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ORIENTATIONS OF GRAPHS

AND

SUBMODULAR FLOWS

András Frank*

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Dedicated to C.St.J.A. Nash-Williams on the occasion of his retirement, with grateful thanks for the joy his classical graph-orientation theorem generated.

ABSTRACT Generalizing an earlier result of H.E. Robbins [1939], C.St.J.A. Nash-Williams [1960] proved that an undirected graph G has a k-edge-connected orientation if and only if G is $2k$-edge-connected. In a recent paper Nash-Williams [1995] found a necessary and sufficient condition for the existence of a strongly-connected orientation of a mixed graph so that every node v has at least a prescribed number of newly oriented edges entering v. It was known earlier how the first of these theorems derives from the theory of submodular flows. In this paper we describe how (a generaliztion of) the second does. As a main device, we prove a simplified feasibility theorem for submodular flows constrained by crossing submodular functions.

I. INTRODUCTION

Let $G = (V, E)$ be an undirected graph and $h : 2^V \to \mathbf{Z} \cup \{-\infty\}$ an integer-valued set-function with $h(\emptyset) = h(V) = 0$. The general form of the orientation problem we consider consists of finding an orientation of the edges of G so that in the resulting digraph $\vec{G}$ there are at least $h(X)$ edges entering X for every subset $X \subseteq V$. More precisely, the goal is to find necessary and sufficiant conditions for the existence of such an orientation. (Sometimes the problem is formulated in an equivalent form where h is defined only on a family $\mathcal{F}$ of subsets of V and h is finite-valued. In this case h may be extended to each subset of V by defining $h(X) = -\infty$ for subsets not in $\mathcal{F}$.)

We will also consider orientation problems when the input is a mixed graph $M = (V, E + \vec{A})$ composed from an undirected graph $G = (V, E)$ and from a directed graph $\vec{D} = (V, \vec{A})$. In this case the elements of E has to be oriented and $\vec{D}$ will serve only to express the requirement for the orientation.

This problem formulation is too general in the sense that NP-complete problems may be formulated as a special case. Therefore we restrict our attention to a special class of functions, namely, when h is **crossing G-supermodular**, that is,

$$h(A) + h(B) \leq h(A \cup B) + h(A \cap B) + d(A, B) \tag{1.1}$$

holds for every pair $\{A, B\}$ of subsets of V for which none of $A - B, B - A, A \cap B, V - (A \cup B)$ is empty where $d(A, B)$ denotes the number of edges in G with one and in $A - B$ and the other end in $B - A$.

* Department of Operations Research, Eötvös University, Múzeum krt. 6-8, Budapest, Hungary, H-1088. e-mail: `frank at cs.elte.hu` . Research supported by the Hungarian National Foundation for Scientific Research Grant, OTKA T17580.

All orientation problems to be considered in this paper may be described by such a function. There are some other orientation theorems which do not fit this framework, most notably, Nash-Williams' [1960] difficult theorem (which may be called the strong orientation theorem) on the existence of well-balanced orientations. This easily implies Nash-Williams' weak orientation Theorem 1.1 (mentioned in the abstract).

The earliest orientation result is due to H.E. Robbins [1939] who proved that *G has a strongly connected orientation if and only if G is 2-edge-connected.*

(A digraph is called **strongly connected** if there is a directed path from every node to every other. More generally, a digraph is called **k-edge-connected** if there are k edge-disjoint paths from every node to every other node. By Menger's theorem this is equivalent to requiring that every non-empty, proper subset of nodes has at least k entering edges. An undirected graph is called **k-edge-connected** if every cut contains at least k edges.)

Nash-Williams generalized Robbins' theorem, as follows.

THEOREM 1.1 [Nash-Williams, 1960] *For any positive integer k an undirected graph $G = (V, E)$ has a k-edge-connected orientation if and only if G is $2k$-edge-connected.*

F. Boesch and R. Tindell [1980] found another extension of Robbins' theorem concerning orientations of mixed graphs. Let $M = (V, E + \vec{A})$ be a mixed graph. We say that a path P from u to v in M is **correct** if P may use undirected edges arbitrarily and directed edges pointing forward along the path. M is called **traversable** if there is a correct path from u to v for every ordered pair of nodes (u, v). This is easily seen to be equivalent to requiring that M has no directed cuts and the undirected graph arising from M by de-orienting the directed edges is connected. (A **directed cut** of a mixed graph is the set of directed edges entering some $\emptyset \subset X \subset V$ provided that there are no directed edge leaving X and that there are no undirected edges connecting X and $V - X$.) Note that in case M is undirected (that is, $\vec{A} = \emptyset$) traversable is the same as connected, while if M is directed, traversable is the same as srongly connected.

Boesch and Tindell proved: *A mixed graph M has a strongly connected orientation if and only if M is traversable and has no undirected cut edges.*

(A short proof of this result consists of a greedy-type procedure that considers the undirected edges in an arbitrary order and orient them one by one in such a way that no directed cut arises, that is, the traversability is preserved. It can rather easily be shown that, at every step, among the two possible orientations of the current edge at least one will always do).

The question naturally emerges: when does there exist a k-edge-connected orientation of a mixed graph? This can be answered with the use of submodular flows. The notion of submodular flows was introduced by J. Edmonds and R. Giles [1977]. They proved (among others) that the submodular flow polyhedron is integral. It was observed in [Frank 1982] that there is a strong link between $0-1$-valued submodular flows and orientations of graphs. For example, the integrality of the submodular flow polyhedron easily implies Nash-Williams' result. (It is a great challenge to relate Nash-Williams' strong orientation theorem to submodular flows or more general integral polyhedra.)

The generality of the notion of submodular flows made it possible to derive several extensions of the weak orientation theorem that will be accounted in the next section. One of the most general problem of this type concerns degree-constrained k-edge-connected orientations of mixed graphs. Unfortunately the necessary and sufficient condition is pretty complicated (due to the fact that the corresponding feasibility theorem for submodular flows is complicated.)

Recently, however, Nash-Williams [1995] found a much simpler characterization for the special case $k = 1$ (:strong connectivity). To formulate his result we need some notions and notation. For an undirected graph G, $d_G(X)$ denotes the number of edges between X and $V - X$. Let $e(X) = e_G(X)$ (respectively, $i(X) = i_G(X)$) denote the number of edges with at least one end (with both ends) in X. For a digraph $\vec{D} = (V, \vec{A})$, $\varrho(X) := \varrho_{\vec{A}}(X) := \varrho_{\vec{D}}(X)$ denotes the number of edges entering X and is called the **in-degree** of X (in $\vec{D}$). The function $\varrho_{\vec{D}}$ is called the **in-degree function** of $\vec{D}$. Similarly, $\delta(X) := \delta_{\vec{A}}(X) := \delta_{\vec{D}}(X)$ denotes the number of edges leaving X.

Let $M = (V, E + \vec{A})$ be a mixed graph. It is clearly an equivalence relation of the nodes when two nodes u and v are in relation if there are correct paths from u to v and from v to u. An equivalence class C is called a **di-component.** If no directed edge enters C, it is called an **initial di-component.** Clearly, M is traversable precisely if there is only one equivalence class.

For a subset $Z \subset V$, let $c(M, Z)$ denote the number of those initial di-components C of $M - Z$ which are not entered by any directed edge with tail in Z. Since no undirected edge connects C and $V - (Z \cup C)$, for any strongly connected orientation of M there must be at least one newly oriented edge with head in C and tail in Z, and therefore there are at least $c(M, Z)$ newly oriented edges leaving Z.

Let $f : V \to \mathbf{Z}$ be a non-negative, integer-valued function and let $f(Z) := \sum_{v \in Z} f(v)$.

THEOREM 1.2 [Nash-Williams, 1995] *A mixed graph $M = (V, E + \vec{A})$ has a strongly connected orientation $(V, \vec{E} + \vec{A})$ satisfying*

$$\varrho_{\vec{G}}(v) \geq f(v) \text{ for every } v \in V \tag{1.2}$$

if and only if M is traversable, M has no undirected cut-edge, and

$$e_G(Z) \geq f(Z) + c(M, Z) \tag{1.3}$$

holds for every non-empty subset $Z \subsetneq V$.

From the above considerations the necessity of (1.3) is easy. Indeed, for an orientation $(\vec{G} + \vec{D})$ of M satisfying the requirements, one has $e_G(Z) = \sum_{v \in Z} \varrho_{\vec{G}}(v) + \delta_{\vec{G}}(Z) \geq f(Z) + c(M, Z)$.

Actually, Nash-Williams considered this orientation problem in a slightly different form. He wanted to find a partial orientation of the undirected edges of M (that is, not necessarily all undirected edges have to get oriented) so that the resulting mixed graph is traversable and $(*)$ there are exactly $f(v)$ newly oriented edges entering every node v. This problem, however, is equivalent to the one in Theorem 1.2. Indeed, if a required partial orientation exists, then, by the theorem of Boesch and Tindell, the remaining undirected edges can be oriented so as to obtain a strongly connected digraph. This orientation clearly satisfies (1.2). Conversely, if there is a strongly connected orientation satisfying (1.2), then de-orienting a sufficient number of newly oriented edges we obtain a traversable mixed graph in which $(*)$ holds.

In the present paper we will show how a generalization of Theorem 1.2 can be derived via submodular flows. As a main tool, we prove a simplified feasibility theorem for submodular flows constrained by crossing submodular functions. The simplification is based on the notion of full-truncation of a set-function, a new form of what was called earlier bi-truncation.

Let s be a specified node of a digraph D. We say that D is **k-edge-connected from s** if there are k edge-disjoint paths from s to every other node. D is **k-edge-connected toward s** if there are k edge-disjoint paths from every node to s. For non-negative integers k, l we say that D is **(k, l)-edge-connected (at s)** if D is k-edge-connected from s and l-edge-connected toward s. ((k, k)-edge-connectivity at s is obviously equivalent to the k-edge-connectivity of D.) A node v of a digraph is called a **source-node (sink-node)** if no edge enters (leaves) v.

Let V be a finite ground-set. Two sets X, Y are called **disjoint** if $X \cap Y = \emptyset$ and **co-disjoint** if $X \cup Y = V$ (that is, their complements are disjoint). X, Y are **intersecting** if none of $X - Y, Y - X, X \cap Y$ is empty. If, in addition, $V \neq X \cup Y$, then X and Y are **crossing.** Function h is called **crossing** (respectively, **intersecting) G-supermodular** if (1.1) holds for every crossing (intersecting) pair $\{A, B\}$ of subsets.

If a family $\mathcal{F}$ of sets contains no two crossing (respectively, intersecting) members, $\mathcal{F}$ is called **cross-free (laminar).** $\mathcal{F}$ is called a **partition** (respectively, **co-partition)** of $A \subseteq V$ if it consists of pairwise disjoint (co-disjoint) sets whose union (intersection) is A. That is, a co-partition of A arises from a partition of $V - A$ by complementing its members. (Note that a co-partition of a set A, unlike a partition, is defined with respect to the ground-set V.) A partition of a subset of V is called a **sub-partition** of V. We say that a family $X_{ij} (i = 1, 2, \ldots, j = 1, 2 \ldots)$ of subsets of V form a **double-partition** of A if the sets $X_i := \cap_j X_{ij}$ $(i = 1, 2, \ldots)$ form a partition of A and, for any fixed i, the sets $X_{ij} (j = 1, 2, \ldots)$ form a co-partition of X_i.

For a sub-partition $\mathcal{P}$ of X, let $e(\mathcal{P}) = e_G(\mathcal{P})$ denote the number of edges of G which either connect two distinct members of $\mathcal{P}$ or a member of $\mathcal{P}$ and $V - X$. Let $i(\mathcal{P}) = i_G(\mathcal{P}))$ denote the number of edges of G which connect two distinct members of $\mathcal{P}$. (Note that $i(\mathcal{F}) = e(\mathcal{F}$ for a partition $\mathcal{P}$ of V). For a co-partition $\bar{\mathcal{P}}$ of X let $e_G(\bar{\mathcal{P}}) := e_G(\mathcal{P})$ where $\mathcal{P}$ denotes the partition of $V - X$ arising from $\bar{\mathcal{P}}$ by complementig each of its members.

Given two elements s, t and a subset X of a ground-set, we say that X is an **$s\bar{t}$-set** if $s \in X, t \notin X$.

II. ORIENTATION RESULTS: OLD AND NEW

In this section we classify some known and new orientation theorems according to the simplicity of the necessary and sufficient conditions.

1. Cut-type conditions

In the introduction we mentioned already Robbins' theorem and its two generalizations by Nash-Williams (Theorem 1.1) and by Boesch and Tindell. In each of these theorems the necessary and sufficient condition required an inequality for every cut.

Another old orientation result when a cut-type condition is sufficient was discussed in the book of Ford and Fulkerson [1962]. They investigated the orientability of a mixed graph so as to obtain a di-Eulerian digraph. Actually, the approach of Ford and Fulkerson (that is, the application of Hoffman's circulation theorem) gives rise to the following slightly more general form.

THEOREM 2.1 *Let $G = (V, E)$ be an undirected graph and $m : V \to \mathbf{Z}_+$ a function for which $m(V) = |E|$. The following are equivalent:*

(1) There exists an orientation of G whose in-degree function ϱ satisfies

$$\varrho(v) = m(v) \text{ for every } v \in V \tag{2.1}$$

(2)

$$e_G(Z) \geq m(Z) \text{ for every } Z \subseteq V, \tag{2.2a}$$

(3)

$$i_G(Z) \leq m(Z) \text{ for every } Z \subseteq V \tag{2.2b}$$

where $e_G(Z)$ (respectively, $i_G(Z)$) denotes the number of edges with at least one end (with both ends) in Z.

There is a generalization concerning orientations satisfying upper and lower bounds on the in-degrees. Let $f : V \to \mathbf{Z}_+$, $g : V \to \mathbf{Z}_+ \cup \{+\infty\}$ be two functions so that $f \leq g$. (Later we will use the same graph G and bounds f, g.)

THEOREM 2.2 (a) *There exists an orientation of G whose in-degree function ϱ satisfies*

$$\varrho(v) \geq f(v) \text{ for every } v \in V \tag{2.3a}$$

if and only if

$$e_G(Z) \geq f(Z) \text{ for every } Z \subseteq V. \tag{2.4a}$$

(b) *There exists an orientation of G for which*

$$\varrho(v) \leq g(v) \text{ for every } v \in V \tag{2.3b}$$

if and only if

$$i_G(Z) \leq g(Z) \text{ for every } Z \subseteq V. \tag{2.4b}$$

(c) *There exists an orientation of G satisfying both (2.3a) and (2.3b) if and only if there is one satisfying (2.3a) and there is one satisfying (2.3b) (or equivalently, both (2.4a) and (2.4b) hold).*

This result was proved directly in [Frank-Gyárfás, 1976] but it can easily be deduced from Hoffman's circulation theorem, as well. The phenomenon formulated in Part (c) of this theorem may be called the **linking principle.** As we will see it occurs in some orientation problems and not in others.

2. Component-type conditions

These results concern the case when the connectivity requirement is 1. It is tempting to try to combine Theorem 2.2 and Robbins' theorem. This task was accomplished in [Frank and Gyárfás, 1976].

Let $G = (V, E)$ be a 2-edge-connected undirected graph. For $Z \subseteq V$ let $c(Z)$ denote the number of components of $G - Z$.

THEOREM 2.3 (a) *There exists a strongly connected orientation of G whose in-degree function ϱ satisfies*

$$\varrho(v) \geq f(v) \text{ for every } v \in V \tag{2.5a}$$

if and only if

$$e_G(Z) \geq c(Z) + f(Z) \text{ for every } \emptyset \neq Z \subseteq V. \tag{2.6a}$$

(b) *There exists a strongly connected orientation of G for which*

$$\varrho(v) \leq g(v) \text{ for every } v \in V \tag{2.5b}$$

if and only if

$$i_G(Z) + c(Z) \leq g(Z) \text{ for every } \emptyset \neq Z \subseteq V. \tag{2.6b}$$

(c) *There exists a strongly-connected orientation of G satisfying both (2.5a) and (2.5b) if and only if there is one satisfying (2.5a) and there is one satisfying (2.5b).*

The following result of Frank and Gyárfás [1976] is a characterization of similar type for the case when, rather than strong connectivity, the reachibility of every node from a specified node s is required.

THEOREM 2.4 *Let s be a specified node of G.*
(a) There exists an orientation of G in which (*) every node is reachable from s and the in-degree function ϱ satisfies

$$\varrho(v) \geq f(v) \text{ for every } v \in V \tag{2.7a}$$

if and only if

$$e_G(Z) \geq c(Z) + f(Z) - \varepsilon(Z) \text{ for every } \emptyset \neq Z \subseteq V, \tag{2.8a}$$

where $\varepsilon(Z) = 0$ if $s \in Z$ and $\varepsilon(X) = 1$ if $s \in V - Z$.

(b) *There exists an orientation of G satisfying (*) for which*

$$\varrho(v) \leq g(v) \text{ for every } v \in V \tag{2.7b}$$

if and only if

$$i_G(Z) + c(Z) \leq g(Z) + \varepsilon(Z) \text{ for every } \emptyset \neq Z \subseteq V. \tag{2.8b}$$

(c) *There exists an orientation of G satisfying $(*)$ and both of (2.7a) and (2.7b) if and only if there is one satisfying (2.7a) and there is one satisfying (2.7b).*

Note that the linking property holds in both cases. The condition in Theorem 1.2 of Nash-Williams is also of component-type. Theorem 1.2 is a generalization of Part (a) of Theorem 2.3. Part (a) of Theorem 2.4 is also immediately follows from Theorem 1.2 when it is applied to $M := G + \vec{D}$ where digraph $\vec{D}$ consists of $|V| - 1$ edges, one from v to s for each node v.

It is not difficult to formulate a counter-part of Nash-Williams' result when, instead of lower bound, one has upper bound prescription on the in-degrees. However, unlike the undirected case in Theorem 2.3, the linking principle is not true anymore. This is shown by the following example found by É. Tardos. Let $V := \{v_1, v_2, v_3, v_4\}$, $E := \{v_1v_4, v_2v_3\}$, $\vec{A} := \{v_1v_2, v_2v_1, v_3v_4, v_4v_3\}$. Let $f(v_1) := 1, f(v_2) := f(v_3) := f(v_4) := 0$, $g(v_1) := g(v_2) := g(v_4) := 1, g(v_3) := 0$. Here M has an a strongly connected orientation satisfying the lower-bound conditions (namely, $\vec{E} := \{v_4v_1, v_2v_3\}$). M also has a strongly connected orientation satisfying the upper bound conditions (namely, $\vec{E} := \{v_1v_4, v_3v_2\}$) but M has no strongly connected orientation satisfying both conditions.

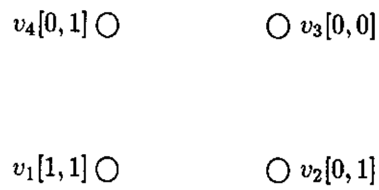

Figure 2.1

Note that Theorem 1.2 easily implies Part (a) of Theorem 2.4 as well. Indeed, apply Theorem 1.2 to the mixed graph arising from G by adding a directed edge from every node to node s and observe that an orientation of the arising mixed graph is strongly connected if and only if the same orientation of G satisfies $(*)$.

3. Partition-type conditons

In the following two theorems partitions are required to formulate the necessary and sufficient conditions. Let $h : 2^V \to \mathbf{Z} \cup \{-\infty\}$ be a set-function with $h(\emptyset) = h(V) = 0$.

THEOREM 2.5 [Frank, 1980] *Suppose that h is non-negative and crossing G-supermodular. There exists an orientation of G for which*

$$\varrho(X) \geq h(X) \text{ for every } X \subseteq V \tag{2.9}$$

if and only if both

$$e_G(\mathcal{P}) \geq \sum h(P_i) \tag{2.10a}$$

and

$$e_G(\mathcal{P}) \geq \sum h(V - P_i) \tag{2.10b}$$

hold for every partition $\mathcal{P}= \{P_1, \ldots, P_p\}$ of V. If, in addition, h is symmetric (that is, $h(X) = h(V - X)$ for every $X \subseteq V$), then it suffices to require only (2.10a) and only for partitions of two parts (which is the same as requiring $d_G(X) \geq 2h(X)$ for every $X \subseteq V$).

Note that (2.10b) is equivalent to requiring (2.10a) for every co-partition $\mathcal{P}=\{P_1,\dots,P_p\}$ of V. Theorem 1.1 of Nash-Williams is a special case of the second part of Theorem 2.5, which actually includes a cut-type condition. In [Frank, 1980] it was also shown that the linking property holds whenever h is non-negative and crossing G-supermodular. In the following result h is unrestricted in sign but is G-supermodular only on intersecting pairs.

THEOREM 2.6 [Frank, 1978] *Suppose that h is intersecting G-supermodular. There exists an orientation of G satisfying (2.9) if and only if*

$$e_G(\mathcal{P}) \geq \sum h(P_i) \tag{2.11}$$

holds for every sub-partition $\mathcal{P}$ of V.

Theorem 2.5 was used in [Frank, 1980] to derive the following characterization of undirected graphs having a k-edge-connected orientation satisfying upper and lower bound prescriptions on the indegree of nodes.

THEOREM 2.7 (a) *There exists a k-edge-connected orientation of G whose in-degree function ϱ satisfies*

$$\varrho(v) \geq f(v) \text{ for every } v \in V \tag{2.12a}$$

if and only if

$$e_G(\mathcal{F}) + i_G(Z) \geq kt + f(Z) \tag{2.13a}$$

holds for every partition $\mathcal{F}=\{Z, V_1,\dots,V_t\}$ of V where only Z may be empty.

(b) *There exists a k-edge-connected orientation of G whose in-degree function ϱ satisfies*

$$\varrho(v) \leq g(v) \text{ for every } v \in V \tag{2.12b}$$

if and only if

$$i_G(\mathcal{F}) - e_G(Z) \geq kt - g(Z) \tag{2.13b}$$

holds for every partition $\mathcal{F}=\{Z, V_1,\dots,V_t\}$ of V where only Z may be empty.

(c) *There exists a k-edge-connected orientation of G satisfying both (2.12a) and (2.12b) if and only if there is one satisfying (2.13a) and there is one satisfying (2.13b).*

Theorem 2.7 may be derived from Theorem 2.5 by defining h as follows. For $X \subset V$ let $h(X) :\equiv k$ if $2 \leq |X|, |V - X|$. If $X = \{v\}$ for some $v \in V$, let $h(X) := \max(k, f(v))$. If $X = \{V - v\}$ for some $v \in V$, let $h(X) := \max(k, d(v) - g(v))$. The derivation consists of showing that for this choice of h condition (2.10a) is equivalent to (2.13a) and that (2.10b) is equivalent to (2.13b).

Using an analogous reduction, it is not difficult to see that Theorem 2.7 extends to the case when the resulted orientation of G is required to be (k, l)-edge-connected at a specified node s.

Note that Theorem 2.3 follows immediately from Theorem 2.7. Indeed, one has to observe that in case $k = 1$ it suffices to require (2.13) only for such partitions for which there is no edge connecting distinct V_i's $(i \geq 1)$. Then the sets V_i are the components of $G - Z$ and hence the number t of these sets is just $c(Z)$.

A consequence of Theorem 2.6 concerning mixed graphs is as follows. Let $M = (V, E + \vec{A})$ be a mixed graph composed from a graph $G = (V, E)$ and from a digraph $\vec{D} = (V, \vec{A})$ and let s be a node of M.

THEOREM 2.8 *There is an orientation of M which is k-edge-connected from s and satisfies $\varrho_{\vec{G}}(v) \geq f(v)$ for every node v if and only if*

$$e_G(\mathcal{F}) + i_G(Z) \geq f(Z) + \sum_i (k - \varrho_{\vec{D}}(V_i)) \tag{2.14}$$

holds for every sub-partition $\mathcal{F} = \{Z, V_1, \ldots, V_t\}$ *of* V *where only* Z *may be empty and* $s \notin \cup_i(V_i)$.

The result immediately follows when Theorem 2.6 is applied to the following function h. Let $h(X) := k - \varrho_{\vec{A}}(X)$ if $s \notin X$ and $|X| \geq 2$, let $h(X) := f(s)$ if $X = \{s\}$, let $h(X) := -\infty$ if $s \in X$ and $2 \leq |X| < |V|$, and let $h(X) := \max(k, f(u))$ if $X = \{u\}$ for any $u \in V - s$.

One of the new results of the present paper is the following generalization of Theorem 2.6. It concerns functions h which are between intersecting and crossing G-supermodularity. The proof will be presented in Section 6.

THEOREM 2.9 *Suppose that* h *is crossing* G*-supermodular and that* h *satisfies*

$$h(A) + h(B) \leq h(A \cap B) + d_G(A, B) \text{ whenever} \tag{2.15}$$

$A \cup B = V, A \cap B \neq \emptyset$ *and* $d_G(A, B) > 0$. *Then* G *has an orientation satisfying (2.9) if and only if (2.11) holds for every sub-partition* $\mathcal{P}$ *of* V.

Note that an intersecting G-supermodular function satisfies (2.15) if $A \cup B = V, A \cap B \neq \emptyset$ irrespective whether $d_G(A, B)$ is positive or zero.

As a direct consequence of Theorem 2.9, in Section 6 we will derive the following result which, in turn, is a generalization of Theorem 1.2 of Nash-Williams.

THEOREM 2.10 *Let* $M = (V, E + \vec{A})$ *be a mixed graph consisting of an undirected graph* $G = (V, E)$ *and a digraph* $\vec{D} = (V, \vec{A})$ *and let* s *be a specified node of* M. *Let* $k \geq 1$ *be an integer and* $f : V \to \mathbf{Z}_+$ *an integer-valued function.* M *has a strongly connected orientation* $\vec{M} = (V, \vec{E} + \vec{A})$ *which is* k*-edge-connected from* s *and satisfies* $\varrho_{\vec{G}}(v) \geq f(v)$ *for every node* v *if and only if for every sub-partition* $\mathcal{F} = \{Z, V_1, \ldots, V_t\}$ *of* V, *where only* Z *may be empty,*

$$e_{\mathcal{F}} + i_G(Z) \geq f(Z) + \sum_{i=1}^{t}(k - \varrho_{\vec{D}}(V_i)) - \varepsilon(k-1) \tag{2.16}$$

holds where $\varepsilon = 1$ *if* $s \in \cup_i V_i$ *and* $\varepsilon = 0$ *otherwise.*

4. Conditions including more complicated families

In Theorems 2.5 and 2.9, beside crossing G-supermodularity, function h was assumed to be non-negative in the first case and to satisfy condition (2.15) in the second. What can be said if we drop the extra requirements and h is simply crossing G-supermodular? Such functions arise when one wants to solve the k-edge-connectivity orientation problem for mixed graphs. It was indicated in [Frank, 1984] how to reduce the problem to the feasibility theorem of submodular flows and the corresponding orientation theorem was explicitly formulated in [Frank, 1993].

THEOREM 2.11 *Let* $G = (V, E)$ *be an undirected graph and* $h : 2^V \to \mathbf{Z} \cup \{-\infty\}$ *a crossing* G*-supermodular set-function with* $h(V) = h(\emptyset) = 0$. G *has an orientation satisfying (2.9) if and only if*

$$s_t \geq \sum_{i=1}^{t}(\sum_j h(V_i^j) - e_i) \tag{2.17}$$

for every sub-partition $\{V_1, V_2, \ldots, V_t\}$ *of* V *where each* V_i *is the intersection of a family of pairwise co-disjoint sets* $V_i^1, V_i^2, \ldots,$ *number* s_t *denotes the number of edges entering a* V_i, *and, for a given* i, e_i *denotes the number of edges connecting different sets* V_i^j $(j = 1, 2, \ldots)$.

Theorem 2.11 may be used for finding a necessary and sufficient condition for the existence of a k-edge-connected orientation of a mixed graph which, in addition, satisfies

$$\varrho_{\vec{G}}(v) \geq f(v) \text{ for every } v \in V, \tag{2.18a}$$

$$\varrho_{\vec{G}}(v) \leq g(v) \text{ for every } v \in V. \tag{2.18b}$$

Define h_1 as follows. $h_1(\emptyset) := h_1(V) := 0$. For $\emptyset \subset X \subset V$ let $h_1(X) := k - \varrho_{\vec{D}}(X)$. The submodularity of $\varrho_{\vec{D}}$ implies that h_1 is crossing supermodular. The crossing supermodularity is preserved if the value of h_1 is increased on singletons and/or on complements of singletons. Hence the following function h is crossing supermodular. Let $h(X) := h_1(X)$ if $2 \leq |X| \leq |V| - 2$. For every $v \in V$ let $h(X) := \max(h_1(v), f(v))$ when $X = \{v\}$ and let $h(X) := \max(h_1(X), d_G(v) - g(v))$. If we apply Theorem 2.11 to this h, we obtain the required condition of the k-edge-connected orientability with (2.18). This condition may be expressed in terms of M, f, g but we omit the details since no extra benefit seems to arise from this translation.

With an easy trick, this orientation problem can be slightly further generalized. To this end let s be a node of M and l, k integers with $0 \leq l \leq k$. Find an orientation of M which is (k,l)-edge-conencted at s and satisfies (2.18). To reduce this problem to the k-edge-connected orientation problem, modify the directed part $\vec{D}$ of M by adding $k-l$ parallel edges from v to s for every node v. Let $\vec{D}^+$ denote the resulting digraph and let $M^+ := G + \vec{D}^+$. Clearly, an orientation $\vec{G} + \vec{D}$ of M is (k,l)-edge-connected at s if and only if the orientation $\vec{G} + \vec{D}^+$ is k-edge-connected.

One may be wondering whether the complicated condition in Theorem 2.11 may possibly be replaced by simpler, partition-type conditions: after all, this was the case in the special case formulated by Nash-Williams in Theorem 1.2. The following example, however, shows that this is not always possible.

Let $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{v_1v_2, v_3v_4\}$. Define h as follows. $h(V) := h(\emptyset) := 0$, $h(X_1) := h(X_2) :=:= h(X_3) := 1$ where $X_1 = \{v_1\}, X_2 = \{v_2, v_4\}, X_3 = \{v_1, v_2, v_3\}$, and let $h(X) := -\infty$ otherwise. This function is crossing h-supermodular since there are no two crossing sets X, Y with $h(X) > -\infty, h(Y) > \infty$. By inspection one sees that no required orientation exists. On the other hand if we reduce $h(X_i)$ by one for any $i = 1,2,3$, then an orientation exists with respect to the revised h_i. This shows that any certificate for the non-existence of the originally required orientation must include each of the sets X_1, X_2 and X_3. But this family of three sets is not a sub-partition, nor their complements form a sub-partition.

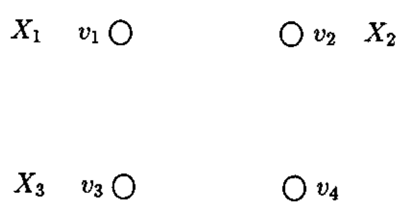

Figure 2.2

By Theorem 2.9, condition (2.15) must be violated and, indeed, sets $A := X_2$ and $B := X_3$ violate (2.15). This example can also be used to show that the same type of trouble occurs when the goal is finding a 2-edge-connected orientation of a mixed graph. Let $M = (V, E + \vec{A})$ be a mixed graph where V and E are the same as before and $\vec{A}$ consists of v_3v_1, v_4v_2, v_3v_2 and two parallel copies of each of v_1v_3, v_2v_3, v_2v_4.

Now $\varrho_{\vec{A}}(X) \geq 2$ whenever $\emptyset \subset X \subset V, X \neq X_i$. Therefore the problem of finding a 2-edge-connected orientation of M is equivalent to the orientation problem in the previous example.

Figure 2.3

Though the necessary and sufficient condition (2.17) in Theorem 2.11 provides a good characterization for the required orientability, it is too complicated to allow a straightforward derivation of Theorems 2.5 and 2.9. It is a natural demand to find such a derivation since the constraining function h in both of these theorems is crossing G-supermodular, with some additional restrictions. The examples above show that one cannot expect to replace condition (2.17) by simple partition-type conditions. However some simplification can be done by introducing the notion of cross-free compositions of sets, a common generalization of partitions and co-partitions. The simplified form (Theorem 6.4) of Theorem 2.11 will allow us to derive Theorems 2.5 and 2.9. In the next preparatory section we exhibit some basic properties of cross-free compositions.

III. CROSS-FREE COMPOSITIONS OF SETS

Intuitively, by a family $\mathcal{F}$ of non-empty subsets of V we mean a collection or list of subsets of V where repetition is allowed. This can be described formally by a function $f : 2^V \to \mathbf{Z}_+$ that assigns a non-negative integer number to every subset of V (meaning that $f(X)$ copies of X belong to $\mathcal{F}$). f is called the **incidence vector** of $\mathcal{F}$. By the **union** of two families given by f, g we mean the family of incidence vector $f + g$

For any element s of V belonging to α members of $\mathcal{F}$ we say that the **degree** $\deg(s) := \deg_{\mathcal{F}}(s)$ of s is α or that s **is covered** α times by $\mathcal{F}$. Suppose that $\pi_0 < \pi_1 < \ldots < \pi_h$ denote the distinct values of degrees of $\mathcal{F}$. If $\pi_i = \pi_{i-1} + 1$ $(1 \leq i \leq h)$, we say that $\mathcal{F}$ **is consecutive.**

We say that the **height** of $\mathcal{F}$ is h. $\mathcal{F}$ is called **regular** if $h = 0$, that is, the degree of every element is the same. A regular family will also be called a **composition** of V. For example, both a partition and a co-partition of V are compositions of V. $\mathcal{F}$ will be called **primitive** if it includes no regular (proper) sub-family. Clearly, a sub-family of a primitive family is also primitive.

When $h \geq 1$ let $L_i := \{s : deg(s) = \pi_i\}$ for $i = 0, 1, \ldots, h$. The sets L_i are called the **degree-levels** of $\mathcal{F}$. If $h = 1$ and $\pi_1 - \pi_0 = 1$, we say that $\mathcal{F}$ is a **composition** of L_1. (That is, every element s of V belongs to $t+1$ or t members of $\mathcal{F}$ according to s belongs to L_1 or not where $t \geq 0$ is an integer.) For example, if A is a non-empty, proper subset of V then both a partition and a co-partition are cross-free compositions of A. In particular, $\{A\}$ is a composition of A. A double-partition of A can also be seen to be a composition of A, though not-necessarily cross-free.

LEMMA 3.1 *If $\mathcal{A}_1, \mathcal{A}_2, \ldots$ is an infinite sequence of compositions of a proper subset A of V, then there are indices $i < j$ so that $\mathcal{A}_j$ arises from $\mathcal{A}_i$ by adding a regular family.*

Proof. Let $f_1, f_2, \ldots$ be an infinite sequence of non-negative integer vectors of dimension m. We claim that there are two indices $i < j$ for which $f_i \leq f_j$. To see this we use induction on m. The claim is trivial for $m = 1$ and hence we suppose that $m > 1$. If there is an upper bound for the first coordinates, then there is an infinite subsequence for which the first coordinates are constant. If no such an upper bound exists, then there is an infinite subsequence in which the first coordinates form a monotonously increasing sequence. In either case delete the first coordinate and apply the inductive hypothesis to this infinite subsequence. For the the resulting indices i, j we have $f_i \leq f_j$.

Let f_i be the incidence vector of $\mathcal{A}_i$. By the claim there are indices i, j so that $f_j - f_i$ is non-negative and integer-valued. Let $\mathcal{R}$ denote the family whose incidence vector is $f_j - f_i$. Since both $\mathcal{A}_i$ and $\mathcal{A}_j$ are compositions of the same set A, $\mathcal{R}$ is regular and $\mathcal{A}_j$ arises from $\mathcal{A}$ by adding $\mathcal{R}$. •

Let $\mathcal{C}$ be an arbitrary family of subsets of V. The **uncrossing procedure** means that, as long as there are two crossing members X, Y in the current family, we replace X and Y by $X \cap Y$ and $X \cup Y$. Because this operation strictly increases the sum $\sum_{Z\in\mathcal{C}}|Z|^2$, the procedure terminates after a finite number of steps. It results in a cross-free family $\mathcal{C}'$ which may depend on the sequence of the uncrossing steps but the degrees do not change and hence the degree-levels are also unaffected. In particular, if $\mathcal{C}$ is a composition of a set A, then $\mathcal{C}'$ is a cross-free composition of A.

Let $T = (U, F)$ be a directed tree with node-set U and edge-set F. Let V be a set disjont from U and $\varphi : V \to U$ a mapping. For $Z \subseteq U$ we use the notation $\varphi^{-1}(Z) := \{s \in V : \varphi(s) \in Z\}$ and call this set the **pre-image** of Z. We call a node of T **empty** if its pre-image is the empty set. Also, for a family $\mathcal{X}$ of subsets of U let the **pre-image** of $\mathcal{X}$, denoted by $\varphi^{-1}(\mathcal{X})$, be the family of non-empty pre-images of the members of $\mathcal{X}$. Note that if $\mathcal{X}'$ is the pre-image of $\mathcal{X}$, then

$$\deg_{\mathcal{X}'}(s) = \deg_{\mathcal{X}}(\varphi(s)) \text{ for every } s \in V. \tag{3.3}$$

In particular, we have:

CLAIM 3.2 *If $\mathcal{X}$ is a cross-free composition of a set $Z \subseteq U$, then its pre-image $\mathcal{X}'$ is a cross-free composition of $\varphi^{-1}(Z)$.* •

The deletion of any edge f of T results in two components. Let T_f denote the node-set of the component of $T - f$ entered by f. Clearly, $\mathcal{T} := \{T_f : f \in F\}$ is a cross-free family on U (depending only on T) and its pre-image $\varphi^{-1}(\mathcal{T})$ is a cross-free family on V. We will say that the pair (T, φ) is a **tree-representation** of $\varphi^{-1}(\mathcal{T})$. It is not difficult to prove (see, Edmonds and Giles [1977]) that any cross-free family on V has such a tree-representation.

(For example, if $\mathcal{A} = \{A_1, \ldots, A_k\}$ is a partition of a subset A of V, then T is an out-directed star, that is, T has node-set $\{u_0, u_1, \ldots, u_k\}$ and edge-set $\{u_0u_1, \ldots, u_0u_k\}$. Furthermore, $\varphi(s) := u_i$ if $s \in A_i$ and $\varphi(s) := u_0$ if $s \in V - A$. If $A = V$, then the center u_0 of T is an empty node of T. Analogously, if $\mathcal{A}$ consists of pairwise co-disjoint sets whose intersection is A, then the representing tree is an in-directed star. Its center is empty if $A = \emptyset$.)

The degree-levels of $\mathcal{T}$, denoted by $\{U_0, U_1, \ldots, U_h\}$, will be called the **levels** of T. Obviously, $\deg_{\mathcal{T}}(v) = \deg_{\mathcal{T}}(u) + 1$ for every edge uv of T and hence every edge of T with tail in U_i has its head in U_{i+1} ($i = 0, \ldots, h-1$). Let F_i denote the set of edges of T with tail in U_{i-1} and head in U_i ($1 \leq i \leq h$). Then the F_i's form a partition of F.

Another consequence of (3.3) is that the non-empty subsets of V of form $\varphi^{-1}(U_i)$ are precisely the degree-levels of $\varphi^{-1}(\mathcal{T})$ and, in particular, the number of these degree-levels is the number of U_i's containing non-empty nodes.

For any subset $X \subseteq F$ let $\mathcal{X} := \{T_e : e \in X\}$. The following claim is obvious.

CLAIM 3.3 *If X is the set of edges leaving (entering) an empty source-node (sink-node) of T, then $\mathcal{X}$ is a partition (co-partition) of U and $\varphi^{-1}(\mathcal{X})$ is a partition (co-partition) of V.* •

LEMMA 3.4 *A regular, cross-free family $\mathcal{R}$ decomposes into partitions and co-partitions of V.*

Proof. It suffices to prove that $\mathcal{R}$ includes a partition or a co-partition of V since leaving out of $\mathcal{R}$ a partition or co-partition leaves a smaller regular cross-free family which, inductively, decomposes into partitions and co-partitions.

Since $\mathcal{R}$ is regular, every non-empty node of T belongs to the same level-set of T. Hence there is an empty source-node or sink-node of T. By Claim 3.3, $\mathcal{R}$ includes a partition or a co-partition. •

Let A be a proper subset of V. As a common generalization of partition and co-partition of A, we introduce the notion of a tree-composition of A. Let $\{A_1, \ldots, A_k\}$ be a partition of A and $\{B_1, \ldots, B_l\}$ a partition of $V - A$ ($k, l \geq 1$). Let $T = (U, F)$ be a directed tree such that $U := \{a_1, \ldots, a_k, b_1, \ldots, b_l\}$ and each directed edge goes from a b_j to an a_i. The family $\mathcal{A} := \{\varphi^{-1}(T_f) : f \in F\}$ is called a **tree-composition of** A where $\varphi(v) = a_i$ if $v \in A_i$ and $\varphi(v) = b_j$ if $v \in B_j$. This is equivalent to saying that the composition $\mathcal{A}$ has a tree-representation consisting of two levels so that every node of the tree is non-empty. We will also say that a partition or a co-partition of V is a **tree-composition of** V.

(If $k = l = 1$, then $\mathcal{A}$ consists of A. If $l = 1 < k$, then $\mathcal{A}$ is a partition of A. If $k = 1 < l$, then $\mathcal{A}$ is a co-partition of A.) Note that a tree-composition $\mathcal{A}$ of A is a cross-free double-partition of A. Indeed, the cross-freeness comes from the definition. For each $i = 1, \ldots, k$) let $\mathcal{A}_i := \{\varphi^{-1}(T_f) : f \in F \text{ enters } a_i\}$. Now $\mathcal{A}_i$ is a co-partition of A_i and hence $\mathcal{A} = \cup_{i=1}^{k} \mathcal{A}_i$ is a double-partition of A.

LEMMA 3.5 *A primitive and cross-free composition of $A \subset V$ is a tree-composition.*

Proof. Let (T, φ) be a tree representation of $\mathcal{A}$. By Claim 3.3 no source-node and sink-node of T is empty and hence T has just two level-sets and no empty node, that is, $\mathcal{A}$ is a tree-composition. •

Let now (T, φ) be a tree-representation of a cross-free family and let $X \subseteq F$ be a non-empty subset. Let P denote the edge set of the unique path of T connecting two nodes u and v of T. Walking from u to v along P, any edge is directed forward or backward. It follows from the definitions that $\deg_{\mathcal{X}}(v) - \deg_{\mathcal{X}}(u)$ is equal to the difference of the forward and backward elements of $P \cap X$. For $X := F_i$ this difference is 0 if $Z := U_i \cup \ldots \cup U_h$ contains both of u and v or if Z contains none of them. The difference is 1 if $v \in Z, u \in U - Z$. This and Claim 3.2 imply:

CLAIM 3.6 *Let $X := F_i$ for some $i = 1, \ldots, h$ and let $Z := U_i \cup \ldots \cup U_h$. Then $\mathcal{X}$ is a cross-free composition of Z. If $\varphi^{-1}(Z)$ is a non-empty, proper subset of V, then $\varphi^{-1}(\mathcal{X})$ is a cross-free composition of $\varphi^{-1}(Z)$. If $\varphi^{-1}(Z)$ is empty or is V, then $\varphi^{-1}(\mathcal{X})$ is a cross-free composition of V.* •

LEMMA 3.7 *Let $\mathcal{C}$ be a consecutive, cross-free family on V with height $h' \geq 1$ and degree-levels $L_0, \ldots, L_{h'}$. Then there is a decomposition $\{\mathcal{R}, \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{h'}\}$ of $\mathcal{C}$ (where $\mathcal{R}$ may be empty) so that $\mathcal{R}$ is regular and $\mathcal{C}_j$ is a cross-free composition of the set $L_j \cup \ldots \cup L_{h'}$ for each $j = 1, \ldots, h'$.*

Proof. Let (T, φ) be a tree-representation of $\mathcal{C}$ with levels $U_0, \ldots, U_h$. There are exactly $h' + 1$ levels of T which contain a non-empty node and the indices of these levels are consecutive integers, denoted by $l, \ldots, l + h'$, as $\mathcal{C}$ is consecutive. Let $I_1 := \{l + 1, \ldots, l + h'\}$ and $I_0 := \{1, \ldots, l, l + h' + 1, \ldots, h\}$ (I_0 may be empty).

Let $\mathcal{C}'_i$ ($i = 1, \ldots, h$) consist of those members of $\mathcal{C}$ whose representing edge belongs to F_i. Let $\mathcal{R} := \cup(\mathcal{C}'_i : i \in I_0)$ and $\mathcal{C}_j := \mathcal{C}'_{l+j}$ ($j = 1, \ldots, h'$). By applying Claim 3.6 to each $i = 1, \ldots, h$, we obtain that $\mathcal{C}_j$ is a composition of $L_j \cup \ldots \cup L_{h'}$ that each $\mathcal{C}_i$ ($i \in I_0$) is regular and hence so is $\mathcal{R}$ •

(We will use Lemma 3.7 only in the special case when $h' = 2$).

IV. FULL-TRUNCATION OF SUBMODULAR FUNCTIONS

Throughout this section we will assume that every set-function is integer-valued, zero on the empty set and finite on the ground-set. (In particular, while defining a set-function f, $f(\emptyset)$ will always be meant 0 even if we do not specify this explicitly.) Let V be a finite ground-set and $b : 2^V \to \mathbf{Z} \cup \{\infty\}$ be an arbitrary set-function. If $\mathcal{F}$ is a family of subsets of V, the sum $\sum(b(X) : X \in \mathcal{F})$ will be abbreviated by $b(\mathcal{F})$.

The **base polyhedron of** b is defined by $B := B(b) := \{x \in R^V : x(V) = b(V), x(A) \leq b(A) \text{ for every } A \subseteq V\}$. (The name refers to a theorem of J. Edmonds asserting that the base-polyhedron of a matroid rank-function is the convex hull of the incidence vectors of bases of the matroid. Base polyhedra for more general fully submodular functions were introduced and investigated by S. Fujishige [1984]).

In this section we will use the same letter b to denote several set-functions. In each case, except the last paragraph, we will **assume that** $b(V) = 0$.

LEMMA 4.1 *If $x \in B(b)$ and $\mathcal{A}$ is a composition of a subset $\emptyset \subset A \subseteq V$, then $x(A) \leq b(\mathcal{A})$. In particular, if $B(b)$ is non-empty, then*

$$b(\mathcal{R}) \geq 0 \tag{4.1}$$

holds for every composition $\mathcal{R}$ of V.

Proof. Suppose that $\deg_{\mathcal{A}}(v) = \alpha$ if $v \in V - A$ and $\deg_{\mathcal{A}}(v) = \alpha + 1$ if $v \in A$. Then $x(A) = \sum(x(X) : X \in \mathcal{A}) - \alpha x(V) = \sum(x(X) : X \in \mathcal{A}) - \alpha b(V) = \sum(x(X) : X \in \mathcal{A}) \leq \sum(b(X) : X \in \mathcal{A}) = b(\mathcal{A})$. •

Suppose now that (4.1) holds for every composition $\mathcal{R}$ of V. (It can be shown with the help of Farkas' lemma that this condition is actually sufficient for the non-emptyness of $B(b)$ but we will not use this fact.) With b we associate a set-function $b^{\downarrow}$, called the **lower full-truncation** or, in short, the **full-truncation** of b, as follows.

$$b^{\downarrow}(A) := \min(b(\mathcal{A}) : \mathcal{A} \text{ a composition of } A). \tag{4.2}$$

In particular, this means that $b^{\downarrow}(A) = \infty$ if every composition $\mathcal{A}$ of A contains a set X with $b(X) = \infty$. (4.1) implies that $b^{\downarrow}(V) \geq 0$ and since $\{V\}$ is a composition of V, we have actually $b^{\downarrow}(V) = 0$. (4.1) also implies that the minimum in (4.2) makes sense at all, that is, $b(\mathcal{A})$ is bounded from below. Indeed, if there were an infinite sequence $\mathcal{A}_1, \mathcal{A}_2, \ldots$ of compositions of A for which $b(\mathcal{A}_i)$ $(i = 1, 2, \ldots)$ is (strictly) monotonously increasing, then by Lemma 3.1 there would be two indices $i < j$ so that $\mathcal{A}_j$ arises from $\mathcal{A}_i$ by adding a regular family $\mathcal{R}$. But then $b(\mathcal{A}_j) < b(\mathcal{A}_i)$ would imply that $b(\mathcal{R}) < 0$, contradiciting (4.1).

(In the definition "lower" refers to that minimum is used in (4.2). We may, instead, use maximum and call then the full-truncation "upper". Typically, minimization is used for submodular functions and maximization for supermodular functions. Since it will be clear from the context which one is meant, we will leave out the adjective.)

LEMMA 4.2 *If b satisfies (4.1), then $B(b^{\downarrow}) = B(b)$, that is, b and $b^{\downarrow}$ define the same base polyhedron.*

Proof. We have already mentioned that $b^{\downarrow}(V) = 0 = b(V)$. Since $\{A\}$ is a composition of A, we have $b^{\downarrow}(A) \leq b(A)$ and hence $B(b) \supseteq B(b^{\downarrow})$.

Conversely, let $x \in B(b)$ and let $\mathcal{A}$ be a composition of a subset $A \subset V$ for which $b^{\downarrow}(A) = b(\mathcal{A})$. By Lemma 4.1 $x(A) \leq b(\mathcal{A}) \leq b^{\downarrow}(A)$ and hence $x \in B(b^{\downarrow})$, that is, $B(b) \subseteq B(b^{\downarrow})$. •

We will say that a composition $\mathcal{A}$ of $A \subset V$ is **b-extreme** if $b(\mathcal{A})$ is minimum (that is, $b^{\downarrow}(A) = b(\mathcal{A})$), subject to this, $|\mathcal{A}|$ is minimum, and subject to this, $\sum(|Z|^2 : Z \in \mathcal{A})$ is maximum.

LEMMA 4.3 *If b satisfies (4.1) for every regular $\mathcal{R}$, then a b-extreme composition $\mathcal{A}$ of A is primitive and*

$$b(X) + b(Y) < b(X \cap Y) + b(X \cup Y) \tag{4.3}$$

whenever $X, Y \in \mathcal{A}$, $X \not\subseteq Y, Y \not\subseteq X$.

Proof. If $\mathcal{A}$, indirectly, includes a regular family $\mathcal{R}$, then $\mathcal{A}' := \mathcal{A} - \mathcal{R}$ is also a composition $\mathcal{A}$ for which $b(A) = b(\mathcal{A}')$, contradicting the minimality of $|\mathcal{A}|$. Hence $\mathcal{A}$ is indeed primitive.

Suppose now that $X, Y \in \mathcal{A}$ violate (4.3), Let $\mathcal{A}' := \mathcal{A} - \{X, Y\} \cup \{X \cap Y, X \cup Y\}$. Then $\mathcal{A}'$ is a composition of A, $|\mathcal{A}'| = |\mathcal{A}|$ and $\sum(|Z|^2 : Z \in \mathcal{A}') > \sum(|Z|^2 : Z \in \mathcal{A})$. We have $b^{\downarrow}(A) \leq b(\mathcal{A}') \leq b(\mathcal{A} = b^{\downarrow}(A)$, that is, $b^{\downarrow}(A) = b(\mathcal{A}')$, contradicting the extremal choice of $\mathcal{A}$. •

We call a set-function b **fully (intersecting, crossing) submodular** if

$$b(A) + b(B) \geq b(A \cap B) + b(A \cup B) \tag{4.4}$$

holds for every (intersecting, crossing) $A, B \subseteq V$.

LEMMA 4.4 *Let b be a crossing (respectively, intersecting) submodular function. If (4.1) holds for every tree-composition (resp., partition) of V, then (4.1) holds for every composition of V.*

Proof. Suppose, indirectly, that $b(\mathcal{R}') < 0$ for a composition $\mathcal{R}'$ of V. Apply the uncrossing procedure as long as the current family includes two members for which the submodular inequality holds. The resulting family $\mathcal{R}$ is cross-free regular and $b(\mathcal{R}) \leq b(\mathcal{R}') < 0$. By Lemma 3.4 $\mathcal{R}$ decomposes into partitions and co-partitions of V. (When b is intersecting submodular, then $\mathcal{R}$ must not contain two co-disjoint sets with non-empty intersection and hence the decomposition of $\mathcal{R}$ consists of partitions only.) By the hypothesis, (4.1) must hold for $\mathcal{R}$, a contradiction. •

LEMMA 4.5 *Let b be a crossing (respectively, intersecting) submodular function satisfying (4.1) for every regular $\mathcal{R}$. Then a b-extreme composition $\mathcal{A}$ of A is a tree-composition (resp., partition) of A. In particular, the minimum in (4.2) is attained at a tree-composition (resp. partition) of A.*

Proof. By Lemma 4.3 $\mathcal{A}$ is primitive and cross-free, that is, by Lemma 3.5, $\mathcal{A}$ is a tree-composition. (When b is intersecting submodular, then $\mathcal{A}$ contains no two co-intersecting sets and hence it must be a partition of A). •

From Lemma 4.3 it follows that $b^{\downarrow} = b$ when b is fully submodular.

THEOREM 4.6 *Let b be a crossing submodular function with $b(V) = 0$ satisfying (4.1) for every partition and co-partition of V. Then $b^{\downarrow}$ is fully submodular.*

Proof. Let A and B be two subsets of V for which $A \not\subseteq B, B \not\subseteq A$ and let $\mathcal{A}$ and $\mathcal{B}$ be compositions of A and B, respectively, so that $b^{\downarrow}(A) = b(\mathcal{A})$ and $b^{\downarrow}(B) = b(\mathcal{B})$. (If $A \subseteq B$ or $B \subseteq A$, then (4.4) automatically holds.)

Let $\mathcal{C}'$ denote the union of $\mathcal{A}$ and $\mathcal{B}$ (recall the definition of union of two families). Since A and B are distinct, $\mathcal{C}'$ is consecutive. Apply the uncrossing procedure and let $\mathcal{C}$ denote the final cross-free family. Since b is crossing submodular, $b(\mathcal{C}) \leq b(\mathcal{C}')$. Since the uncrossing procedure does not affect the degrees, $\mathcal{C}$ is also consecutive and the degree-levels of $\mathcal{C}$ and $\mathcal{C}'$ are the same.

If $A \cap B = \emptyset$, then the degree-levels of $\mathcal{C}'$ is $L_0 = V - (A \cup B), L_1 = A \cup B$ and hence $\mathcal{C}$ is a composition of $A \cup B$. We have $b^{\downarrow}(A \cap B) + b^{\downarrow}(A \cup B) = 0 + b^{\downarrow}(A \cup B) \leq b(\mathcal{C}) \leq b(\mathcal{A}) + b(\mathcal{B}) = b^{\downarrow}(A) + b^{\downarrow}(B)$, as required.

If $A \cup B = V$, then the degree-levels of $\mathcal{C}'$ is $L_0 = V - (A \cap B), L_1 = A \cap B$ and hence $\mathcal{C}$ is a composition of $A \cap B$. We have $b^{\downarrow}(A \cup B) + b^{\downarrow}(A \cap B) = 0 + b^{\downarrow}(A \cap B) \leq b(\mathcal{C}) \leq b(\mathcal{A}) + b(\mathcal{B}) = b^{\downarrow}(A) + b^{\downarrow}(B)$, as required.

Finally, assume that A and B are crossing. Then the height of $\mathcal{C}$ is 2 and the degree-levels of $\mathcal{C}'$ (and of $\mathcal{C}$) are $L_0 = V - (A \cup B), L_1 = (A - B) \cup (B - A), L_2 = A \cap B$. By Lemma 3.7, $\mathcal{C}$ partitions into a (possibly empty) regular family $\mathcal{R}$, a composition $\mathcal{C}_1$ of $L_1 \cup L_2 = A \cup B$ and a composition $\mathcal{C}_2$ of $L_2 = A \cap B$. We have $b(\mathcal{R}) \geq 0$, $b^{\downarrow}(A \cup B) \leq b(\mathcal{C}_1)$ and $b^{\downarrow}(A \cap B) \leq b(\mathcal{C}_2)$ and therefore $b^{\downarrow}(A \cup B) + b^{\downarrow}(A \cap B) \leq b(\mathcal{R}) + b(\mathcal{C}_1) + b(\mathcal{C}_2) = b(\mathcal{C}) \leq b(\mathcal{C}') = b(\mathcal{A}) + b(\mathcal{B}) = b^{\downarrow}(A) + b^{\downarrow}(B)$, as required. •

REMARKS The notion of the (lower) truncation b_1 of a set-function b has been introduced and used earlier (for a survey, see [Lovász 1983]). It is defined by

$$b_1(A) := \min(b(\mathcal{A}) : \mathcal{A} \text{ a partition of } A).$$

The bi-truncation of a set-function b (with $b(V) = 0$) was introduced in [Frank and Tardos, 1988] by

$$b_2(A) := \min(b(\mathcal{A}) : \mathcal{A} \text{ a double-partition of } A).$$

It was proved there that if b is crossing submodular, then b_2 is fully submodular and $B(b_2) = B(b)$ provided that (4.1) holds for every partition and co-partiton of V. It is known from the theory of submodular functions (see, e.g. [Fujishige, 1991]) that a base polyhedron $B(b_0)$ of a fully submodular function b_0 (with $b_0(V) = 0$) uniquely determines b_0, namely, $b_0(A) = \max(x(A) : x \in B(b_0))$, or in other words, the base polyhedra belonging to different fully submodular functions are different. From this it follows that the bi-truncation function and the full-truncation function of a crossing submodular function b (with $b(V) = 0$) is the same, that is, $b_2 = b^{\downarrow}$. We also mention a theorem of Fujishige [1984] stating that a base polyhedron defined by such a b is non-empty if and only if (4.1) is satisfied by every partition and co-partition of V. That is, the hypothesis of Theorem 4.6 is equivalent to the non-emptyness of $B(b)$.

Actually, these results were proved in a more general form when b is crossing submodular and $b(V)$ is arbitrary (but finite). There is however an easy trick to handle such functions. Pick up an arbitrary element s of V and define b_s by $b_s(X) := b(X)$ if $s \in V - X$ and $b_s(X) := b(X) - b(V)$ if $s \in X$. Then b_s is crossing submodular, $b_s(V) = 0$, and $B(b_s)$ is empty if and only if $B(b)$ is empty. Furthermore, $B(b_s)$ is a translation of $B(b)$ (where all but one components of the translating vector are 0 and the exceptional component corresponding to s has value $-b(V)$). Based on this, one can derive that the unique fully submodular function, denoted by $b^{\downarrow}$ and called the full-truncation of b, determining $B(b)$ is $b^{\downarrow}(A) = \min(b(\mathcal{A}) - (l-1)b(S))$ where the minimum is taken over all tree-compositions $\mathcal{A}$ of A and l denotes the number of the sets B_i in the definition of $\mathcal{A}$.

V. FEASIBILITY OF SUBMODULAR FLOWS

Let $\vec{G} = (V, \vec{E})$ be a directed graph. Let $f : \vec{E} \to \mathbf{Z} \cup \{-\infty\}$ and $g : \vec{E} \to \mathbf{Z} \cup \{+\infty\}$ be such that $f \leq g$. For a function $z : \vec{E} \to \mathbf{R}$ let $\varrho_z(A) := \sum(z(e) : e$ enters $A)$ and $\delta_z(A) := \sum(z(e) : e$ leaves A and $\lambda_z(A) := \varrho_z(A) - \delta_z(A)$. Note that λ_z is modular, that is, $\lambda_z(A) = \sum_{v \in A}(\lambda_z(v))$ and therefore we may consider λ_z as a function on V. Furthermore, let $b : 2^V \to \mathbf{Z} \cup \{\infty\}$ be a crossing submodular function with $b(V) = 0$. We call $z : \vec{E} \to \mathbf{R}$ a **submodular flow** (with respect to b) if

$$\lambda_z(A) \leq b(A) \text{ for every } A \subseteq V. \tag{5.1}$$

A submodular flow z is **feasible** if $f \leq z \leq g$. The set of feasible submodular flows is called a **submodular flow polyhedron** and is denoted by $Q(f, g; b)$. Submodular flows were introduced and investigated by R. Giles and J. Edmonds [1977]. They have not assumed that $b(V) = 0$. But this assumption does not really restrict generality since if $b(V) < 0$, then $Q(f, g; b)$ is empty, an uninteresting case, while if $b(V) > 0$, then we can reduce $b(V)$ to 0 since this revision preserves crossing submodularity and does not affect polyhedron $Q(f, g; b)$.

LEMMA 5.1 *z is a submodular flow if and only if λ_z belongs to the base polyhedron $B(b)$. If (4.1) holds for every partition and co-partition $\mathcal{R}$ of V, then $B(b) = B(b^{\downarrow})$ and hence $Q(f, g; b) = Q(f, g; b^{\downarrow})$.*

Proof. The first part is immediate from the definitions The second part follows from the first and from Lemmas 4.4 and 4.2. •

The following two theorems were proved in Frank [1982]. (For a short proof of Theorem 5.2, see [Frank, 1993]).

THEOREM 5.2 *Let b be fully submodular. There exists an integer-valued feasible submodular flow if and only if*

$$\varrho_f(A) - \delta_g(A) \leq b(A) \tag{5.2}$$

holds for every $A \subseteq V$.

By using the properties of truncation and bi-truncation, Theorem 5.2 was used to derive the following:

THEOREM 5.3 *Let b be (A) a crossing or (B) an intersecting submodular function. There exists an integer-valued feasible submodular flow if and only if*

$$\varrho_f(A) - \delta_g(A) \leq b(\mathcal{A}) \tag{5.3}$$

holds for every $A \subseteq V$ and for every double-partition $\mathcal{A}$ of A in case (A) and for every partition $\mathcal{A}$ of A in case (B).

By combining the results of the preceding section, as a new result we have the following refinement of Case (A):

THEOREM 5.4 *Let b be a crossing submodular function. There exists an integer-valued feasible submodular flow if and only if (5.3) holds for every $A \subseteq V$ and for every composition $\mathcal{A}$ of A. Moreover, if (5.3) is violated by a subset $A \subseteq V$, then there is a tree-composition of A violating (5.3).*

Proof. Necessity. Let $z \in Q(f,g;b)$ and apply Lemma 4.1 to $x := \lambda_z$. We have $\varrho_f(A) - \delta_g(A) \leq \varrho_z(A) - \delta_z(A) = \lambda_z(\mathcal{A}) \leq b(\mathcal{A})$.
Sufficiency. Let us assume that (5.3) holds for every $A \subseteq V$ and for every composition of A. For $A := V$ this implies that (4.1) holds so we may consider the full-truncation $b^{\downarrow}$ of b. By Theorem 4.6 $b^{\downarrow}$ is fully submodular. Now (5.3) holds if and only if (5.2) holds for $b^{\downarrow}$ in place of b and hence by Theorem 5.2 the required submodular flow exists. •

The main content of this result is that for the non-emptyness of $Q(f,g;b)$ it suffices to require (5.2) only for tree-compositions of A (which are special double-partitions of A). To demonstrate the the advantage of this simplified characterization, we present two special cases where the essential compositions can be further restricted. In the first case the essential compositions are sub-partitions of V, while in the second, the essential components are partitions and co-partitions of V. In the next section these results will be used to derive Theorems 2.5 and 2.9.

Let b be a crossing submodular function with $b(V) = 0$ such that

$$b(A) + b(B) \geq b(A \cap B) \tag{5.4}$$

whenever $A \cup B = V, A \cap B \neq \emptyset$ and $d_{g-f}(A,B) > 0$, that is, as $b(V) = 0$, the submodular inequality (4.4) holds for such pairs. Note that an intersecting submodular function satisfies (4.4) for every pair with $A \cup B = V$, $A \cap B \neq \emptyset$ therefore the next result is a generalization of Case (B) in Theorem 5.3.

THEOREM 5.5 *Suppose that b is crossing submodular (with $b(V) = 0$) satisfing (5.4). There exists an integer-valued feasible submodular flow if and only if (5.3) holds for every $A \subseteq V$ and for every partition $\mathcal{A}$ of A, which is equivalent to requiring that*

$$\varrho_f(\cup_i A_i) - \delta_g(\cup_i A_i) \leq \sum_i b(A_i) \tag{5.5}$$

for every sub-partition $\{A_1, \ldots, A_t\}$ of V.

Proof. We have seen the necessity of the condition therefore we prove only its sufficiency. By Theorem 5.4 it suffices to show that (5.3) holds for every tree-composition of $A \subseteq V$.
First let $A := V$. By the hypothesis, (5.3) holds for every partition of V. Let $\mathcal{R} := \{X_1, \ldots, X_t\}$ be a co-partition of V which, indirectly, violates (5.3), that is, $b(\mathcal{R}) < 0$, and assume that t is minimal. Since $\mathcal{R}$ is not a partition, $t \geq 3$. The submodular inequality cannot hold for any two members of $\mathcal{R}$, for otherwise by replacing these two sets by their intersection (which is non-empty as $t \geq 3$), we would obtain another co-partition of V violating (5.3), contradicting the minimality of t. By (5.4) $d_{g-f}(A,B) = 0$ for every $A, B \in \mathcal{R}$. It follows that the sets $V - X_i$ $(i = 1, \ldots, t)$ form a partition of V so that $g(e) = f(e)$ for every edge e

connecting different parts. Using (5.3) for $\{X_i\}$ as a partition of X_i, we have $0 = \varrho_f(X_i) - \delta_g(X_i) \leq b(X_i)$ from which $b(\mathcal{R}) = \sum_i b(X_i) \geq 0$ follows, a contradiction.

Second, let $A \subset V$ and assume, indirectly, that there is a composition of A violating (5.3). Then, by Lemma 4.5, a b-extreme composition $\mathcal{A}$ of A also violates (5.3). We know from Lemma 4.5 that $\mathcal{A}$ is a tree-composition and, by Lemma 4.3, that no two co-disjoint members of $\mathcal{A}$ satisfy the submodular inequality. (5.4) implies that $d_{g-f}(X, Y) = 0$ for every two co-disjoint members of $\mathcal{A}$. Let (T, φ) be a tree-representation of $\mathcal{A}$. Let $U := \{a_1, \ldots, a_k, b_1, \ldots, b_l\}$ be the node-set of T, $\{A_1, \ldots, A_k\}$ a partition of A, $\{B_1, \ldots, B_l\}$ a partition of $V - A$ so that $\varphi(v) = a_i$ if $v \in A_i$ and $\varphi(v) = b_j$ if $v \in B_j$.

CLAIM 5.6 *Let $e = uv$ be an edge of $\vec{G}$ for which $u \in B_i, v \in B_j$ $(i, j \in \{1, \ldots, l\}, i \neq j)$. Then $f(e) = g(e)$.*

Proof. Consider the unique path of T connecting $\varphi(u)$ and $\varphi(v)$ and let X_u and X_v be the two members of $\mathcal{A}$ corresponding to the first and last edges of P, respectively. Then $X_u \cup X_v = V$, and $X_u \cap X_v \neq \emptyset$ and hence, by using (5.5), $d_{g-f}(X_u, X_v) = 0$ from which $f(e) = g(e)$ follows. •

For each B_i $(i = 1, \ldots, l)$ let $\mathcal{B}_i$ denote the sub-family of $\mathcal{A}$ consisting of those sets corresponding to the edges of T exiting b_i. Then $\mathcal{B}_i$ is a partition of $V - B_i$ and $\{\mathcal{B}_1, \ldots, \mathcal{B}_l\}$ is a partition of $\mathcal{A}$. By the hypothesis of the Theorem, (5.3) holds for partitions and hence $(*)$ $\varrho_f(V - B_i) - \delta_g(V - B_i) \leq b(\mathcal{B}_i)$ $(i = 1, \ldots, l)$.

Since $B_1, \ldots, B_l$ is a partition of $V - A$, Claim 5.6 and $(*)$ imply that $\varrho_f(A) - \delta_g(A) = \sum_i(\delta_f(B_i) - \varrho_g(B_i)) + \sum(g(e) - f(e) : e$ is an edge from a B_i to $B - B_i) = \sum_i(\delta_f(B_i) - \varrho_g(B_i)) = \sum_i(\varrho_f(V - B_i) - \delta_g(V - B_i)) \leq \sum_i b(\mathcal{B}_i) = b(\mathcal{A})$, contradicting the assumption that $\mathcal{A}$ violates (5.3). Therefore (5.3) is satisfied for every composition and hence, by Theorem 5.4 the required submodular flow exists. • •

Let us turn to the other special case and suppose now that b is a crossing submodular function with $b(V) = 0$ that satisfies

$$\varrho_g(B) - \delta_f(B) \geq b(B) \text{ for every } B \subset V. \tag{5.6}$$

THEOREM 5.7 *Suppose that b satisfies (5.6). There exists an integer-valued feasible submodular flow if and only if*

$$b(\mathcal{R}) \geq 0 \textit{ for every partition and co-partition of } V. \tag{5.7}$$

Proof. We prove the sufficiency only. By Theorem 5.4, it suffices to show that (5.3) holds for every composition. For a composition $\mathcal{F}$ of V, (5.3) is equivalent to $b(\mathcal{F}) \geq 0$ but this holds by (5.7) and by Lemma 4.4.

Suppose now that there is a subset $A \subset V$ and a composition $\mathcal{A}$ of A violating (5.3), that is, $(*)$ $\varrho_f(A) - \delta_g(A) > b(\mathcal{A})$. Let $B := V - A$ and $\mathcal{F} := \mathcal{A} \cup \{B\}$. Then $\mathcal{F}$ is a composition of V and hence $b(\mathcal{F}) \geq 0$. On the other hand, (5.6) for this B is equivalent to $-\varrho_f(A) + \delta_g(A) \geq b(B)$ and, by adding this inequality to $(*)$, we obtain that $b(\mathcal{F}) < 0$, a contradiction. •

VI. BACK TO ORIENTATIONS

Let $G = (V, E)$ be an undirected graph and $h : 2^V \to \mathbf{Z} \cup \{-\infty\}$ a crossing G-supermodular set-function with $h(V) = h(\emptyset) = 0$ and consider again the problem of finding an orientation of G so that the in-degree function $\varrho_{\vec{G}}$ of the resulting digraph $\vec{G} = (V, \vec{E})$ satisfies:

$$\varrho_{\vec{G}}(X) \geq h(X) \text{ for every } X \subseteq V. \tag{6.1}$$

We show first how this orientation problem may be formulated in terms of submodular flows [Frank, 1984]. Let us choose an arbitrary orientation $\vec{G}_r = (V, \vec{E}_r)$ of G whose in-degree function is denoted by $\varrho_r := \varrho_{\vec{G}_r}$. $\vec{G}_r$ will serve as a reference orientation to specify other orientations $\vec{G}$ of G. Define

$$b(X) := \varrho_r(X) - h(X).$$

Any other orientation of G will be defined by a vector $x : \vec{E} \to \{0,1\}$ so that $x(a) = 0$ means that we leave a alone while $x(a) = 1$ means that we reverse the orientation of a. The revised orientation of G defined this way satisfies (6.1) if and only if $\varrho_r(X) - \varrho_x(X) + \delta_x(X) \geq h(X)$ for every $X \subseteq V$. Equivalently, $\varrho_x(X) - \delta_x(X) \leq b(X)$.

LEMMA 6.1 *For any subsets $A, B \subseteq V$ inequalities (4.4) and (1.1) (that is, the submodularity of b and the G-supermodularity of h) are equivalent. Furthermore, $h \geq 0$ if and only if (5.6) holds.*

Proof. The equivalence follows from the identity $\varrho_r(A) + \varrho_r(B) = \varrho_r(A \cap B) + \varrho_r(A \cup B) + d_G(A, B)$. Since $f \equiv 0$ and $g \equiv 1$, (5.6) requires that $\varrho_r(B) \geq b(B)$ (for every $B \subset V$) which is equivalent to $h(B) \geq 0$. •

We say that the submodular flow polyhedron $Q := Q(f, g; b)$ where $f \equiv 0, g \equiv 1$ is **associated with** the orientation problem. By this construction there is a one-to-one correspondence between the orientations of G satisfying (6.1) and the integer points (which are actually $0-1$ points) in Q. Therefore, in order to obtain good characterizations for the existence of the required orientations all we have to do is to translate (5.3) back to terms of G and h.

To this end let $\mathcal{A}$ be a composition of an arbitrary subset $A \subseteq V$ and $j = uv$ an edge of G. Let $e_{u\bar{v}}(\mathcal{A})$ denote the number of $u\bar{v}$-sets in $\mathcal{A}$. That is, $e_{u\bar{v}}(\mathcal{A})$ is the number of sets in $\mathcal{A}$ entered by the directed edge with tail v and head u. Let $e_j(\mathcal{A}) := \max(e_{u\bar{v}}(\mathcal{A}), e_{\bar{u}v}(\mathcal{A}))$ and $e_G(\mathcal{A}) := \sum_{j \in E} e_j(\mathcal{A})$. (This notation has already been used in the special case when $\mathcal{A}$ is a partition or a co-partition of A.) Note that $|e_{u\bar{v}}(\mathcal{A}) - e_{v\bar{u}}(\mathcal{A})| \leq 1$ with equality if and only if $|A \cap \{u, v\}| = 1$.

The quantity $e_j(\mathcal{A})$ indicates the possible (maximal) contribution of an edge $j = uv$ to the sum $\sum(\varrho_{\vec{G}}(X) : X \in \mathcal{A})$ for any orientation $\vec{G}$ of G. Hence $e_G(\mathcal{A})$ measures the total of these contributions and we have:

CLAIM 6.2 *For any orientation $\vec{G}$ of G*

$$\sum_{X \in \mathcal{A}} \varrho_{\vec{G}}(X) \leq e_G(\mathcal{A}) \tag{6.2}$$

and equality holds if $\mathcal{A}$ is regular. •

LEMMA 6.3 *(5.3) holds for a composition $\mathcal{A}$ of $A \subseteq V$ if*

$$h(\mathcal{A}) \leq e_G(\mathcal{A}) \tag{6.3}$$

holds for $\mathcal{A}$.

Proof. Suppose first that $\mathcal{A}$ is a composition of V, that is, $\mathcal{A}$ is regular. Then the left-hand side of (5.3) is 0 and, by the second half of Claim 6.2, $\sum_{X \in \mathcal{A}} \varrho_r(X) = \sum_{j \in E} e_j(\mathcal{A})$. From (6.3) we have $b(\mathcal{A}) = \sum_{X \in \mathcal{A}} b(X) = \sum_{X \in \mathcal{A}} \varrho_r(X) - \sum_{X \in \mathcal{A}} h(X) = e_G(\mathcal{A}) - h(\mathcal{A}) \geq 0$, that is, (5.3) holds in this case.

Second, let $\mathcal{A}$ be a composition of a subset $A \subset V$. As $f \equiv 0$, the left-hand side of (5.3) is $-\delta_{\vec{G}}(A)$. Now $\mathcal{R} := \mathcal{A} \cup \{V - A\}$ is regular (so (6.3) holds for $\mathcal{R}$ in place of $\mathcal{A}$) and $e_j(\mathcal{A}) = e_j(\mathcal{R})$ for every edge j. Applying the second part of Claim 6.2 to $\mathcal{R}$ we obtain from (6.3) that $\sum_{X \in \mathcal{A}} h(X) \leq \sum_{j \in E} e_j(\mathcal{A}) = \sum_{j \in E} e_j(\mathcal{R}) = \sum_{X \in \mathcal{R}} \varrho_r(X) = \sum_{X \in \mathcal{A}} \varrho_r(X) + \delta_r(A)$ and hence $-\delta_r(A) \leq \sum_{X \in \mathcal{A}} \varrho_r(X) - \sum_{X \in \mathcal{A}} h(X) = \sum_{X \in \mathcal{A}} b(X)$ from which (5.3) follows. •

The next result provides a simplified characterization for the orientation problem when h is crossing G-supermodular. Note that the old characterization (2.17) in Theorem 2.11 is equivalent to requiring (6.3) for all double-partitions of every subset A of V while in Theorem 6.4 inequality (6.3) is required only for tree-compositions.

THEOREM 6.4 *G has an orientation $\vec{G}$ satisfying (6.1) if and only if (6.3) holds for every subset $A \subseteq V$ and for every tree-composition $\mathcal{A}$ of A.*

Proof. Necessity. Suppose that there is an orientation $\vec{G}$ of G satisfying (6.1) and let ϱ denote its in-degree function. Let $\mathcal{A}$ be a tree-composition. Using (6.2) we have $\sum(h(X) : X \in \mathcal{A}) \leq \sum(\varrho(X) : X \in \mathcal{A}) \leq e_G(\mathcal{A})$ and (6.3) follows.

Sufficiency. Let Q be a submodular flow polyhedron associated with the orientation problem. By Lemma 6.3 and Theorem 5.4, Q contains a $0-1$ point and hence the required orientation exists. •

Proof *of Theorem 2.5.* By the correspondence between orientations and submodular flows the theorem follows from the second half of Claim 6.2 and from Theorem 5.7. •

We remark that there is a much shorter proof of Theorem 2.5 in [Frank 1993] (relying on Fujishige's [1984] characterization for non-emptyness of the base polyhedron of a crossing submodular function.) The point in the present derivation was to show that Theorem 2.5 does follow from the general approach.

Proof *of Theorem 2.9.* It follows from $f \equiv 0, g \equiv 1$ that $d_{g-f}(A,B) = d_G(A,B) > 0$. By Lemma 6.1, properties (2.15) and (5.4) are equivalent. Hence Theorem 5.5 applies and implies the existence of the required orientation. •

Proof *of Theorem 2.10.* Define h as follows. Let $h(\emptyset) := h(V) := 0$. Let $h(X) := k - \varrho_{\vec{D}}(X)$ if $X \subseteq V - s$ and $|X| \geq 2$. Let $h(X) := 1 - \varrho_{\vec{D}}(X)$ if $s \in X \subset V$ and $|X| \geq 2$. Let $h(X) := \max(k - \varrho_{\vec{D}}(v), f(v))$ if $X = \{v\}$ for some $v \in V - s$. Let $h(s) := \max(1 - \varrho_{\vec{D}}(s), f(s))$.

Since the in-degree function is submodular, h is easily seen to be crossing G-supermodular (even crossing supermodular).

CLAIM *h satisfies (2.15).*

Proof. Suppose that $A \cup B = V, A \cap B \neq \emptyset$ and $d_G(A,B) > 0$. Then $|A| \geq 2, |B| \geq 2$ and at least one of the two sets A and B, say A, contains s. Let $k' := 1$ if $s \in B$ and $k' := k$ if $s \notin B$. Note that $s \in B$ if and only if $s \in A \cap B$. Hence $h(A \cap B) \geq k'$ and $h(A) + h(B) = 1 - \varrho_{\vec{D}}(A) + k' - \varrho_{\vec{D}}(B) \leq 1 + k' - \varrho_{\vec{D}}(A \cap B) \leq d_G(A,B) + h(A \cap B)$, as required. •

By the claim we may apply Theorem 2.9. We have to show that (2.11) is satisfied. Suppose indirectly that a sub-partition $\mathcal{P}$ of V violates (2.11). Let $V_1, \ldots, V_t$ denote those members X of $\mathcal{P}$ for which either $|X| > 2$ or $|X| = 1$ and $h(X) > f(v)$ where $X = \{v\}$. Then each other member of $\mathcal{P}$ is a singleton. Let Z denote the set of these singletons and let $\mathcal{F} := \{Z, V_1, \ldots, V_t\}$. An easy calculation shows that $\mathcal{F}$ violates (2.16). • •

Proof *of Theorem 1.2.* We show here only the sufficiency of the conditions on Theorem 1.2, so suppose them to hold. For the special case $k = 1$ the requirement for the orientation in Theorem 2.10 is the same as that in Theorem 1.2. When $k = 1$, condition (2.16) transforms into:

$$e_{\mathcal{F}} + i_G(Z) \geq f(Z) + \sum_i (1 - \varrho_{\vec{D}}(V_i)) \tag{6.4}$$

holds whenever $\mathcal{F} = \{Z, V_1, \dots, V_t\}$ is a sub-partition of V where only Z may be empty. Therefore Theorem 1.2 will follow from Theorem 2.10 once we show that the conditions in Theorem 1.2 imply (6.4). Suppose that (6.4) is violated by a partition $\mathcal{F} = \{Z, V_1, \dots, V_t\}$, that is,

$$e_{\mathcal{F}} + i_G(Z) < f(Z) + \sum_i (1 - \varrho_{\vec{D}}(V_i)) \tag{6.5}$$

and assume that $|\mathcal{F}|$ is minimal. Then obviously $1 - \varrho_{\vec{D}}(V_i) > 0$ since otherwise by leaving out V_i from $\mathcal{F}$ we would obtain a smaller sub-partition violating (6.4). Therefore $\varrho_{\vec{D}}(V_i) = 0$ for each i and hence (6.5) is equivalent to

$$e_{\mathcal{F}} + i_G(Z) < f(Z) + |\mathcal{F}| - 1 \tag{6.6}$$

and $\mathcal{F}$ is a minimal sub-partition satisfying (6.6).

It is not possible that $t = 2$ and $V_1 \cup V_2 = V$ because this would imply $Z = \emptyset$ and $e_{\mathcal{F}} = d(V_1)(= (d(V_2))$ and in this case (2.18) transforms to $d_G(V_1) < 2$ which, along with $\varrho_{\vec{D}}(V_1) = \varrho_{\vec{D}}(V_2) = 0$, contradicts the assumption that M is traversable and has no undirected cut edge.

Next we show that there is no edge of G connecting a V_j to $R := V - (Z \cup \cup_i V_i)$. Indeed, if there were such an edge, then $(*)$ $e_{\mathcal{F}'} + i_G(Z) \leq e_{\mathcal{F}} - 1 + i_G(Z) < f(Z) + (|\mathcal{F}| - 1) - 1 = f(Z) + |\mathcal{F}'| - 1$ holds for the sub-partition $\mathcal{F}' := \mathcal{F} - \{V_i\}$ contradicting the minimality of $\mathcal{F}$.

Finally, we claim that no edge of G connects two members X, Y of $\mathcal{F} - \{Z\}$, that is, $d_G(X, Y) = 0$. Suppose to the contrary that $d_G(X, Y) \geq 1$. We have seen that $X \cup Y \neq V$. Let $\mathcal{F}' := \mathcal{F} - \{X, Y\} \cup \{X \cup Y\}$. Then $(*)$ holds for this $\mathcal{F}'$ contradicting again the minimality of $\mathcal{F}$.

Since no directed edge of M enters any V_i each V_i includes a di-component of $M - Z$ and hence $c(M, Z) \geq |\mathcal{F}| - 1$. Hence we have $e_G(Z) = e_{\mathcal{F}} + i_G(Z) < f(Z) + |\mathcal{F}| - 1 \leq f(Z) + c(M, Z)$, contradicting (1.3). • •

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