ is odd. Then $d_G(s) \geq 5$. Let M' denote a mixed graph arising from M by adding a new node x and three parallel undirected edges between x and s. Property (3.1) holds for M' and hence Theorem 3.3 applies to M'. Since $d_G(s) \geq 5$, among the $(d_G(s) + 3)/2$ splittable pairs in M' provided by Theorem 3.3, at least one pair $\{e, f\}$ must consist of original edges. Clearly, $\{e, f\}$ is splittable in M, as well. □

Proof of Theorem 3.3. By Lemma 3.1 it suffices to prove that there is one splittable pair. We may suppose that every undirected edge h of M is incident to s,

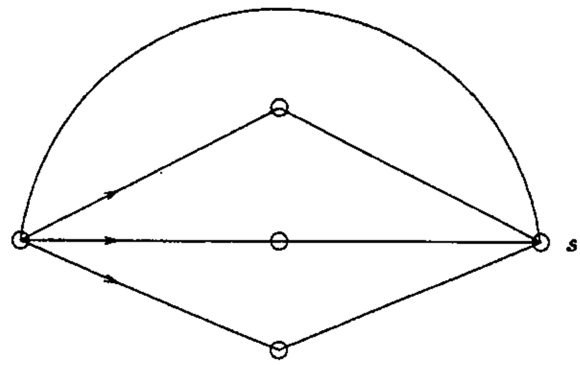

FIG. 3. *There is no way to match the edges at s so that the mixed graph remains strongly connected.*

otherwise we could replace h by two oppositely directed edges. Let M denote a counterexample in which every edge not incident to s is directed and the total number of nodes and edges is minimum. It is clear that M is connected (in the undirected sense). Note that for $X \subseteq V$ and edges $e = su, f = st$ one has $\beta_{M^{ef}(X)} = \beta_M(X) - 2$ if $u, t \in X$ and $\beta_{M^{ef}}(X) = \beta_M(X)$ otherwise. □

CLAIM 3.2. *A pair $\{e, f\}$ of edges $e = su, f = st$ is splittable if and only if there is no dangerous set X containing t and u.*

Proof. First suppose that X is a dangerous set containing u and t. Then $\beta_{M^{ef}}(X) + 2 = \beta_M(X) \leq R_M(X) + 1$. There are nodes $x \in X, y \in V - X$ such that $R_M(X) = r_M(x, y)$. By applying Lemma 1.4 to M^{ef}, we obtain $r_{M^{ef}}(x, y) \leq \beta_{M^{ef}}(X) = \beta_M(X) - 2 \leq R_M(X) - 1 = r_M(x, y) - 1$, that is, $\{e, f\}$ is not splittable.

Conversely, suppose that $\{e, f\}$ is not splittable. Then there are nodes $x, y \in V$ such that $\lambda(x, y; M^{ef}) < r_M(x, y) \leq k$. Then $r_{M^{ef}}(x, y) = \lambda(x, y; M^{ef})$, and by applying Lemma 1.4 to M^{ef} we see that there is a set $X \subset V$ separating x and y, for which $\beta_{M^{ef}}(X) = r_{M^{ef}}(x, y)$. Using (1.12a) we have $\beta_M(X) - 2 \leq \beta_{M^{ef}}(X) = r_{M^{ef}}(x, y) \leq r_M(x, y) - 1 \leq R_M(X) - 1 \leq \beta_M(X) - 1$. Hence $\beta_M(X) \leq R_M(X) + 1$, that is, X is dangerous. Furthermore, $\beta_M(X) > \beta_{M^{ef}}(X)$, that is, $u, t \in X$. □

CLAIM 3.3. *Let $X \subseteq V$ be a tight set. A pair $\{e = su, f = st\}$ of edges is splittable in M if the corresponding pair $\{e', f'\}$ is splittable in the contracted mixed graph $M' := M/X$.*

Proof. For a subset Z of nodes of M for which either $Z \subseteq V - X$ or $X \subseteq Z$, let Z' denote the subset of nodes of M' corresponding to Z. For such a Z, clearly $R_{M'}(Z') \geq R_M(Z)$ and $\rho_{M'}(Z') = \rho_M(Z)$. Therefore, if Z is dangerous in M, then Z' is dangerous in M'. This fact and Claim 3.2 imply that

(*) there is no dangerous set Z in M containing u and t such that $X \subseteq Z$ or $Z \subseteq V - X$.

By claim 3.2, if $\{e, f\}$ is not splittable in M, there is a dangerous set Y containing u and t. If (1.13a) holds for X and Y, then $0 + 1 \geq s_M(X) + s_M(Y) \geq s_M(X \cap Y) + s_M(X \cup Y) \geq s_M(X \cup Y)$, that is, $Z := X \cup Y$ is dangerous, contradicting (*).

If (1.13b) holds, then $0 + 1 \geq s_M(X) + s_M(Y) \geq s_M(X - Y) + s_M(Y - X) + 2\bar{d}_G(X, Y) \geq s_M(Y - X) + 2\bar{d}_G(X, Y)$. Hence $s_M(Y - X) \leq 1$ and $\bar{d}_G(X, Y) = 0$, in particular, $u, t \notin X \cap Y$. That is, $Z := Y - X$ is dangerous and contains u and t, contradicting (*) □

We call a tight set X *trivial* if $|X| = 1$. Since M is a minimal counterexample, Claim 3.3 shows that

(3.3) in M every tight set is trivial.

CLAIM 3.4. *For every $u, v \in V$,*

$$r_M(u,v) = \min(\beta_M(u), \beta_M(v), k). \tag{3.4}$$

Proof. By (1.12a), $\min(\beta_M(u), \beta_M(v)) \geq r_M(u,v)$, so if $\lambda(u,v;M) \geq k$, then (3.4) follows. If $\lambda(u,v;M) < k$, then by Lemma 1.4 there is a set $X \subset V$ separating u and v, such that $\beta_M(X) = R_M(X) = r_M(u,v)$. That is, X is tight and by (3.3), $|X| = 1$, from which the claim follows. □

CLAIM 3.5. *$V - T(M)$ as nonempty.*

Proof. Let $Y := \{v \in V : \rho_D(v) > \delta_D(v)\}, X := \{v \in V : \rho_D(v) < \delta_D(v)\}$. If the claim is false, then X and Y form a partition of V. By the definition of X and Y there is a directed edge $h := xy$ from X to Y.

Let $M' := M - h$. Then $T(M') \subseteq T(M)$ and, since $k \geq 2$, (3.1) is valid for M'. Since $\rho_M(y) > \delta_M(y) \geq k$, the set $\{y\}$ is not in-tight. Similarly, $\{x\}$ is not out-tight. Using this fact and (3.3) we get that h does not enter any in-tight set. Therefore, $\lambda(u,v;M) = \lambda(u,v;M')$ for every $u, v \in V$. Since $T(M') \subseteq T(M)$, (1.7) holds for M'. Because of the minimality of M, M' is not a counterexample and there is a pair $\{e, f\}$ of edges splittable in M'. Hence $\{e, f\}$ is splittable in M as well, a contradiction. □

Let $t_0 \in V - T(M)$ be a node for which

$$\beta_M(t_0) \text{ is minimum.} \tag{3.5}$$

We distinguish between two cases.

Case 1. $d_G(s, t_0) \geq 1$.

In order to be consistent with the notation in earlier claims, for Case 1 let us rename t_0 by t. That is, $d_G(s,t) \geq 1$.

CLAIM 3.6. *For every $t\bar{s}$-set $X (X \neq \{t\})$,*

$$R_M(X - t) \geq R_M(X). \tag{3.6}$$

Proof. There is a pair of nodes x, y such that $x \in X, y \in V - X$, and $R_M(X) = r_M(x,y)$. If $x \neq t$, then $R_M(X - t) \geq r_M(x,y) = R_M(X)$ and (3.6) follows. Assume that $x = t$ and let $u \in X - t$ be an arbitrary node. By (3.4) we have $R_M(X) = r_M(t,y) = \min(\beta_M(t), \beta_M(y), k)$ and $R_M(X - t) \geq r_M(u,t) = \min(\beta_M(u), \beta_M(t), k)$. Hence (3.6) follows if $\beta_M(u) \geq \beta_M(t)$ or if $\beta_M(u) \geq k$. So assume that $\beta_M(u) < \beta_M(t)$ and $\beta_M(u) < k$. By (3.5), u must be in $T(M)$. Since $T(M)$ never consists of a single node, (1.7) implies that $\beta_M(u) \geq k$, a contradiction. □

CLAIM 3.7. *If $X \subseteq V$ is dangerous, then $d_G(s,X) \leq d_G(s, V - X)$.*

Proof. Let $\alpha := d_G(s,X)$ and $\beta := d_G(s, V - X)$, and assume that X is, say, in-dangerous. We have $R_M(V - X) = R_M(X + s) = R_M(X) \geq \rho_M(X) - 1 = \delta_M(V - X) - \beta + \alpha - 1 \geq R_M(V - X) - \beta + \alpha - 1$, from which $\alpha \leq \beta + 1$ follows. However, we cannot have equality, otherwise $d_G(s) = 2\beta + 1$ would follow, contradicting the hypothesis of the theorem that $d_G(s)$ is even. □

Let S denote the set of neighbours of s. Since no pair $\{su, st\}$ is splittable in M, Claim 3.2 implies that every element of S belongs to a dangerous $t\bar{s}$-set. Let $\mathcal{L}$ be a minimal family of such dangerous sets so that $\cup(X : X \in \mathcal{L}) \supseteq S$. By Claim 3.7, $|\mathcal{L}| \geq 2$. We may assume that the members of $\mathcal{L}$ are maximal dangerous $t\bar{s}$-sets.

CLAIM 3.8. $|\mathcal{L}| \geq 3$.

Proof. By Claim 3.7, $|\mathcal{L}| \geq 2$. Assume that $\mathcal{L}$ has just two members, X and Y. Since $S \subseteq X \cup Y$, by Claim 3.7 we have $d_G(s,X) \leq d_G(s, V - X) < d_G(s,Y) \leq$

$d_G(s, V-Y) < d_G(s, X)$, a contradiction. Here the two strict inequalities hold since $S \subseteq X \cup Y$ and $t \in S \cap X \cap Y$. □

CLAIM 3.9. *For every two members* X, Y *of* $\mathcal{L}$, $|X-Y| = |Y-X| = 1, \bar{d}_D(X,Y) = 0$, *and* $\bar{d}_G(X,Y) = 1$.

Proof. If (1.13b) holds for X and Y, then $1+1 \geq s_M(X) + s_M(Y) \geq s_M(X-Y) + s_M(Y-X) + 2\bar{d}_G(X,Y) \geq 0+0+2$, and hence $\bar{d}_G(X,Y) = 1$ and both $X-Y$ and $Y-X$ are tight. By (3.3) the claim follows.

If (1.13b) does not hold, then (1.13a) does. Since X and Y are maximal dangerous sets, $s_M(X \cup Y) \geq 2$, and hence $1+1 \geq s_M(X)+s_M(Y) \geq s_M(X \cap Y)+s_M(X \cup Y) \geq 0+2$. Therefore, $s_M(X \cap Y) = 0$ and by (3.3) $|X \cap Y| = 1$, that is, $X \cap Y = \{t\}$. By Claim 3.6, $R_M(X-t) \geq R_M(X)$ and $R_M(Y-t) \geq R_M(Y)$. Therefore, $R_M(X) + R_M(Y) \leq R_M(X-t) + R_M(Y-t) = R_M(X-Y) + R_M(Y-X)$. That is, (1.5b) holds for R_M.

Since $X \cap Y = \{t\}$, (1.3d) holds and, therefore, (1.13b) holds, a contradiction. □

Let X_1, X_2, X_3 be three members of $\mathcal{L}$ and $Z := X_1 \cap X_2 \cap X_3$. By the minimality of $\mathcal{L}$, each X_i has an element x_i not in any other member of $\mathcal{L}$. By Claim 3.9 it follows that $X_i = Z + x_i$ $(i = 1,2,3)$ and $\bar{d}_D(X_i, X_j) = 0, \bar{d}_G(X_i, X_j) = 1$ for $(1 \leq i < j \leq 3)$. Hence only one edge leaves or enters Z, namely, the edge st. That is, st is a cut-edge, contradicting (3.1). This contradiction shows that Case 1 cannot occur.

Case 2. $d_G(s, t_0) = 0$.

Since M is connected, $\rho_M(t_0 = \delta_M(t_0) > 0$. Let at_0 and t_0b be arbitrary edges in M so that, if possible, $a \neq b$. Note that $a = b$ only if t_0 has just one neighbour in M. Let M' denote the mixed graph arising from M by splitting off at_0 and t_0b.

Clearly, $T(M) = T(M')$ and $\lambda(t_0, v; M') \geq \lambda(t_0, v; M) - 1$ holds for every $v \in V - t_0$. Moreover, we claim that $\lambda(u, v; M') = \lambda(u, v; M)$ for every $u, v \in V - t_0$. This is straightforward if $a = b$, and follows from (3.3) and Claim 2.1 if $a \neq b$. Therefore, we have

$$r_{M'}(t_0, v) \geq r_M(t_0, v) - 1 \quad \text{for every } v \in V - t_0, \tag{3.7a}$$

$$r_{M'}(u, v) = r_M(u, v) \quad \text{for every } u, v \in V - t_0. \tag{3.7b}$$

CLAIM 3.10. M' *satisfies* (3.1).

Proof. If (3.1) is not true for M', there is a set $C \subseteq V$ separating t_0 from a and b with $\rho_M(C) = \delta_M(C) = 2$ such that there is just one undirected edge $h = sz$ entering C and one of the edges at_0 and t_0b enters C while the other one leaves C.

Let $h' = su$ be another undirected edge of M. Then $u \neq z$ and we claim that $\lambda(u, z; M) \geq 2$, otherwise there is a $z\bar{u}$-set X with $\rho_M(X) = 1$. Since $d_G(X) \geq 1, \rho_D(X) = 0$. Since $k \geq 2, X$ cannot separate any two members of $T(M)$ and hence $\delta_D(X) = \rho_D(X) = 0$. But then h or h' is a cut-edge violating (3.1).

$\lambda(u, z; M) \geq 2$ and $\rho_M(C) = 2$ imply that C is tight. By (3.3), $|C| = 1$, that is, $C = \{z\} = \{t_0\}$. The existence of edge h contradicts the assumption that $d_G(s, t_0) = 0$. □

By the minimality of M, M' is not a counterexample of Theorem 3.3. Since (3.1) holds for M', there is a pair $\{e := su, f := st\}$ of undirected edges splittable in M'. Since we are at Case 2, $t \neq t_0$.

CLAIM 3.11. $\{e, f\}$ *is splittable in* M.

Proof. By claim 3.2, if the pair $\{e, f\}$ is not splittable in M, then there is a dangerous set $X \subseteq V$ containing u and t. By (3.7), $R_{M'}(X) \geq R_M(X) - 1$. Using the fact that X is not dangerous in M', that is, $\beta_{M'}(X) > R_{M'}(X) + 2$, we obtain

$$(3.8) \qquad \beta_M(X) \geq \beta_{M'}(X) \geq R_{M'}(X) + 2 \geq R_M(X) + 1 \geq \beta_M(X).$$

Hence equality follows everywhere, in particular, $R_{M'}(X) = R_M(X) - 1$. This and (3.7) imply that $R_M(X) = r_M(t_0, y)$ for some $y \in V$, which is separated from t_0 by X, and

$$(3.9) \qquad r_M(t_0, y) > r_M(x, y)$$

for any $x \in V$, which is separated from y by X.

If $t_0 \in X$, then choose $x := t$, an element different from t_0. If $t_0 \in V - X$, then there must be an element $x \in V - X - t_0$; otherwise, $V - X = \{t_0\}$, from which $\beta_M(X) \geq \beta_M(t_0) + 2$ follows. Using (3.4) we have $R_M(X) = r_M(t_0, y) = \min(\beta_M(y), \beta_M(t_0), k) \leq \beta_M(t_0) \leq \beta_M(X) - 2$, contradicting the hypothesis that X is dangerous.

In both cases, from (3.9) and (3.4) we have $\min(\beta_M(t_0), \beta_M(y), k) = r_M(t_0, y) > r_M(x, y) = \min(\beta_M(x), \beta_M(y), k)$, which implies $\beta_M(x) < \beta_M(t_0)$ and $\beta_M(x) < k$. The first inequality shows, by (3.5), that $x \in T(M)$, while the second one implies, by (1.7), that $x \notin T(M)$, a contradiction. □

Claim 3.11 contradicts the fact that M is a counterexample. Thus Case 2 is also impossible and the proof of Theorem 3.3 is complete. □

We mention two special cases. In the first, $r_M(x, y) \equiv k$ is assumed, while the second concerns mixed graphs with all di-Eulerian nodes.

COROLLARY 3.1. *Suppose that in a mixed graph $M = (V + s, A \cup E)$, node s is incident only with undirected edges, $0 < d(s) \neq 3$, and there is no cut-edge incident to s. Let $k \geq 2$ be an integer such that $\lambda(x, y; M) \geq k$ for every $x, y \in V$. Then there is a pair of edges $e = su, f = st$ such that $\lambda(x, y; M^{ef}) \geq k$ for every $x, y \in V$.*

COROLLARY 3.2. *Suppose that in a mixed graph $M = (V + s, A \cup E)$, node s is incident only with undirected edges, $0 < d(s) \neq 3$, there is no cut-edge incident to s, and $\rho_M(v) = \delta_M(v)$ for every node $v \in V$. Then there is a pair of edges $e = su, f = st$ such that $\lambda(x, y; M^{ef}) = \lambda(x, y; M)$ for every $x, y \in V$.*

Note that this corollary is already a generalization of Mader's Theorem 3.1.

We close this section by pointing out that for a mixed graph $M = (V + s, A \cup E)$, one cannot always split away a pair of edges incident to s in such a way that for every pair of vertices $x, y \in V$ $\min(\lambda(x, y; M), \lambda(y, x; M))$ is preserved. Such an example is given in Fig. 4.

4. Increasing edge-connectivity. This section is offered to exhibit two new edge-connectivity augmentation results according to whether only directed or undirected edges are allowed to be added. We will formulate the results for mixed starting graphs, but these forms are clearly equivalent to the cases when the starting graph is a directed graph. Therefore, our first theorem is basically a directed augmentation theorem in which both the starting graph and the new edges to be added are directed, while in the second theorem the starting graph is directed and the new edges are undirected.

In both theorems we have the same requirement for the demand function r, namely, $r(x, y)$ is symmetric, not larger than a specified positive integer k, and precisely k for pairs of non-di-Eulerian nodes x, y. An interesting phenomenon in the case

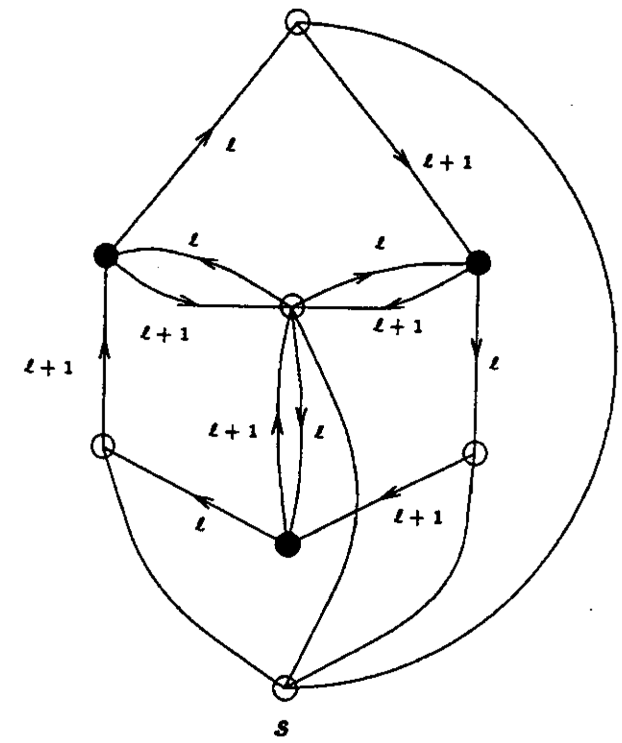

FIG. 4. *Here ℓ and $\ell+1$ denote the multiplicity of the arcs. It is not difficult to see that* $\min(\lambda(x,y;M),\lambda(y,x;M)) = 2\ell+1$ *for any choice of $x,y \in Z$, and none of the two possible splittings at s preserves this.*

of undirected augmentation is that for $k=1$, the necessary and sufficient condition is different from the one given for $k>1$.

Our proof method strongly follows that of Frank [6], which had two ingredients. The first was the splitting theorems of W. Mader, while the second was an observation that the set of degree vectors of possible augmentation forms a so-called contrapolymatroid, a matroid-like structure. The idea of using a splitting theorem for augmentation problems dates back to as early as 1976 (Plesnik [22]). Cai and Sun [1] also use splitting theorems. Actually, this approach was our main motivation in developing stronger splitting theorems in §§2 and 3.

To be more specific, let N be a mixed graph composed from a directed graph $D=(V,A)$ and an undirected graph $G=(V,E)$. Let $T(D) := \{v \in V : \rho_D(v) \neq \delta_D(v)\}$ be the set of non-di-Eulerian nodes of D. Let k be a positive integer and let $r(x,y)(x,y \in V)$ be a nonnegative integer-valued demand function satisfying

(4.1a) $$r(x,y) = r(y,x) \le k \quad \text{for every } x,y \in V, \quad \text{and}$$

(4.1b) $$r(x,y) \equiv k \quad \text{for every } x,y \in T(D).$$

Let $R(\varnothing) = R(V) = 0$, and for $X \subseteq V$ let

(4.2) $$R(X) := \max(r(x,y) : X \text{ separates } x \text{ and } y).$$

Let us define

(4.3a) $$q_{\mathrm{in}}(X) := R(X) - \rho_N(X),$$

(4.3b) $$q_{\mathrm{out}}(X) := R(X) - \delta_N(X).$$

THEOREM 4.1. *Given a mixed graph $N = (V, A \cup E)$, positive integers k, γ, and a demand function $r(x, y)$ satisfying (4.1), N can be extended to a mixed graph N^+ by adding γ new directed edges so that*

$$\lambda(x, y; N^+) \geq r(x, y) \quad \textit{for every } x, y \in V \tag{4.4}$$

if and only if

$$\sum q_{\text{in}}(X_i) \leq \gamma \tag{4.5a}$$

and

$$\sum q_{\text{out}}(X_i) \leq \gamma \tag{4.5b}$$

hold for every subpartition $\{X_1, \ldots, X_t\}$ of V.

Proof. It can be assumed that N is a directed graph because undirected edge of N can be replaced by a pair of two oppositely directed edges, and this operation does not affect the local edge-connectivity. That is, $N = D$. Let N^+ denote an augmentation of D with γ new edges.

CLAIM 4.1. *N^+ satisfies (4.4) if and only if*

$$\rho_{N^+}(X) \geq R(X) \quad \textit{and} \quad \delta_{N^+}(X) \geq R(X) \tag{4.6}$$

hold for every $X \subseteq V$.

Proof. If N^+ satisfies (4.4), then for any subset X separating x and y, $\rho_{N^+}(X) \geq \lambda(x, y; N^+) \geq r(x, y)$. Hence $\rho_{N^+}(X) \geq R(X)$ for every $X \subseteq V$. The second inequality in (4.6) follows analogously. Conversely, assume that (4.6) is satisfied. By Menger's theorem there is a $y\bar{x}$-set X for which $\lambda(x, y; N^+) = \rho_{N^+}(X)$. Hence $\lambda(x, y; N^+) = \rho_{N^+}(X) \geq R(X) \geq r(x, y)$ as required. □

We first examine the proof of necessity. By (4.6) we have $\gamma \geq \sum \rho_{N^+}(X_i) - \sum \rho_D(X_i) \geq \sum R(X_i) - \sum \rho_D(X_i) = \sum q_{\text{in}}(X_i)$, that is, (4.5a) holds. Inequality (4.5b) follows analogously.

The proof of sufficiency is structured as follows. First, we extend D by adding a new node s together with new directed and undirected edges incident to s. Secondly, we get rid of some new edges. Finally, we replace each remaining undirected edge by a pair of oppositely directed edges and apply Theorem 2.4.

To be more specific, extend D by adding a new node s, k parallel undirected edges connecting s and x for every $x \in V - T(D)$, and k parallel directed edges from s to x and x to s for every $x \in T(D)$. The resulting mixed graph M satisfies

$$\rho_M(X) \geq R(X), \tag{4.7a}$$
$$\delta_M(X) \geq R(X) \tag{4.7b}$$

for every $X \subseteq V$.

Let $s(X) = \beta_M(X) - R(X), s_{\text{in}}(X) = \rho_M(X) - R(X)$, and $s_{\text{out}}(X) = \delta_M(X) - R(X)$ for $X \subset V$. By (4.7) these "surplus" functions are nonnegative. We say that X is *R-tight, in-R-tight,* and *out-R-tight* if $s(X) = 0, s_{\text{in}}(X) = 0$, and $s_{\text{out}}(X) = 0$, respectively.

Secondly, starting with the undirected edges and then continuing with the directed ones, discard new edges from M one by one as long as possible without violating

(4.7). Henceforth we use M to denote the final graph. Recall the notation $\beta_M(X) = \min(\rho_M(X), \delta_M(X))$. During this process new R-tight sets may arise, and if a set becomes R-tight at any moment, it stays so throughout.

LEMMA 4.1. *$\delta_M(S) \le \gamma$ and $\rho_M(s) \le \gamma$.*

Proof. We prove only the first inequality; the second is analogous. In M,

(4.8) every directed edge $e = sx$ enters an in-R-tight subset of V,

since otherwise e could have been discarded without violating (4.7).

We also claim that in M,

(4.9) every undirected edge sx enters an in-R-tight set $X \subseteq V - T(D)$.

Indeed, let M' denote the current graph at the moment of the discarding phase when the last undirected edge has been discarded. The fact that e cannot be discarded means that there exists a set $X \subseteq V$ containing x so that $\beta_{M'}(X) = R(X)$. Since at this moment no new directed edge has yet been discarded, X cannot contain any element of $T(D)$. That is, $X \subseteq V - T(D)$ and hence $\rho_{M'}(X) = \delta_{M'}(X)$, that is, (4.9) follows.

Let $S := \{x \in V - T(D)$: there is an undirected edge sx of $M\}$. Let $S_{\text{in}} := \{x \in T(D)$: there is a directed edge sx of $M\}$. Let us call an in-R-tight set X *extreme* if there is no in-R-tight set Y with $X \cap (S \cup S_{\text{in}}) \subset Y \cap (S \cup S_{\text{in}})$, and if $X \cap (S \cup S_{\text{in}}) = Y \cap (S \cup S_{\text{in}})$ for an in-R-tight set Y, then $X \subseteq Y$. Thus X is as large inside $S \cup S_{\text{in}}$ as possible, and subject to this, X is as small outside $S \cup S_{\text{in}}$ as possible. □

CLAIM 4.2. *For any two in-R-tight sets X, Y, at least one of the following holds:*

(a) *$X \cup Y$ is in-R-tight;*

(b) *both $X - Y$ and $Y - X$ are in-R-tight and $X \cap Y \cap (S \cup S_{\text{in}}) = \varnothing$;*

(c) *$T(D) \subseteq X \cup Y$ and $X \cap Y \cap T(D) \ne \varnothing$.*

Proof. If (1.5a) holds, then by (1.3a), $0 + 0 = s_{\text{in}}(X) + s_{\text{in}}(Y) \ge s_{\text{in}}(X \cap Y) + s_{\text{in}}(X \cup Y) \ge 0$. It follows that $s_{\text{in}}(X \cup Y) = 0$, that is, (a) holds.

Now suppose that (1.5a) does not hold. If $X \cap Y \cap T(D) \ne \varnothing$, then $T(D) \subseteq X \cup Y$, otherwise $R(X) = R(Y) = R(X \cap Y) = R(X \cup Y) = k$ and (1.5a) would hold. That is, we are at alternative (c).

So assume that $X \cap Y \cap T(D) = \varnothing$. Now (1.3b) applies to M and by Lemma 1.1 inequality (1.5b) holds. We obtain $\rho_M(X) + \rho_M(Y) = \rho_M(X - Y) + \rho_M(Y - X) + \bar{d}_{D'}(X, Y) + 2\bar{d}_{G'}(X, Y)$, where D' and G' denote the directed and undirected part of M, respectively. Combining this inequality with (1.5b) we get $0 + 0 = s_{\text{in}}(X) + s_{\text{in}}(Y) \ge s_{\text{in}}(X - Y) + s_{\text{in}}(Y - X) + \bar{d}_{D'}(X, Y) + 2\bar{d}_{G'}(X, Y) \ge 0$. It follows that $s_{\text{in}}(X - Y) = 0 = s_{\text{in}}(Y - X)$ and $\bar{d}_{D'}(X, Y) = 0 = 2\bar{d}_{G'}(X, Y)$, that is, (b) holds. □

By (4.8), there is a family $\mathcal{F}_1$ of in-R-tight sets whose union includes S_{in}. We may choose $\mathcal{F}_1$ so that its members are extreme sets and $|\mathcal{F}_1|$ is minimum.

CLAIM 4.3. *$\mathcal{F}_1$ is either a subpartition or consists of two members whose union includes $T(D)$.*

Proof. If $\mathcal{F}_1$ is not a subpartition, then it has two members X, Y with $X \cap Y \ne \varnothing$. Since the members of $\mathcal{F}_1$ are extreme and $|\mathcal{F}_1|$ is minimal, alternatives (a) and (b) cannot occur in Claim 4.2. Therefore, (c) must hold. □

Let $Z := T(D) \cup \bigcup(X : X \in \mathcal{F}_1)$. By (4.9), for every $y \in S - Z$ there is an in-tight set $Y \subseteq V - T(D)$ containing y. We claim that $Y \cap Z = \varnothing$. If not, then $X \cap Y \ne \varnothing$ for a member X of $\mathcal{F}_1$. But this contradicts Claim 4.2 because alternatives (a) and (b) cannot hold since X is extreme; since (c) cannot hold either since $Y \cap T(D) = \varnothing$.

Therefore, there is a family $\mathcal{F}_2$ of in-R-tight subsets of $V-Z$ whose union includes $S-Z$. Assume that $|\mathcal{F}_2|$ is miniumum and, subject to this, $\sum(|X| : X \in \mathcal{F}_2)$ is minimum.

CLAIM 4.4. *$\mathcal{F}_2$ is a subpartition of $V-Z$.*

Proof. Indirectly, let X, Y be two members of $\mathcal{F}_2$ with $X \cap Y \neq \emptyset$. This contradicts Claim 4.2 because the minimal choice of $\mathcal{F}_2$ implies that neither alternative (a) nor (b) may hold, and (c) is also impossible since $X, Y \subseteq V - Z \subseteq V - T(D)$. □

Let $\mathcal{F} := \mathcal{F}_1 \cup \mathcal{F}_2$. By Claim 4.4, if $\mathcal{F}_1$ is a subpartition so is $\mathcal{F}$. By (4.5a), $\gamma \geq \sum(R(X) - \rho_D(X) : X \in \mathcal{F}) = \sum(R(X) - \rho_M(X) : X \in \mathcal{F}) + \delta_M(s) = \delta_M(s)$ as required for the lemma.

If $\mathcal{F}_1$ is not a subpartition, then by Claim 4.3 it consists of two members A, B with $T(D) \subseteq A \cup B$. Now $R(A-B) = R(A) = R(B-A) = R(B) = k$ and $\rho_D(X) = \delta_D(X)$ for every $X \in \mathcal{F}_2$. Furthermore, (1.2c) applies to A and B, from which $\delta_D(A-B) + \delta_D(B-A) \leq \rho_D(A) + \rho_D(B)$.

By applying (4.5b) to the subpartition consisting of $A-B, B-A$, and the members of $\mathcal{F}_2$ we get $\gamma \geq [R(A-B) - \delta_D(A-B)] + [R(B-A) - \delta_D(B-A)] + \sum(R(X) - \delta_D(X) : X \in \mathcal{F}_2) \geq [R(A) - \rho_D(A)] + [R(B) - \rho_D(B)] + \sum(R(X) - \rho_D(X) : X \in \mathcal{F}_2) \geq [R(A) - \rho_M(A)] + [R(B) - \rho_M(B)] + \sum(R(X) - \rho_M(X) : X \in \mathcal{F}_2) + \delta_M(s) = \delta_M(s)$, and the proof of Lemma 4.1 is complete. □

By adding back some discarded new edges, if necessary, we may assume that $\delta_M(s) = \rho_M(s) = \gamma$. Now replace each undirected edge of M by a pair of oppositely directed edges. By our construction, the resulting digraph D' satisfies $T(D') \subseteq T(D)$. Therefore, we can apply Theorem 2.4 to $M := D'$. The resulting digraph $N^+ := M^+$ satisfies (4.4). □

Remark. It is interesting to note that if a node v is di-Eulerian in D (that is, $\rho_D(v) = \delta_D(v)$), then v is di-Eulerian in the augmented D^+ as well.

Let us mention two corollaries. For simplicity we formulate them for directed starting graphs. In the first one we assume that T is empty, that is, D is di-Eulerian. Then k may be chosen arbitrarily large and hence r is not bounded above.

COROLLARY 4.1. *Given a di-Eulerian digraph $D = (V, A)$ and a symmetric demand function r, D can be extended to a digraph D^+ by adding γ new edges so that $\lambda(x, y; D^+) \geq r(x, y)$ for every $x, y \in V$ if and only if $\sum q_{\text{in}}(X_i) \leq \gamma$ and $\sum q_{\text{out}}(X_i) \leq \gamma$ hold for every subpartition $\{X_1, \ldots, X_t\}$ of V. Furthermore, D^+ may be chosen to be di-Eulerian.*

In the second application we do not have any positive demand for di-Eulerian nodes.

COROLLARY 4.2. *We are given a digraph $D = (V, A)$, positive integers k, γ, and a subset $\bar{T} \subseteq V$ so that $\rho_D(v) = \delta_D(v)$ holds for every $v \in V - T$. D can be extended to a digraph D^+ by adding γ new directed edges so that $\lambda(x, y; D^+) \geq k$ for every $x, y \in T$ if and only if*

$$\sum(k - \rho_D(X_i)) \leq \gamma \quad \textit{and} \quad \sum(k - \delta_D(X_i)) \leq \gamma \tag{4.10}$$

hold for every subpartition $\{X_1, \ldots, X_t\}$ of V for which $X_i \cap T, T - X_i \neq \emptyset \, (i = 1, \ldots, t)$.

The special case $T = V$ of this corollary was proven in Frank [6].

Our next goal is to prove an augmentation theorem when only undirected edges are allowed to be added. Our result is a generalization of Theorem 5.5 in [6], where the starting graph is an undirected graph. Let N be a mixed graph composed from a directed graph $D = (V, A)$ and an undirected graph $G = (V, E)$, and let $r(x, y)$ be a demand function satisfying (4.1). We say that a component C of N is *marginal* (with respect to r) if $r(u,v) \leq \lambda(u,v;N)$ for every $u,v \in C$, and $r(u,v) \leq \lambda(u,v;N)+1$ for every u,v separated by C. In other words, C is marginal if we do not want to increase the local edge-connectivity between the element C, and the demand for increased local edge-connectivity between a node in C and a node outside C is 0 or 1.

THEOREM 4.2. *We are given a mixed graph N, integers $k \geq 2, \gamma \geq 0$, and a demand function $r(x,y)$ satisfying* (4.1) *so that there are no marginal components. N can be extended to a mixed graph N^+ by adding γ new undirected edges so that*

$$\lambda(x,y;N^+) \geq r(x,y) \quad \textit{for every } x,y \in V \tag{4.11}$$

if and only if

$$\sum (R(X_i) - \beta_N(X_i)) \leq 2\gamma \tag{4.12}$$

holds for every subpartition $\{X_1,\ldots,X_t\}$ of V.

Remark. If N has a marginal component, then the above min-max theorem is not true as is shown by the empty graph on four nodes (taking $r(u,v) \equiv 1$). For the special case when N is undirected, in [6, Thm. 5.3] a very simple reduction method was used to get rid of marginal components. The same method easily generalizes to mixed graphs. Since no new idea is required, we leave out the details.

Proof. Again we may assume that N is a directed graph, that is, $N = D$. Let N^+ denote an augmentation of D with γ new undirected edges.

CLAIM 4.5. *N^+ satisfies* (4.11) *if and only if*

$$\beta_{N^+}(X) \geq R(X) \tag{4.13}$$

holds for every $X \subseteq V$.

Proof. First suppose that N^+ satisfies (4.11). By applying Lemma 1.4 to N^+, we obtain that $\beta_{N^+}(X) \geq r_{N^+}(x,y) \geq r(x,y)$ for any subset X separating x and y. Hence (4.11) follows.

Conversely, assume that (4.11) is satisfied. By Menger's theorem there is a $y\bar{x}$-set X for which $\lambda(x,y;N^+) = \rho_{N^+}(X)$. Hence $\lambda(x,y;N^+) = \rho_{N^+}(X) \geq \beta_{N^+}(X) \geq R(X) \geq r(x,y)$ as required. □

We first examine the proof of necessity. If N^+ satisfies (4.11), then by Claim 4.5 there are at least $R(X) - \beta_N(X)$ new edges between X and $V - X$. Therefore, the number γ of new edges is at least half of $\sum_{i=1}^{t}(R(X_i) - \beta_N(X_i))$.

We now examine the proof of sufficiency. First, extend D by adding a new node s and k parallel edges connecting s and x for every $x \in V$. The resulting mixed graph M satisfies

$$\beta_M(X) \geq R(X) \tag{4.14}$$

for every $X \subseteq V$.

Second, discard new edges one by one as long as possible without violating (4.14). Henceforth we use M to denote the final mixed graph and let $S := \{x \in V :$ there is an edge in M between s and $x\}$. We call an R-tight set X *extreme* if there is no R-tight set Y with $X \cap S \subset Y \cap S$, and if, in addition, $X \cap S = Y \cap S$ for an R-tight set Y, then $X \subseteq Y$.

LEMMA 4.2. $d_M(s) \leq 2\gamma$.

Proof. Since no further new edge can be left out of M without violating (4.14), there is a family $\mathcal{F}$ of R-tight sets whose union includes S. We may choose $\mathcal{F}$ so that its members are extreme and $|\mathcal{F}|$ is minimum. □

CLAIM 4.6. *$\mathcal{F}$ is a subpartition of V.*

Proof. Assume indirectly that $X \cap Y \neq \varnothing$ for some $X, Y \in \mathcal{F}$. By Lemma 1.2, at least one of the following inequalities holds:

$$0 + 0 = s(X) + s(Y) \geq s(X \cap Y) + s(X \cup Y) + 2d_{G'}(X,Y) \tag{4.15a}$$
$$+ d_D(X,Y) \geq 0,$$

$$0 + 0 = s(X) + s(Y) \geq s(X - Y) + s(Y - X) + 2\bar{d}_{G'}(X,Y) \tag{4.15b}$$
$$+ \bar{d}_D(X,Y) \geq 0,$$

where G' denotes the undirected part of M.

If (4.15a) holds, then $s(X \cup Y) = 0$, that is, $X \cup Y$ is R-tight, contradicting the fact that X, Y are extreme. If (4.15b) holds, then $s(X - Y) = s(Y - X) = 0$ and $\bar{d}_{G'}(X,Y) = 0$. Therefore, both $X - Y$ and $Y - X$ are R-tight and $X \cap Y \cap S = \varnothing$, which again contradicts the extremality of X and Y. □

By (4.12), $2\gamma \geq \sum(R(X) - \beta_D(X) : X \in \mathcal{F}) = \sum(R(X) - \beta_M(X) : X \in \mathcal{F}) + d_M(s) = d_M(s)$, and Lemma 4.2 follows. □

By adding back new edges which are parallel to existing new edges, we may assume that $d_M(s) = 2\gamma$. We claim that there is no cut-edge of M incident to s. Indeed, if $e = st$ were such an edge, then let C denote the component of $M - e$ containing t. Since e is the only edge of M leaving C, C is a marginal component contradicting the hypothesis.

The theorem now immediately follows from Theorem 3.3. □

If N is an undirected graph in Theorem 4.2, then every node is di-Eulerian, and hence r may be an arbitrary symmetric function. Therefore, Theorem 4.2 is a generalization of the following result from [6].

COROLLARY 4.3. *Given an undirected graph $G = (V, E)$ and a symmetric demand function $r(x,y) \geq 2$, it is possible to add γ new undirected edges to G so that in the resulting graph $G^+, \lambda(x,y;G^+) \geq r(x,y)$ holds for every pair of nodes x, y if and only if $\sum(R(X_i) - d_G(X_i)) \leq 2\gamma$ holds for every subpartition $\{X_i\}$ of V.*

In another special case, $r \equiv k \geq 2$.

COROLLARY 4.4. *Let $N = (V, A \cup E)$ be a mixed graph and let $k \geq 2, \gamma \geq 1$ be integers. N can be made k-edge connected by adding γ new undirected edges if and only if*

$$\sum(k - \beta_N(X_i)) \leq 2\gamma$$

holds for every subpartition $\{X_1, \ldots, X_t\}$ of V.

The example in Fig. 5 shows that for $k = 1$, Corollary 4.4 (and hence Theorem 4.2) is not true in general. However, we can prove the following theorem.

THEOREM 4.3. *A mixed graph N with connected underlying graph can be made 1-edge-connected (= strongly connected) by adding γ new undirected edges if and only if (*) every family $\mathcal{F}$ of $\gamma + 1$ disjoint subsets of nodes contains (not necessarily distinct) members X, Y for which $\rho_N(X) > 0$ and $\delta_N(Y) > 0$.*

Proof. First, suppose that E' is a set of new undirected edges whose addition makes N strongly connected, and there is a family $\mathcal{F}$ of $\gamma + 1$ disjoint subsets of V so that for each member of $\mathcal{F}$, say, $\rho_N(X) = 0$. Since the underlying graph is connected, $X_0 := V - \bigcup(X : X \in \mathcal{F})$ is nonempty. Moreover, since there is no edge (undirected or directed) connecting distinct members of $\mathcal{F}, N - X_0$ has at least $\gamma + 1$ components.

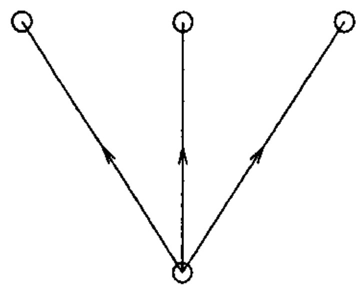

FIG. 5. *It is easy to see that we need three edges here, but $k = 1$ implies $\sum(k - \beta_N(X_i)) \leq 4$, which suggests that two edges would suffice.*

Since the union of any j members of $\mathcal{F}(j = 1, \ldots, \gamma + 1)$ must be connected to the rest by an element of E', we get $|E'| \geq \gamma + 1$, showing that $(*)$ is necessary.

To see the sufficiency we may assume again that N is directed. Now $(*)$ implies (4.10) for $k = 1$ and $T = V$, and hence, by Corollary 4.2, there are γ directed edges whose addition makes N strongly connected. If we leave out the orientations of the newly added edges we get the required undirected augmentations. □

5. Variations, polyhedra, and algorithms. In this section we briefly outline some variations of the augmentation problem, the polyhedral background, and some algorithmic aspects. We are concerned here with the case when only undirected edges are allowed to be added. A similar approach was discussed in detail in Frank [6]. Since no new idea is required, we refer the reader to that paper for definitions and details. For the directed augmentation problem, we have not yet found analogous methods to handle minimum node-cost and degree-constrained versions. This is a possible subject of future research.

Let N, k, γ, r be the same as in Theorem 4.2, but this time, rather than finding a minimum cardinality augmentation, we are interested in an augmentation satisfying (4.11) in which the degree of every node is a prescribed value.

THEOREM 5.1. *Given a mixed graph $N = (V, A + E)$ and an integer-valued vector $m : V \to \mathbf{Z}^+$ for which*

$$m(V) \text{ is even,} \tag{5.1}$$

N can be extended to N^+ satisfying (4.11) by adding a set F of new undirected edges for which $d_F(v) = m(v)$ for every $v \in V$ if and only if

$$m(X) + \beta_N(X) \geq R(X) \text{ for every } X \subseteq V. \tag{5.2}$$

Proof. Add a new node s and $m(v)$ parallel undirected edges between s and v for every $v \in V$. Since there is no marginal component of N, in the extended graph there is no cut-edge incident to s. We may apply Theorem 3.3 and the result immediately follows. □

From this easy derivation one should realize that Theorem 5.1 is nothing but a reinterpretation of Theorem 3.3. Let us call an integral-valued vector m satisfying (5.1) and (5.2) an *augmentation vector*.

Let $q : 2^V \to \mathbf{Z}$ be an integer-valued set function. We call q *skew supermodular* if $q(\varnothing) = 0$ and

$$q(X) + q(Y) \leq q(X \cap Y) + q(X \cup Y) \quad \text{or} \tag{5.3a}$$

$$q(X) + q(Y) \leq q(X - Y) + q(Y - X) \tag{5.3b}$$

hold for every pair of subsets $X, Y \subseteq V$ (we point out that other names were used earlier instead of skew supermodular, for example, weakly supermodular [10] and X-supermodular [8]). The following theorem was proved in [6, Thm. 7.1] for set functions of form $q(X) = R(X) - d_G(X)$. However, its proof relied only on one feature of q, namely, that q is skew supermodular. Hence we state the theorem in this more general form.

THEOREM 5.2. *Where q is a skew supermodular function, the polyhedron*

$$C(q) := \{z : \mathbf{R}^V : z \geq 0, z(X) \geq q(X) \text{ for every } X \subseteq V\} \tag{5.4}$$

is a contrapolymatroid $C(p)$, where the unique fully supermodular function p defining $C(q)$ is given by

$$p(A) := \max\left(\sum q(A_i) : \{A_1, \ldots, A_t\} \text{ a subpartition of } V\right). \tag{5.5}$$

By Lemma 1.2, $q := R - \beta_N$ is skew supermodular and hence Theorem 5.2 applies. We find that the augmentation vectors m are precisely the integer-valued elements of $C(q)$ satisfying (5.1).

We briefly indicate how this fact can be used for degree-constrained and minimum node-cost augmentations. Suppose first that we have a nonnegative cost function $c : V \to \mathbf{R}_+$, and we are interested in an augmentation of a mixed graph N that satisfies (4.11). Also, the total cost of new edges is minimum. Here the cost of an edge uv is defined to be $c(u) + c(v)$.

It was shown in Frank [6] that with a slight modification of the greedy algorithm we can find a minimum cost integer element m of a contrapolymatroid for which $m(V)$ is even. That is, with the help of the greedy algorithm, first find an integer vector m' that minimizes cx over $C(q)$. If $m'(V)$ is even, define $m = m'$. If $m'(V)$ is odd, define $m(v_n)$ by adding 1 to $m'(v_n)$, where v_n is an element of V of least cost, while $m(x) := m'(x)$ for $x \in V - v_n$. This way we obtain a minimum cost augmentation vector m and, by Theorem 5.1, m determines a minimum node-cost augmentation satisfying (4.11).

To consider the degree-constrained augmentation, let $f : V \to \mathbf{Z}$ and $g : V \to \mathbf{Z} \cup \{\infty\}$ be two functions with $f \leq g$. When does there exist an augmentation of N satisfying (4.11) for which $f(v) \leq d_F(v) \leq g(v)$ holds for every $v \in V$? Let $B := \{x \in \mathbf{R} : f \leq x \leq g\}$ denote a box. By Theorem 5.1 the desired F exists if and only if there is an integer element m of $B \cap C(q)$ with the additional property that $m(V)$ is even. The intersection of a box and a contrapolymatroid is a generalized polymatroid. It was shown in [6, Prop. 6.10] that a g-polymatroid defined by a strong pair (p, b) has no integer element m for which $m(V)$ is even if and only if $p(V) = b(V)$ is odd (a submodular function b and a supermodular function p form a *strong pair* if $b(X) - p(Y) \geq b(X - Y) - p(Y - X)$ holds for all $X, Y \subset V$, where V is the groundset for b and p). From this one can derive the following theorem.

THEOREM 5.3. *Given N, k, r as in Theorem 4.2 and integer-valued vector f, g, N can be extended to N^+ satisfying* (4.11) *by adding a set F of new undirected edges for which $f(v) \leq d_F(v) \leq g(v)$ for every $v \in V$ if and only if $q(X) \leq g(X)$ for every $\emptyset \subset X \subset V$, and there is no partition $\mathcal{F} := \{X_0, X_1, \ldots, X_t\}$, where only X_0 may be empty, with the following properties: $f(X_0) = g(X_0), g(X_i) = q(X_i)(i = 1, \ldots, t)$, and $g(V)$ is odd.*

Minimum node-cost degree-constrained augmentation problems can also be handled with the same technique.

To conclude, let's briefly say something about the algorithmic aspects. The proof of Theorem 4.2 consisted of two parts: the edge-deletion phase and the splitting-off phase. An argument analogous to the one used in [6] shows that the edge-deletion phase can be carried out on a graph with n vertices by performing $2n^2$ MFMC computations. The splitting-off phase requires no more than n^3 MFMC calculations. Since one MFMC calculation can be carried out in $O(n^3)$ steps, the overall complexity of the algorithm is $O(n^6)$. Actually, these bounds are valid for the more general problem when the starting graph is endowed with integer capacities on the edges and we are allowed to add a new edge in any number of copies. Theoretically, this problem is not more general since we can replace an edge by as many parallel edges as its capacity will hold. But from a computational point of view such a reduction is not satisfactory. Fortunately, the MFMC algorithm is strongly polynomial, and hence the approach outlined above gives rise to strongly polynomial time algorithm in the capacitated case as well.

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