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ON INTEGER MULTIFLOW MAXIMIZATION*

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Abstract. Generalizing the two-commodity flow theorem of Rothschild and Whinston [*Oper. Res.*, 14 (1966), pp. 377–387] and the multiflow theorem of Lovász [*Acta Mat. Akad. Sci. Hungaricae*, 28 (1976), pp. 129–138] and Cherkasky [*Ekonom.-Mat. Metody*, 13 (1977), pp. 143–151], Karzanov and Lomonosov [*Mathematical Programming*, O. I. Larichev, ed., Institute for System Studies, 1978, pp. 59–66] in 1978 proved a min-max theorem on maximum multiflows. Their original proof is quite long and technical and relies on earlier investigations into metrics. The main purpose of the present paper is to provide a relatively simple proof of this theorem. Our proof relies on the locking theorem, which is another result of Karzanov and Lomonosov, and the polymatroid intersection theorem of Edmonds [*Combinatorial Structures and Their Applications*, R. Guy, H. Hanani, N. Sauer, and J. Schönheim, eds., Gordon and Breach, 1970, pp. 69–87]. For completeness, we also provide a simplified proof of the locking theorem. Finally, we introduce the notion of a node demand problem and, as another application of the locking theorem, we derive a feasibility theorem concerning it.

The presented approach gives rise to (combinatorial) polynomial-time algorithms.

Key words. multiflow, polymatroid, network flows, locking, node demands

AMS subject classifications. 05C38, 90B10, 90C27, 05C85

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1. Introduction. Let $G = (V, E)$ and $H = (T, F)$ be two undirected graphs so that $T \subseteq V$. We call a path of G *H-admissible* if it connects two nodes x, y of T so that $xy \in F$. G will be called a *supply graph*, H a *demand graph*, and the elements of T *terminals* while the other elements of V are called *inner nodes*. The *maximization problem* consists of finding a maximum number of edge-disjoint H-admissible paths. If H consists of one edge, then Menger's theorem gives an answer.

In general, the problem is NP-complete even in the special case when G is Eulerian. (A graph is called *Eulerian* if the degree of every node is even.) There are, however, important special cases when the problem is tractable. Rothschild and Whinston [13] proved a max-flow-min-cut-type theorem when (G, T) is inner Eulerian and H consists of two edges. (We say that the pair (G, T) is *inner Eulerian* if the degree $d(v)$ is even for every inner node v.) Another result is due, independently, to Lovász [12] and Cherkasky [1]. They solved the maximization problem when H is a complete graph and G is inner Eulerian. In [8] Karzanov and Lomonosov found a common generalization of these two theorems. Their original proof is rather lengthy and technical and it is certainly much more difficult than those of the two special cases mentioned above. Details of these proofs were described in [4] and [10, 11]. Later Karzanov [6, 7] gave another proof which was based on the splitting-off technique and

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gave rise to a strongly polynomial solution algorithm. However, this latter proof was also rather complicated.

The main contribution of this paper is a relatively simple proof of the theorem of Karzanov and Lomonosov. The proof relies on two ingredients: the so-called locking theorem, which is another result of Karzanov and Lomonosov [8], and the polymatroid intersection theorem of Edmonds [2]. For completeness, we will also provide a simplified proof of the locking theorem. Since both of these ingredients can be solved by a (combinatorial) polynomial-time algorithm, the approach gives rise to an alternate strongly polynomial-time algorithm for the (capacitated) maximization problem in question which is faster than that in [6].

In what follows we do not distinguish between a one-element set $\{x\}$ and its only element x. For a set X and an element t let $X+t$ denote the union of X and t. For a vector $m : S \to \mathbf{R}$ we use the notation $m(X) := \sum(m(s) : s \in X)$. A family of pairwise disjoint nonempty subsets of a set S is called a *subpartition* of S. For two elements s, t a set X is called a *$t\bar{s}$-set* if $t \in X, s \notin X$. An integer-valued vector or function is called *even* if each of its values is an even integer. For a polyhedron P we use the notation $P/2 := \{x/2 : x \in P\}$.

For a graph $G = (V, E)$ the cut $[X, V-X]$ denotes the set of edges with precisely one end node in X. Its cardinality is denoted by $d(X)(= d(V-X))$. $d(X)$ is called the *degree function* of G. Let $d(X, Y)$ denote the number of edges with one in $X - Y$ and the other in $Y - X$. Let $\bar{d}(X, Y) := d(X \cap Y, V - (X \cup Y))$. It is easy to prove that d satisfies the following identities for every pair X, Y of subsets of V:

$$d(X) + d(Y) = d(X \cap Y) + d(X \cup Y) + 2d(X, Y), \tag{1.1}$$

$$d(X) + d(Y) = d(X - Y) + d(Y - X) + 2\bar{d}(X, Y). \tag{1.2}$$

Let A and B be two disjoint subsets of V. A path connecting an element of A and an element of B is called an *(A, B)-path.* A path connecting two distinct elements of A is called an *A-path.* $\lambda(A, B; G)$ or simply $\lambda(A, B)$ stands for the maximum number of edge-disjoint (A, B)-paths. By Menger's theorem $\lambda(A, B) = \min(d(X) : A \subseteq X \subseteq V - B)$.

One may consider a fractional version of the edge-disjoint paths problem. Let G and H be as before. By an *H-multiflow* or briefly *multiflow* x we mean a family $\{P_1, P_2, \ldots, P_k\}$ of paths of G along with nonnegative coefficients $\alpha_1, \alpha_2, \ldots, \alpha_k$ so that each P_i connects the end nodes of a demand edge. x is called *integer-valued* if each α_i is an integer.

If each P_i connects an element of A and an element of B (that is, when H is a complete bipartite graph with bipartition (A, B)), we speak of an *(A, B)-flow.* For an H-multiflow x let $x(e) := \sum(\alpha_i : P_i \text{ uses } e)$ $(e \in E)$ and $x(t) := \sum(\alpha_i : P_i \text{ ends at } t)$ $(t \in T)$. For a given capacity function $c : E \to \mathbf{R}_+$, x is called *c-admissible* if $x(e) \le c(e)$ for every $e \in E$.

2. The locking problem. Let $G = (V, E)$ be a graph and $T \subseteq V$ a subset of terminal nodes. For a subset $A \subseteq T$ the notation $\lambda(A, T - A; G)$ will be abbreviated by $\lambda(A; G)$ or by $\lambda(A)$ when no confusion can arise. Throughout the paper we assume that the current (G, T) is inner Eulerian.

Lovász [12] and Cherkasky [1] proved the following theorem.

THEOREM 2.1. *For an inner Eulerian pair (G, T) the maximum number of edge-disjoint T-paths is equal to $(\sum \lambda(t) : t \in T)/2$.*

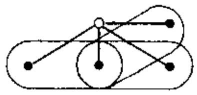


FIG. 1.

An equivalent formulation now follows.

THEOREM 2.1′. *Given an inner Eulerian pair* (G,T), *there is a family* $\mathcal{F}$ *of edge-disjoint* T*-paths in* G *so that* $\mathcal{F}$ *contains* $\lambda(t)$ *paths ending at* t *for each* $t \in T$.

In other words, there is a single family of edge-disjoint T-paths that includes maximum families of edge-disjoint $(t, T-t)$-paths simultaneously for all $t \in T$.

Karzanov and Lomonosov [8] extended this theorem. To formulate their result let us say that a family $\mathcal{F}$ of edge-disjoint T-paths *locks a subset* $A \subseteq T$ if $\mathcal{F}$ contains $\lambda(A)$ $(A, T-A)$-paths. Furthermore, we say that $\mathcal{F}$ *locks a family* $\mathcal{L}$ of subsets of T if $\mathcal{F}$ locks all members of $\mathcal{L}$.

Theorem 2.1′ asserts that there is a family $\mathcal{F}$ of paths that locks all singletons of T. Is it always possible to find a family of edge-disjoint T-paths that locks a specified family $\mathcal{L}$? The answer, in general, is no, as is shown by Figure 1. Here $\mathcal{L}$ consists of three pairwise crossing sets. (Two subsets X, Y of T are called *crossing* if none of $X-Y, Y-X, X\cap Y, T-(X\cup Y)$ is empty.)

Figure 1 indicates why it is natural to require $\mathcal{L}$ to be 3-cross-free. A family $\mathcal{L}$ of subsets of T is called 3*-cross-free* if it has no three pairwise crossing members.

LOCKING THEOREM 2.2 (see [8, 5, 10, 11]). *Let* (G,T) *be inner Eulerian and* $\mathcal{L}$ *a* 3*-cross-free family of subsets of* T. *Then there is a family of edge-disjoint* T*-paths that locks* $\mathcal{L}$.

A proof of a slightly weaker version was sketched in [8]. The present proof relies on an idea of splitting used previously in [5], but is technically simpler. *Splitting off* a pair of adjacent edges $e = st, f = sx$ of a graph G refers to an operation that replaces e and f by a new edge connecting x and t (this way we may introduce parallel edges between x and t). The resulting graph is denoted by G^{ef}.

Proof. We may assume that $T-A \in \mathcal{L}$ for each $A \in \mathcal{L}$ because for $A \in \mathcal{L}$ adding $T-A$ to $\mathcal{L}$ affects neither 3-cross-freeness nor lockability. Also assume that G is connected.

We proceed by induction on the number of edges incident to the elements of $V-T$. If this number is zero, then the statement is trivial. Therefore, there is an edge $e = st$ with $t \in T, s \notin T$. We are going to show that there is an edge $f = sx$ for which

$$\lambda(A;G) = \lambda(A;G^{ef}) \quad \text{for every } A \in \mathcal{L}. \tag{2.1}$$

From this the theorem follows since, by induction, there is a family $\mathcal{F}$ of T-paths of G^{ef} locking $\mathcal{L}$. If a path $P \in \mathcal{F}$ uses the new edge h of G^{ef} having arisen from the splitting of e, f, then revise $\mathcal{F}$ by replacing h in P by e and f. By (2.1) the revised $\mathcal{F}$ locks $\mathcal{L}$ in G.

CLAIM 1. *Suppose for* $X, Y \subseteq V$ *that* $X\cap T \subseteq Y \cap T$ *and that* $d(X) = \lambda(X\cap T), d(Y) = \lambda(Y\cap T)$. *Then* $d(X\cap Y) = \lambda(X\cap T), d(X\cup Y) = \lambda(Y\cap T)$ *and* $d(X,Y) = 0$.

Proof. Since $X\cap T \subseteq Y\cap T$ we have $(X\cap Y)\cap T = X\cap T$ and hence $d(X\cap Y) \geq \lambda(X\cap T)$. Analogously, $(X\cup Y)\cap T = Y\cap T$ and $d(X\cup Y) \geq \lambda(Y\cap T)$. Therefore, by (1.1), $\lambda(X\cap T)+\lambda(Y\cap T) = d(X)+d(Y) = d(X\cap Y)+d(X\cup Y)+2d(X,Y) \geq \lambda(X\cap T)+\lambda(Y\cap T)+2d(X,Y)$, from which the claim follows. □

Call a set $X \subseteq V$ *tight* if $X \cap T \in \mathcal{L}$ and $d(X) = \lambda(X \cap T)$. Since $\mathcal{L}$ is closed under complementation, $V - X$ is tight if X is tight. Because (G, T) is inner Eulerian, a pair of edges $e = st, f = sx$ will satisfy (2.1) precisely if

(2.2) there is no tight set X with $t, x \in X \subseteq V - s$.

CLAIM 2. *There are no three maximal tight $t\bar{s}$-sets.*

Proof. Let X, Y, Z be maximal tight $t\bar{s}$-sets. Since $\mathcal{L}$ is 3-cross-free, two of the three sets $X \cap T, Y \cap T, Z \cap T$, say $X \cap T$ and $Y \cap T$, are noncrossing.

Then either $X \cap T \subseteq Y \cap T$ or $Y \cap T \subseteq X \cap T$ or $T \subseteq X \cup Y$. In the first two cases Claim 1 implies that $X \cup Y$ is tight, contradicting the maximality of X and Y. In the last case, by applying Claim 1 to $X' = V - X$ and Y, we obtain that $d(X', Y) = 0$, contradicting the existence of edge st. □

Let S denote the set of neighbors of s.

CLAIM 3. *It is not possible to cover S by two tight $t\bar{s}$-sets.*

Proof. Suppose that $S \subseteq X \cup Y$, where X and Y are tight $t\bar{s}$-sets. Let $\alpha := d(s, X - Y), \beta := d(s, Y - X), \gamma := d(s, X \cap Y)$. By symmetry we may assume that $\alpha \geq \beta$. $(X + s) \cap T = X \cap T$ implies that $d(X + s) \geq \lambda(X \cap T)$. On the other hand, since γ is positive, we have $d(X + s) = d(X) - \alpha - \gamma + \beta < d(X) = \lambda(X \cap T)$, which is a contradiction. □

By Claims 2 and 3 there is an edge $f = sx$ satisfying (2.2), and then (2.1) holds; the proof of Locking Theorem 2.2 is complete.

Remark. One may be interested in other possible locking theorems when, rather than 3-cross-freeness, some other property is assumed for the family $\mathcal{L} \subseteq 2^T$ to be locked. On the negative side, Karzanov and Pevzner [9] showed that for every $\mathcal{L}$, including three pairwise crossing sets, there is a graph G and a subset T of its nodes so that (G, T) is inner Eulerian and there is no family of T-paths locking all members of $\mathcal{L}$. On the other hand, there are other locking theorems in which some restrictions are imposed on the relationship of G and the family $\mathcal{L}$. For example, let G be a planar Eulerian graph and let $T := \{t_1, \ldots, t_k\}$ denote the nodes of its outer face in the cyclic order. If we define $\mathcal{L}$ to consist of all subsets of T of form $\{t_i, \ldots, t_j\}$ $(1 \leq i \leq j \leq k)$ then, although $\mathcal{L}$ is not 3-cross-free when $k \geq 4$, the locking theorem holds. (This is a theorem equivalent, by planar dualization, to a result of Hurkens, Schrijver, and Tardos [3].

We will need a slight extension of Theorem 2.2. Let $m : T \to \mathbf{Z}$ be a nonnegative integer-valued function on T. A family $\mathcal{F}$ of edge-disjoint T-paths is called *m-independent* if every terminal $t \in T$ is the end of at most $m(t)$ members of $\mathcal{F}$. Let $\lambda_m(A)$ denote the maximum number of edge-disjoint m-independent $(A, T-A)$-paths. We say that a family $\mathcal{F}$ of edge-disjoint T-paths *m-locks a subset* $A \subseteq T$ if $\mathcal{F}$ is m-independent and contains $\lambda_m(A)$ $(A, T - A)$-paths. Furthermore, we say that $\mathcal{F}$ *m-locks a family* $\mathcal{L}$ of subsets of T if $\mathcal{F}$ m-locks all members of $\mathcal{L}$.

The following theorem is a straightforward consequence of Theorem 2.2 and will be used in the proof of Theorem 4.3.

THEOREM 2.3. *Let G be inner Eulerian and $\mathcal{L}$ a 3-cross-free family of subsets of T. Let $m : T \to \mathbf{Z}_+$ be a vector so that $m(t) + d(t)$ is even for $t \in T$. Then there is a family $\mathcal{F}$ of edge-disjoint T-paths that m-locks $\mathcal{L}$.*

Proof. Let G' be a graph arising from G by splitting every node $t \in T$ in the following way: add a new node t' along with $m(t)$ parallel edges between t and t' and replace each edge xt of G by xt'. The result immediately follows when Theorem 2.2 is applied to (G', T). □

3. Flows and polymatroids. A nonnegative set function $b : 2^T \to \mathbf{R}_+$ is called a *polymatroid function* if

1. $b(\emptyset) = 0$,
2. b is monotone increasing, i.e., $b(X) \geq b(Y)$ when $Y \subseteq X \subseteq T$,
3. b is submodular, i.e., $b(X) + b(Y) \geq b(X \cup Y) + b(X \cap Y)$ for $X, Y \subseteq T$.

The degree function d of a graph G satisfies properties 1 and 3 but typically not 2.

A polyhedron $P(b) := \{x \in \mathbf{R}^T, x \geq 0, x(A) \leq b(A) \text{ for every } A \subseteq T\}$ is called a *polymatroid*. It is called *integral* if every vertex of P is integer-valued.

The concept of a polymatroid was introduced by Edmonds [2]. He proved that a polymatroid uniquely determines its defining polymatroid function. Furthermore, a polymatroid is integral if and only if b is integer-valued.

For a polymatroid $P(b)$ the face $B(b) := \{x : x \in P, x(T) = b(T)\}$ of $P(b)$ is called the *basis polyhedron* and its elements are the *bases*. Edmonds also proved the following result.

THEOREM 3.1 (see [2]). *For an (integral) polymatroid $P(b)$ and an (integer-valued) vector $x \in P(b)$ there is an (integer-valued) basis y with $y \geq x$.*

The polymatroid intersection theorem of Edmonds states that the linear system of two polymatroids is totally dual integral (TDI). Here we need only the following consequence.

THEOREM 3.2 (see [2]). *For two polymatroid functions a and b defined on the power set of T*

$$\max(x(T) : x \in P(a) \cap P(b)) = \min(a(X) + b(T - X) : X \subseteq T).$$

Furthermore, if a and b are integer-valued, the maximum is attained by an integer vector.

It follows that there is a vector x in $P(a) \cap P(b)$ and a bipartition $\{A, B\}$ of T so that $x(A) = a(A)$ and $x(B) = b(B)$, and if a and b are integral-valued, then so is x.

Let $G = (V, E)$ be a graph endowed with a capacity function $c : E \to \mathbf{R}_+$. Let T be a subset of nodes and $A \subset T, B := T - A$. Define $P_A := \{m \in \mathbf{R}_+^{\mathbf{A}} :$ there is a c-admissible (A, B)-flow x for which $x(v) = m(v)$ for every $v \in A\}$.

For $X \subseteq A$ let $f_A(X) := \min(\delta_c(Y) : Y \subseteq V, X \subseteq Y \cap T \subseteq A)$. Here $\delta_c(Y) := \sum(c(e) : e \in [Y, V - Y])$. Clearly, f_A is submodular and monotone increasing. By a multiterminal version of the max-flow min-cut (MFMC) theorem a vector $m \in R_+^A$ belongs to P_A if and only if $m(X) \leq f_A(X)$. Therefore, P_A is a polymatroid. Furthermore, if c and m are integer-valued, then there is a c-admissible integer-valued (A, B)-flow x for which $x(v) = m(v)$ for every $v \in A$.

Let $G = (V, E)$ be an Eulerian graph and T a subset of nodes. Define c by $c(e) = 1$ for every $e \in E$. Let $\mathcal{T} := \{T_1, T_k, \ldots, T_k\}$ be a partition of T and $\lambda_i := \lambda(T_i, T - T_i)$. Let P denote the direct sum of polymatroids $P_{T_1}, P_{T_2}, \ldots, P_{T_k}$.

LEMMA 3.3. *Let q be an integer basis of P. Then there is a family $\mathcal{F}$ of edge-disjoint T-paths connecting distinct members of $\mathcal{T}$ so that each $t \in T$ is the end point of exactly $q(t)$ paths of $\mathcal{F}$.*

Proof. For each $T_i \in \mathcal{T}$ let X_i be a minimal subset of V for which $X_i \cap T = T_i$ and $d(X_i) = \lambda_i$. We claim that these sets are disjoint. If, indirectly, $X_i \cap X_j \neq \emptyset$ for some $1 \leq i < j \leq k$, then (1.2) implies $\lambda_i + \lambda_j \leq d(X_i - X_j) + d(X_j - X_i) \leq d(X_i) + d(X_j) = \lambda_i + \lambda_j$. Hence $\lambda_i = d(X_i - X_j)$, contradicting the minimality of X_i.

We claim that there is a family $\mathcal{F}_0$ of edge-disjoint paths in G connecting distinct X_i's and not using edges induced by any X_i so that $\mathcal{F}_0$ contains $\lambda_i = d(X_i)$ paths ending in X_i for each i $(1 \leq i \leq k)$. Indeed, apply Theorem 2.1′ to the pair (G', T'), where the graph G' arises from G by contracting each X_i into a node denoted by t_i and $T' := \{t_1, \dots, t_k\}$.

Since q is a basis of P, for each T_i there is a family $\mathcal{F}'_i$ of $q(T_i)$ $(= \lambda_i = d(X_i))$ edge-disjoint paths in G connecting T_i and $T - T_i$, so that each $t \in T_i$ is the end node of $q(t)$ members of $\mathcal{F}'_i$. For each member of $\mathcal{F}'_i$ erase the edges outside X_i and denote by $\mathcal{F}_i$ the family of the resulting paths. By glueing together the paths in $\mathcal{F}_0$ and the paths in $\mathcal{F}_i$ $(i = 1, \dots, k)$ we obtain a family $\mathcal{F}$ of paths satisfying the requirements. □

LEMMA 3.4. *Let q be an integer basis of P and m an integer vector for which $m \geq q$. Then an m-independent family $\mathcal{F}$ of T-paths that m-locks $\mathcal{T}$ contains at least $q(T)/2$ paths connecting distinct members of $\mathcal{T}$.*

Proof. By Lemma 3.3, $\lambda_q(T_i) = q(T_i)$. The assumption $m \geq q$ implies that $\lambda_m \geq \lambda_q$. Since $\mathcal{F}$ m-locks $\mathcal{T}$, there are $\lambda_m(T_i) \geq \lambda_q(T_i) = q(T_i)$ paths in $\mathcal{F}$ connecting T_i and $T - T_i$ for each $i = 1, \dots, k$, from which the lemma follows. □

Remark. Since q is a basis, $\mathcal{F}$ contains at most $q(T_i)$ $(T_i, T - T_i)$-paths, and therefore $\mathcal{F}$ contains at most $q(T)/2$ paths connecting distinct members of $\mathcal{T}$. That is, the number of such paths in $\mathcal{F}$ is precisely $q(T)/2$, but we will not need this fact.

Let $\mathcal{A}$ and $\mathcal{B}$ be two partitions of T and let $\mathcal{L} := \mathcal{A} \cup \mathcal{B}$. Let H be a demand graph on T so that uv is an edge of H if and only if no $X \in \mathcal{L}$ includes both u and v.

For $A_i \in \mathcal{A}$ let $a_i(X)$ $(X \subseteq A_i)$ be a set function defined by $a_i(X) := \lambda(X, T - A_i)$. We saw above that a_i is a polymatroid function. Define b_j analogously for $\mathcal{B}$. For $X \subseteq T$ let

$$a(X) := \sum a_i(X \cap A_i) \quad \text{and} \quad b(X) := \sum b_j(X \cap B_j). \tag{3.1}$$

Then a and b are polymatroid functions. Let $P(a)$ and $P(b)$ be the polymatroids defined by a and b, respectively.

LEMMA 3.5. *Let m' be an arbitrary even vector in $P(a) \cap P(b)$ and $h := m'(T)/2$. Then there are h edge-disjoint H-admissible paths.*

Proof. Since G is Eulerian, $P(a/2)(= P(a)/2)$ is an integral polymatroid. By applying Theorem 3.1 to $P(a/2)$ and to $x := m'/2$ we find that there is an even basis m_a of $P(a)$ so that $m_a \geq m'$. Analogously, there is an even basis m_b of $P(b)$ so that $m_b \geq m'$. Define a vector m by $m(t) := \max(m_a(t), m_b(t))$ for $t \in T$. Clearly, m is even and $m_a(t) + m_b(t) \geq m(t) + m'(t)$ for each $t \in T$. Hence

$$m_a(T) + m_b(T) - m(T) \geq m'(T).$$

Since $\mathcal{L}$ is 3-cross-free, we can apply Theorem 2.3. Let $\mathcal{F}$ denote the family of m-independent T-paths provided by the theorem. Then $|\mathcal{F}| \leq m(T)/2$.

We are going to prove that the number h' of H-admissible paths in $\mathcal{F}$ is at least h. (Note that a path is not H-admissible precisely if it connects two nodes belonging to the same member of $\mathcal{L}$.)

By applying Lemma 3.4 with the choice $\mathcal{T} := \mathcal{A}$, $P := P(a), q := m_a$, we find that there are at most $|\mathcal{F}| - m_a(T)/2$ paths in $\mathcal{F}$ having both end nodes in the same member of $\mathcal{A}$. Analogously, there are at most $|\mathcal{F}| - m_b(T)/2$ paths in $\mathcal{F}$ having both end nodes in the same member of $\mathcal{B}$.

Hence $h' \geq |\mathcal{F}| - (|\mathcal{F}| - m_a(T)/2) - (|\mathcal{F}| - m_b(T)/2) = (m_a(T) + m_b(T))/2 - |\mathcal{F}| \geq (m_a(T) + m_b(T) - m(T))/2 \geq m'(T)/2 = h$, as required. □

(Note that an element $t \in T$ need not be the end node of exactly $m'(t)$ members of the family assured by Lemma 3.5. For further comments, see the beginning of section 6.)

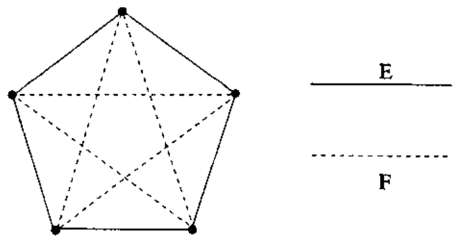


FIG. 2.

4. Maximization. Let $G = (V, E)$ be a supply graph and $H = (T, F)$ a demand graph so that $T \subseteq V$ and $E \cap F = \emptyset$. Throughout this section we assume that the pair (G, T) is inner Eulerian; that is, $d(v)$ is even for every $v \in V - T$, where d stands for the degree function of G.

The *maximization form* of the edge-disjoint paths problem consists of finding a maximum number $\mu = \mu(G, H)$ of edge-disjoint H-admissible paths. We can easily get an upper bound on μ. Let us call a subpartition $\{X_1, X_2, \ldots, X_k\}$ of V *admissible* if $T \subseteq \cup X_i$ and each $X_i \cap T$ is stable in $H (i = 1, \ldots, k)$. Clearly,

$$\mu(G, H) \leq \sum d(X_i)/2. \tag{4.1}$$

The value $\sum d(X_i)/2$ will be called the *value of the subpartition.* Let $\tau = \tau(G, H)$ denote the minimum value of an admissible subpartition. We have $\mu \leq \tau$.

Figure 2 shows that we do not have equality, in general.

There are two known special cases when equality holds. Theorem 2.1 shows that this is the case if H is a complete graph on T. Reformulating Theorem 2.1, we have the following theorem.

THEOREM 4.1. *Suppose that (G, T) is inner Eulerian and the demand graph H is complete. Then $\mu(G, H) = \tau(G, H)$.*

Another special case for which $\mu = \tau$ is when H consists of two edges; that is, $H = 2K_2$.

THEOREM 4.2. *Suppose that (G, T) is inner Eulerian and H consists of two edges $s_i t_i$ $(i = 1, 2)$. Then $\mu(G, H) = \tau(G, H)$.*

This is a theorem of Rothschild and Whinston. Actually, they proved it in the following simpler form.

THEOREM 4.2′ (see [13]). *Suppose that (G, T) is inner Eulerian and H consists of two edges $s_i t_i (i = 1, 2)$. Then $\mu(G, H)$ is the minimum cardinality τ' of a cut $[X, V - X]$ of G for which $\{s_i, t_i\} \cap X = 1$ $(i = 1, 2)$.*

(The equivalence of the two forms, that is, $\tau = \tau'$, may be proven as follows. Since a cut $[X, V - X]$ for which $\{s_i, t_i\} \cap X = 1$ $(i = 1, 2)$ provides an admissible partition of special form, clearly $\tau' \geq \tau$. To see the other direction let $\mathcal{P} := \{X_1, \ldots, X_k\}$ be a minimal admissible subpartition of G for which k is minimum. Then $\tau = \sum d(X_i)/2$, and since $|T| \leq 4$, we have $2 \leq k \leq 4$.

If $k = 2$, then both X_1 and X_2 contain exactly two terminal nodes which are not connected in H. Furthermore, if say $d(X_1) \leq d(X_2)$, then $\{X_1, V - X_1\}$ would also be an admissible partition whose value is not bigger than that of $\{X_1, X_2\}$. Therefore, $\{X_1, V - X_1\}$ is another optimal admissible partition and hence $\tau' = \tau$.

If $k \geq 3$, then there are two members of $\mathcal{P}$, say X_1 and X_2, such that each contains one terminal node and these two terminal nodes, say s_1 and s_2, are not connected in H. But now by replacing X_1 and X_2 by $X_1 \cup X_2$ we obtain another minimal admissible subpartition, contradicting the minimum choice of k. $\square$)

Let us call a graph $H = (T, F)$ *bistable* if there are two partitions $\mathcal{A}$ and $\mathcal{B}$ of T such that for $x, y \in T$ xy is an edge of H precisely if x and y belong to different parts of $\mathcal{A}$ and different parts of $\mathcal{B}$. It is easily seen that a graph is bistable if and only if its complement is the line graph of a bipartite graph. (It can also be shown that bistable graphs are those for which the family of maximal stable sets of H can be partitioned into two parts, each consisting of disjoint sets.)

Clearly, a clique or, more generally, a complete k-partite graph, is bistable and $2K_2$ is also bistable. Therefore, Theorems 2.1 and 2.2 are special cases of the following.

THEOREM 4.3 (see [6; 10, 11]). *Suppose that (G,T) is inner Eulerian and $H = (T, F)$ is bistable. Then $\mu(G,H) = \tau(G,H)$.*

A proof of a slightly weaker, half-integral version was previously sketched in [8]. The reader may feel that bistable demand graphs form a rather peculiar class of graphs and there may be larger, more natural classes of graphs for which $\mu = \tau$ holds. Karzanov and Pevzner [9], however, showed that if $H = (T, F)$ is not bistable and contains no isolated nodes, then there is a supply graph $G = (V, E)$, with $T \subseteq V$ and (G,T) inner Eulerian, so that $\mu(G,H) < \tau(G,H)$.

In section 6 we will outline our original plan of proof, which was intended to use only Theorem 3.2, and we will point out why that attempt failed. This perhaps will help the reader understand how we were led to invoke the locking theorem in the proof below.

Proof. By (4.1) we have $\mu(G,H) \leq \tau(G,H)$. To see the other direction, first we prove that the theorem follows from its special case when the graph is completely Eulerian. So suppose the theorem is true for (G', H') whenever G' is Eulerian and we want to prove it for (G,H) when G is inner Eulerian. Let K denote the set of nodes of G with odd degree. Since (G,T) is inner Eulerian, $K \subseteq T$. If K is empty, we are done. If not, for a new node t, let $T' := T + t$ and $V' := V + t$. Let $E' := E \cup \{xt : x \in K\}$ and $F' := F \cup \{xt : x \in T\}$. Then $G' := (V', E')$ is Eulerian and $H' := (T', F')$ is bistable. Let μ' and τ' denote, respectively, the maximum and minimum in question concerning (G', H'). By the assumption $\mu' = \tau'$.

Obviously, there is an optimal solution to the maximization problem concerning (G', H') in which every edge xt, $x \in K$, is itself a path in the solution. Thus we have $\mu' = \mu + |K|$. Furthermore, let $\mathcal{M}'$ be an optimal admissible subpartition for (G', H') so that $t \in X \in \mathcal{M}$. Since every edge xt, $x \in T$, belongs to H', $X \cap T = \{t\}$. Hence $\mathcal{M} - \{X\}$ is an admissible subpartition for (G,H), and therefore $\tau \leq \tau' - |K|$. We can conclude that $\mu = \mu' - |K| = \tau' - |K| \geq \tau$, as required.

Let $\mathcal{A}$ and $\mathcal{B}$ be the two partitions of T defining the bistable graph H. Note that each stable set of H is a subset of some $S \in \mathcal{A} \cup \mathcal{B}$. Let a and b be defined by (3.1). Since $P(a)/2$ and $P(b)/2$ are integral polymatroids, by Theorem 3.2 there exist an even vector m' in $P(a) \cap P(b)$ and a bipartition $\{A, B\}$ of T so that

$$m'(A) = a(A) \quad \text{and} \quad m'(B) = b(B). \tag{4.2}$$

Hence we have $m'(T) = a(A) + b(B)$. By Lemma 3.5 there are $m'(T)/2$ edge-disjoint H-admissible paths in G. Thus the proof will be complete if we find an admissible subpartition of value $(a(A) + b(B))/2$. To this end let us assume that A is a maximal subset of T for which A and $B := T - A$ satisfy (4.2). We claim that

$$a(A + t) > a(A) \quad \text{for every element } t \in B. \tag{4.3}$$

Indeed, if we have $a(A+t) = a(A)$ for some element $t \in B$, then $a(A) = m'(A) \leq m'(A+t) \leq a(A+t) = a(A)$, from which $m'(A+t) = a(A+t)$ and $m'(t) = 0$. Furthermore, $b(B) = m'(B) = m'(B-t) \leq b(B-t) \leq b(B)$, and hence $m'(B-t) = b(B-t)$; that is, the bipartition $\{A+t, B-t\}$ of T would also satisfy (4.2), contradicting the maximal choice of A. (Note that, because of this choice of A and B, the role of $\mathcal{A}$ and $\mathcal{B}$ will not be fully symmetric.)

For each $A_i \in \mathcal{A}$ for which $A \cap A_i$ is nonempty there exists a set $X_i \subseteq V$ for which $A_i \cap A \subseteq X_i \cap T \subseteq A_i$ and $d(X_i) = a(A_i \cap A) = m'(A_i \cap A)$. Here the last equality follows from (4.2) and the definition of a. Analogously, for each $B_j \in \mathcal{B}$ for which $B \cap B_j$ is nonempty there exists a set Y_j for which $B_j \cap B \subseteq Y_j \cap T \subseteq B_j$ and $d(Y_j) = b(B_j \cap B) = m'(B_j \cap B)$. Assume that both X_i and Y_j are chosen minimal and let $\mathcal{K} := \{X_i : A_i \in \mathcal{A},\ A_i \cap A \text{ nonempty}\} \cup \{Y_j : B_j \in \mathcal{B},\ B_j \cap B \text{ nonempty}\}$.

LEMMA 4.4. *$\mathcal{K}$ is an admissible subpartition of value* $(a(A) + b(B))/2$.

Proof. Clearly, each element of T belongs to at least one member of $\mathcal{K}$, and we show that no more than one. That is, we claim that

$$X_i \cap T \subseteq A \tag{4.4a}$$

and

$$Y_j \cap T \subseteq B. \tag{4.4b}$$

We have $m'(A_i \cap A) = a(A_i \cap A) = d(X_i) \geq a(A_i \cap X_i) \geq m'(A_i \cap X_i) \geq m'(A_i \cap A)$, and hence $a(A_i \cap A) = a(A_i \cap X_i)$. Hence (4.4a) must hold, for otherwise there is an element $t \in (X_i \cap T) - A$ and t would violate (4.3).

Also, $m'(B_j \cap B) = b(B_j \cap B) = d(Y_j) \geq b(B_j \cap Y_j) \geq m'(B_j \cap Y_j) \geq m'(B_j \cap B)$, and hence $m'(t) = 0$ for every $t \in Y_j \cap A$. We have $m'(X_i \cap A) + m'(Y_j \cap B) = m'(A_i \cap A) + m'(B_j \cap B) = d(X_i) + d(Y_j) \geq d(X_i - Y_j) + d(Y_j - X_i) \geq a((X_i - Y_j) \cap A) + b((Y_j - X_i) \cap B) \geq m'((X_i - Y_j) \cap A) + m'((Y_j - X_i) \cap B) = m'(X_i \cap A) + m'(Y_j \cap B)$. Hence $d(Y_j') = b(Y_j' \cap B) = m'(Y_j' \cap B)$ holds for $Y_j' := Y_j - X_i$. Therefore, if (4.4b) is not true and there is an element $t \in (Y_j \cap T) - B$ which belongs to, say X_i, then Y_j' is a proper subset of Y_j, contradicting the minimal choice of Y_j. Hence the proof of (4.4) is complete.

We claim that $\mathcal{K}$ is a subpartition. Assume to the contrary that $L \cap K \neq \emptyset$ for some $K, L \in \mathcal{K}$. By the definition of $\mathcal{K}$ and by (4.4) we have $L \cap K \cap T = \emptyset$. The minimal choice of the members of $\mathcal{K}$ implies that $d(K) < d(K - L)$. But then $d(K) + d(L) \geq d(K - L) + d(L - K) > d(K) + d(L)$, which is a contradiction.

By its definition, $\mathcal{K}$ is admissible and its value is $(\sum_i d(X_i) + \sum_j d(Y_j))/2 = (\sum_i a(A \cap A_i) + \sum_j b(B \cap B_j))/2 = m'(T)/2$, as required. □

By Lemmata 3.5 and 4.4 and by (4.2) we have $\mu \geq m'(T)/2 = (a(A) + b(B))/2 \geq \tau$, and the proof of Theorem 4.3 is complete.

5. Algorithmic aspects. In this section we briefly outline how the proof above gives rise to a strongly polynomial (combinatorial) algorithm in the capacitated case. (Informally, a polynomial-time algorithm is *strongly polynomial* if the number of steps does not depend on the magnitude of the occurring capacities and costs.)

The input of the algorithm consists of two graphs $G = (V, E)$ and $H = (T, F)$, where $T \subseteq V$. G is endowed with a nonnegative rational capacity function $c : E \Rightarrow \mathbf{Q}_+$. We assume that $H = (T, F)$ is given by two partitions $\mathcal{A} = \{A_1, A_2, \ldots, A_h\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_k\}$ of T so that $xy \in F$ if and only if each A_i and each B_j contains at most one of x and y. (Note that if a graph H is given by its incidence matrix, one can test H efficiently for bistability. Namely, decide first by enumeration whether there are more than $2|T|$ maximal stable sets of H. If the answer is yes, then H is not bistable. If the answer is no, then H is bistable if and only if the intersection graph of the maximal stable sets is bipartite.)

The output of the algorithm consists of a c-admissible H-multiflow x, so that $\sum(x(t) : t \in T) = \sum(\delta_c(Z) : Z \in \mathcal{K})$, and an admissible subpartition $\mathcal{K} = \{Z_1, Z_2, \ldots, Z_t\}$ of V. Moreover, if c is integer-valued and *Eulerian* in the sense that $\delta_c(v)$ is even for every node $v \in V$, then the output x is integer-valued as well.

Actually, we will assume that c is integer-valued and Eulerian. If this is not the case, one can multiply through the capacities by $2N$, where N denotes the least common denominator of the capacities. If c is inner Eulerian, we can apply the reduction described in section 4 to obtain a completely Eulerian case.

First, we remark that the proof of Theorem 2.2 immediately provides a polynomial-time algorithm for the set system $\mathcal{L}= \mathcal{A} \cup \mathcal{B}$ when c is identically 1. It is not difficult to show that, for general integer-valued Eulerian c, if in every step one splits off as much capacity as possible, then the algorithm is strongly polynomial (cf. [5]). In what follows we comment on the use of the polymatroid intersection algorithm to construct an even vector m' and an admissible subpartition occurring in the proof of Theorem 4.3.

For disjoint sets $X, Y \subseteq V$ let $\lambda_c(X, Y)$ denote the value of a flow between X and Y. With the help of a MFMC computation $\lambda_c(X, Y)$ can be computed in (strongly) polynomial time.

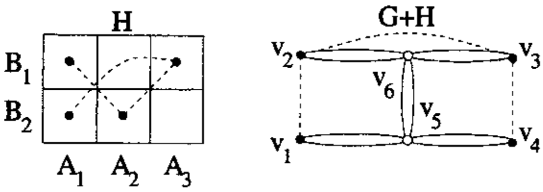
For $A_i \in \mathcal{A}$ let $a_i(X)$ $(X \subseteq A_i)$ be a set function defined by $a_i(X) := \lambda_c(X, T - A_i)$. Define b_j analogously. For $X \subseteq T$ let $a(X) := \sum a_i(X \cap A_i)$ and $b(X) := \sum b_j(X \cap B_j)$. Let $P(a)$ and $P(b)$ be the polymatroids defined by a and b. It is known from polymatroid theory that $P(a/2) = P(a)/2$ (and $P(b/2) = P(b)/2$). Since c is Eulerian, both $a/2$ and $b/2$ are integer-valued, and hence $P(a)/2$ and $P(b)/2$ are integral polymatroids. Therefore, if z is an integer-valued vector in $P(a/2) \cap P(b/2)$ for which $z(V)$ is maximum, then $m' := 2z$ is an even vector in $P(a) \cap P(b)$ for which $m'(V)$ is maximum. By Theorem 3.2 there is a bipartition $\{A, B\}$ of T so that $z(A) = a(A)/2$ and $z(B) = b(B)/2$ holds. Hence $m'(A) = a(A)$ and $m'(B) = b(B)$.

There is a (combinatorial) strongly polynomial algorithm, due to Schönsleben [14], for computing z (and hence m') and $\{A, B\}$. This algorithm works if an oracle is available to minimize $a(A) - z(A)$ and $b(A) - z(A)$ over $A \subseteq T$, where $z : T \to \mathbf{Q}$ is a vector. In our case this oracle can indeed be constructed by invoking the MFMC algorithm, and this way one obtains a purely combinatorial strongly polynomial algorithm for computing m' and A, B satisfying (4.2). Using the proofs of Claims 1 and 2 in the proof of Theorem 4.3, one may compute in strongly polynomial time an integer-valued maximum multiflow and an admissible subpartition of minimum value.

Karzanov [6] described a more direct way to compute m' and a minimal admissible subpartition. His method consists of one MFMC computation on an appropriately defined auxiliary digraph on $|V||T|$ nodes, and its complexity is $O(\varphi(|T||V|))$, where $\varphi(n)$ denotes the complexity of an MFMC computation on a network with n nodes.

Next, the even basis m_a of $P(a)$ (respectively, $m_b \in P(b)$) defined in the proof of Theorem 4.3 can be constructed by $|\mathcal{A}|$ (respectively, $|\mathcal{B}|$) MFMC computations. Thus, vector m defined in (4.4) can be computed from m' in $O(|T|\varphi(|V|))$ steps.

The m-locking problem can be solved by applying at most $O(|V|)$ splitting-off operations at every node $v \in V - T$; each operation consists of finding $|\mathcal{A}| + |\mathcal{B}|$ maximum flows in G. This requires $O(|V|^2(|\mathcal{A}| + |\mathcal{B}|)\varphi(|V|))$ or $O(|V|^2|T|\varphi(|V|))$



$m' = (2,2,2,0) \in P(a) \cap P(b)$
$A = \{v_1, v_2, v_3, v_4\}, B = \emptyset$
$m = (2,2,2,2)$
$P_1 = (v_1, v_5, v_6, v_2)$ $P_2 = (v_2, v_6, v_3)$ $P_3 = (v_4, v_5, v_6, v_3)$ $P_4 = (v_1, v_5, v_4)$

FIG. 3.

operations.

Thus, the overall complexity of the algorithm is $O(|V|^2|T|\varphi(|V|) + \varphi(|T||V|))$. If one uses an MFMC algorithm of complexity $\varphi(n) = O(n^3)$, this gives an $O(|V|^5|T|)$ upper bound for the running time of the algorithm (to be compared with the complexity $O(|V|^6|T|^3)$ of the algorithm in [6]).

6. Node demand problems. The reader might have a feeling that invoking the locking theorem in the proof above contains a seemingly unnecessary twist. In fact, we originally tried to prove Theorem 4.3 by using the following more natural and direct approach, but Figure 3 will show why our attempt failed.

Recall that the polymatroid intersection theorem ensured the existence of a maximum even vector m' in $P(a) \cap P(b)$ for which $m'(T)/2$ is precisely $\tau(G,H)$. Theorem 4.3 would follow if there existed a system of H-feasible paths so that each $t \in T$ is the end node of precisely $m'(t)$ of them. Unfortunately, such a system need not always exist, as is shown by the figure.

Demand graph H is defined by the partitions $\mathcal{A} := \{\{v_1, v_4\}, \{v_2\}, \{v_3\}\}$ and $\mathcal{B} := \{\{v_1, v_3\}, \{v_2, v_4\}\}$. Here $\{\{v_1, v_4, v_5\}, \{v_2\}, \{v_3\}\}$ is an admissible subpartition of value 3; that is, the maximum $\mu(G,H)$ is at most 3. On the other hand, there are three H-admissible edge-disjoint paths in G, namely, $P_1 := (v_1, v_5, v_6, v_2), P_2 := (v_2, v_6, v_3), P_3 := (v_3, v_6, v_5, v_4)$. Hence the value of the primal and dual optima is 3. It can easily be checked that this system of paths is the *only* optimal solution. The bad thing is that two nodes (v_1 and v_4) are the end nodes of just one path (that is, an odd number of them). Therefore, there is no hope to obtain these paths by first determining an optimal *even vector* m' in the intersection of the two polymatroids in question and then finding H-admissible paths so that each node $t \in T$ is the end node of $m'(t)$ of them. Furthermore, one must insist on the evenness of m' since Theorem 2.2 is true only for such vectors.

(Incidentally, vector $m' := (2,2,2,0)$ is an optimal element of the polymatroid intersection and $\{A := T, B := \emptyset\}$ is a bipartition of T satisfying (4.2). Vector m arising in the proof is $m := (2,2,2,2)$. When Theorem 2.3 is applied to this m we obtain a family $\mathcal{F}$ of four paths, namely, $P_1 := (v_1, v_5, v_6, v_2), P_2 := (v_2, v_6, v_3), P_3 := (v_3, v_6, v_5, v_4), P_4 := (v_1, v_5, v_4)$. Among these paths P_4 is the only non-H-admissible, and we obtain P_1, P_2, P_3 as an optimal solution to the maximization problem.)

Although this direct approach to the maximization problem did not prove successful, it led us to the following problem to be considered for its own sake.

Let $G = (V,E)$ be a graph $H = (T,F)$, a demand graph with $T \subseteq V$. Moreover, let $m : T \to \mathbf{Z}_+$ be a *demand* function. The *node demand problem* consists of finding a system of H-admissible paths so that each terminal t is the end node of precisely $m(t)$ paths. We call the problem and also the vector m *feasible* when such a solution exists.

The node demand problem is called *Eulerian* if it is inner Eulerian and $m(t)+d(t)$ is even for each $t \in T$. We call a demand graph H *two-covered* (*one-covered*) if every node $t \in T$ belongs to at most two (exactly one) maximal stable sets of H. Note that bistable graphs are always two-covered but a five-element circuit, for example, is two-covered and not bistable. It can be shown that a graph H is two-covered if and only if H is the complement of the line graph of a triangle-free graph.

THEOREM 6.1. *Suppose that the node demand problem defined by (G, H, m) is Eulerian and H is two-covered. Then it is feasible if and only if the following condition holds:*

$$m(S) - m(X \cap T - S) \leq d(X) \tag{6.1}$$

for every $X \subseteq V$ and $S \subseteq X \cap T$ where S is stable in H.

Proof. Since S is stable in H, in a solution to the node demand problem each path with an end node in S has the other end node in $T - S$. Among these $m(S)$ paths at most $m(X \cap T - S)$ may end in $X - S$, and hence at least $m(S) - m(X \cap T - S)$ must end outside X, from which (6.1) follows.

To prove the sufficiency first observe that the family $\mathcal{L}$ of maximal stable sets of H is 3-cross-free. Indeed, for a contradiction, let S_1, S_2, S_3 be maximal stable sets of H which are pairwise crossing. Since H is two-covered, $S_1 \cap S_2 \cap S_3 = \emptyset$ and there are distinct elements $a \in S_1 \cap S_2, b \in S_2 \cap S_3, c \in S_3 \cap S_1$. Now $\{a, b, c\}$ is stable and a maximal stable set S containing a, b, c is distinct from each S_i. But then the element a (and b, c, as well) would belong to more than two maximal stable sets, contradicting that H is two-covered.

CLAIM 4. *$\lambda_m(S) = m(S)$ for any stable set S of H.*

Proof. Recall that $\lambda_m(S)$ was defined to be the maximum number of edge-disjoint paths connecting S and $T - S$ so that each $x \in T$ is the end node of at most $m(x)$ of them. By a version of the Menger theorem $\lambda_m(S) = \min(d(X) + m(S - X) + m(T - S - X) : X \subseteq V)$. (Indeed, apply the edge-disjoint undirected version of the Menger theorem to the graph arising from G by adding two new nodes s, t so that s (respectively, t) is connected to each node x in S (respectively, in $T - S$) by $m(x)$ new parallel edges.)

If X denotes the set where the minimum is attained, then, by (6.1), we have $\lambda_m(S) = d(X) + m(S - X) + m(X \cap T - S) \geq m(S \cap X) - m(X \cap T - S) + m(S - X) + m(X \cap T - S) = m(S)$, and the claim follows. □

Apply Theorem 2.3 to $G, m, \mathcal{L}$ and consider the path system $\mathcal{F}$ provided by the theorem (where $\mathcal{L}$ is the collection of maximal stable sets of H).

CLAIM 5. *$\mathcal{F}$ is a solution to the node demand problem.*

Proof. Let S be an element of $\mathcal{L}$, that is, a maximal stable set of H. Since $\mathcal{F}$ locks S, $\mathcal{F}$ contains $\lambda_m(S) = m(S)$ paths connecting S and $T - S$. This shows that each node x in S is the end node of precisely $m(x)$ members of $\mathcal{F}$ and that each path in $\mathcal{F}$ having an end node in S must have the other end node in $T - S$.

Because every node x of H belongs to a maximal stable set of H, x is the end node of precisely $m(x)$ members of $\mathcal{F}$. Moreover, since every pair of nonadjacent nodes x, y of H belongs to a maximal stable set of H, no path in $\mathcal{F}$ may connect x and y; that is, $\mathcal{F}$ consists of H-feasible paths.

Remark. The condition in Theorem 6.1 may be formulated in an equivalent form. By taking $S := X \cap Z$ in (6.1), we see that (6.1) implies

$$m(X \cap Z) - m(X \cap T - Z) \leq d(X) \tag{6.1$'$}$$

for every $X \subseteq V$ and every maximal stable set Z of H. Conversely, we claim that (6.1) follows from (6.1′). Indeed, let Z be a maximal stable set of H including S. Then $m(S) \leq m(X \cap Z) \leq d(X) + m(X \cap T - Z) \leq d(X) + m(X \cap T - S)$, and (6.1) follows.

Equation (6.1′) has the advantage that there are only a few maximal stable sets in a two-covered graph (at most $2|T|$). On the other hand, in the proof above it is slightly easier to work with (6.1).

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