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TWO ARC-DISJOINT PATHS IN EULERIAN DIGRAPHS*

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Abstract. Let G be an Eulerian digraph, and let $\{x_1, x_2\}, \{y_1, y_2\}$ be two pairs of vertices in G. A directed path from a vertex s to a vertex t is called an st-path. An instance $(G; \{x_1, x_2\}, \{y_1, y_2\})$ is called feasible if there is a choice of h, i, j, k with $\{h, i\} = \{j, k\} = \{1, 2\}$ such that G has two arc-disjoint $x_h x_i$- and $y_j y_k$-paths. In this paper, we characterize the structure of minimal infeasible instances, based on which an $O(m + n \log n)$ time algorithm is presented to decide whether a given instance is feasible, where n and m are the number of vertices and arcs in the instance, respectively. If the instance is feasible, the corresponding two arc-disjoint paths can be computed in $O(m(m + n \log n))$ time.

Key words. Eulerian digraph, disjoint paths, minimum cut, planar graph, polynomial time algorithm

AMS subject classifications. 05C35, 05C40

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1. Introduction.

Finding a set of edge-disjoint paths connecting pairs of specified vertices (called terminals) in a graph or a digraph is one of the classical and fundamental problems in graph theory (see [6] for a survey), which has a wide variety of applications. A path between terminals s and t (or a directed path from s to t) is called an st-path. If the graph is undirected, an important result by Robertson and Seymour [10] says that edge-disjoint paths for k pairs $\{s_i, t_i\}$ of terminals, $i = 1, 2, \ldots, k$, can be obtained in polynomial time for a fixed k. In the case of $k = 2$, a complete characterization of undirected graphs G that do not have edge-disjoint s_1t_1- and s_2t_2-paths is available (Dinits and Karzanov [2, 3], Seymour [11], and Thomassen [12]). Such G can be reduced to a graph G' that has a planar representation with the following properties (see Fig. 1):

(i) the four terminals have degree 2, and all other vertices are of degree 3, and

(ii) the terminals are located on the outer face in the order of s_1, s_2, t_1, t_2.

Contrary to this, the characterization of arc-disjoint path problems in digraphs seems much more difficult. For example, the weak 2-linking problem (i.e., to decide whether there are arc-disjoint s_1t_1- and s_2t_2-paths) in a general digraph is shown by Fortune, Hopcroft, and Wyllie [4] to be NP-complete. However, if the digraph under consideration is Eulerian, the situation becomes slightly easier. For a given digraph $G = (V, E)$ with ordered terminal pairs (s_i, t_i), $i = 1, 2, \ldots, k$, call $H = (V, \{(t_i, s_i) \mid i = 1, 2, \ldots, k\})$ its demand digraph. The weak 2-linking problem in an Eulerian digraph $G + H$ is known by Frank [5] to be polynomially solvable. Furthermore, Ibaraki and Poljak [9] showed that the weak 3-linking problem for an Eulerian digraph $G + H$ can also be solved in polynomial time. It is based on the observations that the weak 3-linking problem is equivalent to finding arc-disjoint x_1x_2-, x_2x_3-, and x_3x_1-

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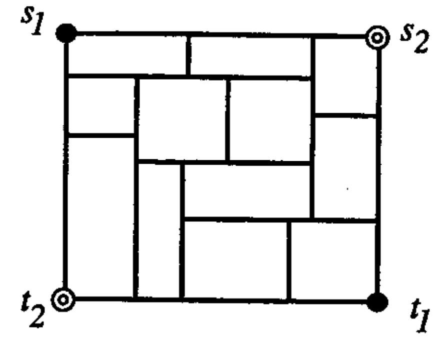


FIG. 1. *An infeasible instance for a two edge-disjoint path problem in an undirected graph G.*

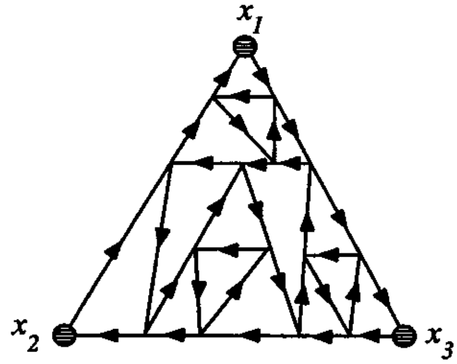


FIG. 2. *An infeasible instance for a weak 3-linking problem in an Eulerian digraph $G+H$.*

paths in an Eulerian digraph with terminals x_1, x_2, x_3 and that the resulting problem is infeasible if and only if it is reducible to a 2-connected Eulerian digraph G', which has a planar representation (see Fig. 2) such that

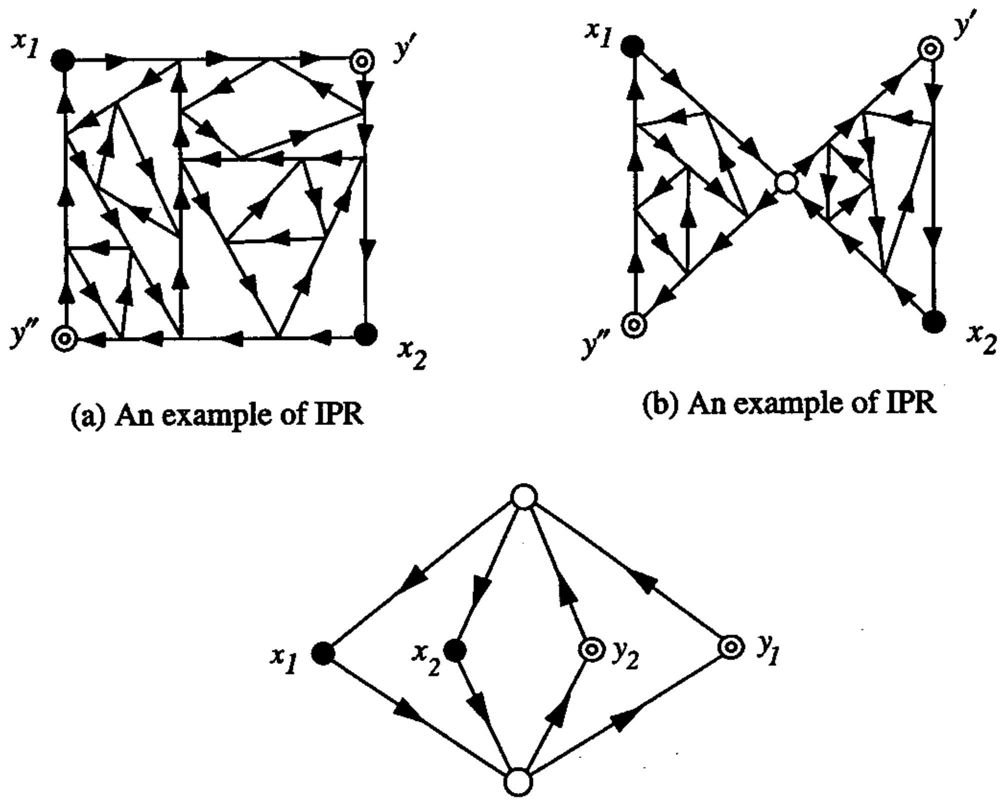
(i) all terminals have degree 2, and all other vertices have degree 4, and

(ii) every face is a directed cycle, and all the terminals are located on the outer face (which is also a directed cycle) in the order of x_3, x_2, x_1 (where the arcs in the outer face are directed clockwise).

In this paper, we generalize the above result to the two arc-disjoint path problem in an Eulerian digraph G (but $G+H$ is not Eulerian), which decides whether there are arc-disjoint paths connecting two *unordered* terminal pairs $\{x_1, x_2\}$ and $\{y_1, y_2\}$ (i.e., $x'x''$- and $y'y''$- paths, where either $x'x'' = x_1x_2$ or x_2x_1 and either $y'y'' = y_1y_2$ or y_2y_1). This problem includes the above weak 3-linking problem as a special case: for a given instance $(G; (s_1, t_1), (s_2, t_2), (s_3, t_3))$ of the 3-linking problem (where $G+H$ is Eulerian), add four new vertices x_1, x_2, y_1, and y_2 together with seven new arcs $(t_1, y_1), (y_1, s_2), (t_2, y_2), (y_2, x_1), (x_1, s_3), (t_3, x_2), (x_2, s_1)$ to obtain an instance $(G'; \{x_1, x_2\}, \{y_1, y_2\})$ of the two arc-disjoint path problem (where G' is Eulerian), which is clearly feasible if and only if the instance $(G; (s_1, t_1), (s_2, t_2), (s_3, t_3))$ is feasible. We show that the problem can be solved in $O(m + n \log n)$ time, where m and n are, respectively, the numbers of arcs and vertices in G, by deriving an analogue of the above structural characterization of infeasible instances: an Eulerian digraph G with four terminals x_1, x_2, y_1, y_2 is infeasible if and only if it is reducible to an Eulerian digraph G' that has a planar representation (see Fig. 3(a),(b)) such that

(i) all terminals have degree 2, and all other vertices have degree 4,

(ii) there is at most one cut vertex, and



(a) An example of IPR

(b) An example of IPR

(c) The minimal infeasible instance which has no IPR

FIG. 3. *Examples of infeasible instances for two arc-disjoint path problems in an Eulerian digraph G.*

(iii) every face is a directed cycle, and all terminals are located on the outer face in the order of x_1, y', x_2, y'', where $\{y', y''\} = \{y_1, y_2\}$.

The proof for this is, however, substantially different from that of [9]. For a feasible instance, we also show that the corresponding two arc-disjoint paths can be computed in $O(m(m + n\log n))$ time.

2. Preliminaries. Let $G = (V, E)$ be a digraph which may have multiple arcs. Denote by $deg(v)$, $indeg(v)$ and $outdeg(v)$ the degree, indegree, and outdegree of a vertex v in G, respectively, where the *degree* of a vertex is the sum of its out- and indegrees. We call a digraph *Eulerian* if the outdegree and indegree of each vertex are equal. Under a *path* or a *cycle*, we always understand a *directed* path or cycle. Repetition of arcs is not allowed, but repetition of vertices is allowed. A cycle that visits every arc exactly once is called Eulerian. A path from s to t is called an st-path. If $\{P_1, P_2, \ldots, P_k\}$ is a collection of arc-disjoint paths such that the last vertex of P_i coincides with the initial vertex of P_{i+1} for each $i = 1, 2, \ldots, k-1$, we denote by $P = \langle P_1, P_2, \cdots, P_k \rangle$ the concatenation of the paths. In the following discussion, digraph G, path P, or cycle C may sometimes be treated either as a vertex set or an arc set, as far as its meaning is unambiguous from the context. If it is necessary to specify, we use $E(G)$ and $V(G)$ to mean the arc set and the vertex set of a digraph G, respectively. For a digraph $G = (V, E)$ and an arc set $E' \subseteq E$, we denote the digraph $(V, E - E')$ by $G - E'$. For a vertex set $Z \subset V$, the subdigraph *induced* by Z is denoted by $G[Z] = (Z, E_Z)$, where $E_Z = \{(u, v) \in E \mid u, v \in Z\}$, and $G[V - Z]$ may be denoted by $G - Z$.

For a subset Z of vertices, $\delta^+(Z)$ denotes the set of arcs from Z to $V-Z$, $\delta^-(Z)$ the set of arcs from $V-Z$ to Z, and $\delta(Z)=\delta^+(Z)\cup\delta^-(Z)$. If G is Eulerian, then $|\delta^+(Z)|=|\delta^-(Z)|$ holds for every Z, where $|A|$ denotes the cardinality of a set A, and therefore $|\delta(Z)|$ is always even. A set $Z\subset V$ is called a *k-cut* if $|\delta(Z)|$ is k. For two disjoint $S,T\subset V$, we say that a cut Z *separates* S and T if $S\subseteq Z$ and $T\subseteq V-Z$ and define $\delta(S,T)$ to be the set of arcs from S to T and arcs from T to S. Throughout this paper, a singleton set $\{v\}$ may also be denoted as v.

Two cuts Z_1 and Z_2 *intersect* each other if $Z_1\cap Z_2\neq\emptyset$, $Z_1-Z_2\neq\emptyset$, and $Z_2-Z_1\neq\emptyset$, and they *cross* each other if, in addition, $V-(Z_1\cup Z_2)\neq\emptyset$ holds. For two crossing cuts Z_1 and Z_2, we easily see that

$$|\delta(Z_1)|+|\delta(Z_2)|\geq|\delta(Z_1\cap Z_2)|+|\delta(Z_1\cup Z_2)| \tag{2.1}$$

and

$$|\delta(Z_1)|+|\delta(Z_2)|=|\delta(Z_1-Z_2)|+|\delta(Z_2-Z_1)|+2|\delta(Z_1\cap Z_2,V-(Z_1\cup Z_2))| \tag{2.2}$$

hold.

Some further notions, such as planarity, edge connectivity, and vertex connectivity, we refer to the unoriented graph $\overline{G}$ obtained from G by ignoring arc orientation. A digraph G is called *connected* if $\overline{G}$ is connected. For a connected digraph $G=(V,E)$, a vertex z is called a *cut vertex* if $G-\{z\}$ has more than one connected component. We call an *undirected* path in $\overline{G}$ a *chain*. A chain with end vertices s and t is called an st-chain, and these s and t are said to be connected (by the chain). A concatenation of a collection of chains is also defined analogously to paths.

Consider an instance $(G=(V,E);X,Y)$ with $X=\{x_1,x_2\}\subseteq V$ and $Y=\{y_1,y_2\}\subseteq V$. Throughout this paper, when we refer to an instance $(G;X,Y)$, we assume that G is Eulerian and is connected (hence strongly connected since G is Eulerian). Each $t\in X\cup Y$ is called a *terminal*. We say that an instance $(G;X,Y)$ is *feasible* if it has two arc-disjoint $x'x''$- and $y'y''$-paths such that $\{x',x''\}=X$ and $\{y',y''\}=Y$; otherwise it is *infeasible*.

LEMMA 2.1. *Let P_X be an $x'x''$-path with $\{x',x''\}=X$ in $(G;X,Y)$. If y_1 and y_2 are connected in $G-E(P_X)$, then $(G;X,Y)$ is feasible.*

Proof. Since G is Eulerian, $G-E(P_X)$ has an $x''x'$-path P'_X and each of the connected components in $G-E(P_X)-E(P'_X)$ is Eulerian (possibly, a single vertex). If y_1 and y_2 are contained in the same connected component in $G-E(P_X)-E(P'_X)$, then the instance is feasible. Assume therefore that y_1 and y_2 are contained in two distinct components H_1 and H_2, respectively. Since y_1 and y_2 are connected in $G-E(P_X)$ but are not connected in $G-E(P_X)-E(P'_X)$, H_1 and H_2 must contain vertices v_1 and v_2 in $V(P'_X)$, respectively. Without loss of generality, assume that P'_X visits v_1 before v_2. Then, H_1 has a y_1v_1-path, P'_X contains a v_1v_2-path, and H_2 has a v_2y_2-path. This implies that the instance is feasible. □

LEMMA 2.2. *If an instance $(G;X,Y)$ satisfies $X\cap Y\neq\emptyset$, then it is feasible.*

Proof. Assume without loss of generality that $x_1=y_1$ for $X=\{x_1,x_2\}$ and $Y=\{y_1,y_2\}$. Since G is connected, there is arc-disjoint x_1x_2-path P_X and x_2x_1-path P'_X. Consider the connected component containing y_2 in $G-E(P_X)-E(P'_X)$. It contains a vertex in $V(P_X)$ or $V(P'_X)$ (say, $V(P'_X)$) since G is connected. Then, y_1 and y_2 are connected in $G-E(P_X)$. Lemma 2.1 then implies that $(G;X,Y)$ is feasible. □

In the following, therefore, we assume $X\cap Y=\emptyset$ for an instance $(G;X,Y)$.

DEFINITION 2.1. *We say that an instance* $(G; X, Y)$ *with* $X = \{x_1, x_2\}$ *and* $Y = \{y_1, y_2\}$ *has an infeasible planar representation (IPR) if the following conditions hold (see Fig.* 3(a),(b)).

(i) G *is planar and has at most one cut vertex.*

(ii) *All the terminals have degree* 2, *and all other vertices have degree* 4.

(iii) G *has a planar representation in which every face is a directed cycle (or equivalently, the arcs incident to a vertex are alternately oriented out and in), and all the terminals lie on the boundary of the outer face (which is also a directed cycle) in the order of* x_1, y', x_2, y'', *where* $Y = \{y', y''\}$. □

We then have the next lemma.

LEMMA 2.3. *Any* $(G; X, Y)$ *which has an IPR is infeasible.*

Proof. If an IPR has arc-disjoint $x'x''$- and $y'y''$-paths, where $\{x', x''\} = X$ and $\{y', y''\} = Y$, then these two paths must cross at some nonterminal vertex in the planar representation (since every terminal has degree 2 and is located on the boundary of the outer face). However, the two paths cannot cross at a nonterminal vertex, because the arcs incident to a vertex are alternately oriented out and in. □

It is easy to see that a feasible instance $(G; X, Y)$ never becomes infeasible by contracting any arc. We say that an instance $(G; X, Y)$ is *minimal infeasible* if it is infeasible, but the instance $(G'; X, Y)$ obtained by contracting *any* arc becomes feasible. The main contribution of this paper is to show that the converse of Lemma 2.3 holds for such minimal infeasible instances. In the case of $|V| = 6$, however, there is a minimal infeasible instance with $V = \{x_1, x_2, y_1, y_2, v, w\}$ and $E = \{(v, x_1), (x_1, w), (v, x_2), (x_2, w), (w, y_1), (y_1, v), (w, y_2), (y_2, v)\}$ (see Fig. 3(c)), which is clearly infeasible but has no IPR. We shall see that this is the only exception.

3. Irreducible instances. Let us consider the following three types of reductions:

(1) Let Z be a 2-cut and $Z \cap (X \cup Y) = \emptyset$. Let u be the tail of the arc from $V - Z$ to Z and v the head of the arc from Z to $V - Z$. Delete Z, and if $u \neq v$ then add the arc (u, v) to $G[V - Z]$. See Fig. 4(1).

(2) Let Z be a 2-cut, $|Z| \geq 2$, and $|Z \cap (X \cup Y)| = 1$. Then contract Z to the terminal $t \in Z$, deleting any resulting loops. (The resulting terminal t has degree 2.) See Fig. 4(2).

(3) Let Z be a 4-cut such that $G[Z]$ is connected, $|Z| \geq 2$, and $Z \cap (X \cup Y) = \emptyset$. Then contract Z into a single vertex. See Fig. 4(3).

The next lemma is immediate from the definition of reductions.

LEMMA 3.1. *An instance* $(G; X, Y)$ *is feasible if and only if it is feasible after performing any of the reductions* (1), (2), *or* (3). □

We say that an instance $(G; X, Y)$ is *reducible* if one of the above reductions (1)–(3) can be applied (in this case, we also call such 2-cuts or 4-cuts *reducible*); otherwise they are *irreducible*. An irreducible instance cannot have a 4-cut W with $W \cap (X \cup Y) = \emptyset$ even if W does not induce a connected subdigraph, because in such a case there is a 2-cut $Z \subset W$ with $Z \cap (X \cup Y) = \emptyset$ (i.e., it is reducible). As will be shown in section 9, an irreducible instance of a given instance $(G; X, Y)$ can be obtained in polynomial time. In this section, we present some properties of infeasible irreducible instances.

LEMMA 3.2. *Any minimal infeasible instance is irreducible.*

Proof. The proof is obvious because any of the reductions (1), (2), and (3) can be performed by an adequate sequence of arc contractions, during which any infeasible instance never becomes feasible by Lemma 3.1. □

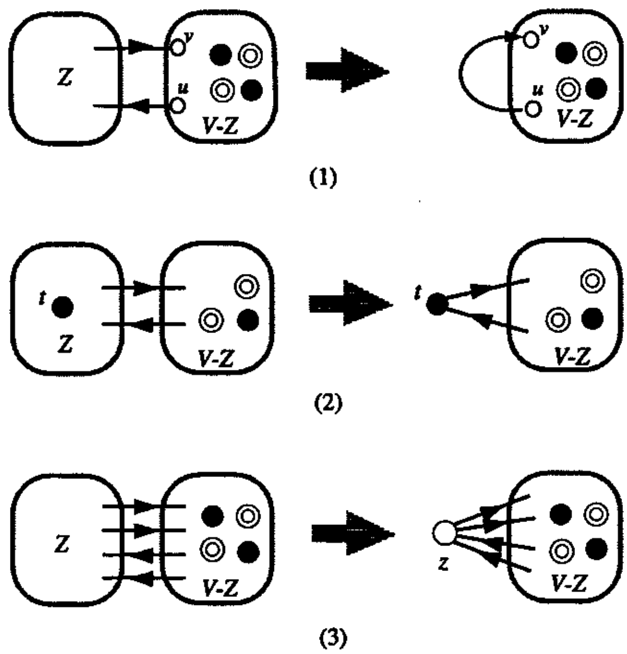


FIG. 4. *Three types of reductions of irreducible cuts Z.*

LEMMA 3.3. *Let $(G=(V,E);X,Y)$ be an infeasible irreducible instance. Then the following hold.*

(i) *There is no 2-cut Z that separates X and Y.*

(ii) *Each terminal has degree 2.*

(iii) *Each nonterminal vertex has degree 4.*

(iv) *For any pair of vertices $u, v \in V$, there is at most one arc connecting them (i.e., at most one of (u,v) and (v,u) exists in E).*

Proof. (i) Assume that a 2-cut Z separates X and Y, where $\delta(Z)=\{e_1,e_2\}$. Since G is connected and Eulerian, there are arc-disjoint x_1x_2-path P_X and x_2x_1-path P'_X. Clearly, one of them (say, P_X) contains no arcs from $\{e_1,e_2\}$. Similarly G has a $y'y''$-path P_Y, where $\{y',y''\}=Y$, such that $E(P_Y)\cap\{e_1,e_2\}=\emptyset$. These P_X and P_Y are arc-disjoint in G, and hence $(G;X,Y)$ is feasible.

(ii) Assume $deg(x_1)\geq 4$ for a terminal $x_1\in X$ without loss of generality. If G has arc-disjoint x_1y_1-path P_1 and x_1y_2-path P_2, then $G-E(P_1)-E(P_2)$ has arc-disjoint y_1x_1-path P_3 and y_2x_1-path P_4, since G is Eulerian. Let H be the connected component in $G-E(P_1)-E(P_2)-E(P_3)-E(P_4)$ that contains x_2. Since G is connected, H must contain a vertex z in $V(P_i)$ for some i. Assume $z\in V(P_1)\cup V(P_2)$ (the case of $z\in V(P_3)\cup V(P_4)$ can be treated similarly). Then $E(P_1)\cup E(P_2)\cup E(H)$ contains a path P_X from x_1 to x_2 via z. However, y_1 and y_2 are connected in $G-E(P_1)-E(P_2)-E(H)$, since $Q_Y=\langle P_3,P_4\rangle$ is a y_1y_2-chain, and the instance would be feasible by Lemma 2.1, contradicting the assumption. Therefore, at least one of the above P_1 and P_2 does not exist; i.e., by Menger's theorem, there must be a 2-cut W such that $x_1\in W$ and $Y\subseteq V-W$. Since $deg(x_1)\geq 4$, we have $|W|\geq 2$. From (i) of this lemma, $x_2\in V-W$ holds. This, however, implies that there is a reducible 2-cut W, which is a contradiction.

(iii) Assume $deg(u)\geq 6$ for a nonterminal vertex u. Let W be a cut that minimizes $|\delta(W)|$ among cuts W such that $u\in W$ and $\{x_1,y_1,y_2\}\subseteq V-W$. By the minimality of $|\delta(W)|$, $G[W]$ is connected. By $deg(u)\geq 6$, $|\delta(W)|=2$ would imply $|W|\geq 2$, and W is reducible. Hence $|\delta(W)|\geq 4$. From this, either (a) $|\delta(W)|\geq 6$ or (b)

$|\delta(W)| = 4$ and $x_2 \in W$ must hold (otherwise, W would be reducible). In the case of (a), by Menger's theorem G has three arc-disjoint paths P_1, P_2, and P_3 from u to some vertices $w_1, w_2, w_3 \in \{x_1, y_1, y_2\}$. By (ii) of this lemma, every terminal has degree 2, and then we can assume that these paths P_1, P_2, and P_3 are ux_1-, uy_1-, and uy_2-paths, respectively. Since G is Eulerian, $G - E(P_1) - E(P_2) - E(P_3)$ has three arc-disjoint x_1u-path P_4, y_1u-path P_5, and y_2u-path P_6. Let H be the connected component in $G - \bigcup_{i=1,\ldots,6} E(P_i)$ that contains x_2. Since G is connected, H must contain a vertex z in $V(P_i)$ for some i. Assume $z \in V(P_1) \cup V(P_4) \cup V(P_2) \cup V(P_3)$ (the case of $z \in V(P_1) \cup V(P_4) \cup V(P_5) \cup V(P_6)$ can be treated analogously). Then $E(P_1) \cup E(P_4) \cup E(P_2) \cup E(P_3) \cup E(H)$ contains a path P_X from x_1 to x_2 via z. However, y_1 and y_2 are connected in $G - E(P_1) - E(P_4) - E(P_2) - E(P_3) - E(H)$, since $Q_Y = \langle P_5, P_6 \rangle$ is a y_1y_2-chain, and the instance would be feasible by Lemma 2.1, contradicting the assumption.

Therefore, we assume (b); i.e., there is a 4-cut W separating $\{u, x_2\}$ and $\{x_1, y_1, y_2\}$. In this case, applying the above argument to u and $\{x_2, y_1, y_2\}$, we can conclude that there is also a 4-cut W' such that $\{u, x_1\} \subseteq W'$ and $\{x_2, y_1, y_2\} \subseteq V - W'$. These two cuts W and W' cross each other, and from (2.1) we have

$$|\delta(W)| + |\delta(W')| \geq |\delta(W \cap W')| + |\delta(W \cup W')|.$$

Here $|\delta(W)| = |\delta(W')| = 4$ and $|\delta(W \cup W')| \geq 4$ by (i). This implies $|\delta(W \cap W')| \leq 4$; i.e., $W \cap W'$ with $(W \cap W') \cap (X \cup Y) = \emptyset$ is a reducible 4-cut (or contains a reducible 2-cut $W'' \subset W \cap W'$), which is a contradiction.

(iv) If there are multiple arcs (u, v) and (u, v) in E, then u and v are nonterminal vertices by (iii) and $deg(u) = deg(v) = 4$ holds. This means that $Z = \{u, v\}$ is a reducible 2- or 4-cut, which is a contradiction. Similarly if there are two arcs (u, v) and (v, u), it is also easy to show that there is a reducible 2- or 4-cut $Z = \{u, v\}$, which is a contradiction. □

LEMMA 3.4. *Let $(G; X, Y)$ be an irreducible infeasible instance. Then G has at most two 2-cuts that separate $\{x_1, y'\}$ and $\{x_2, y''\}$, where $Y = \{y', y''\}$. If there are two such 2-cuts Z and Z', then G has a cut vertex z such that $Z = Z' \cup \{z\}$ or $Z = (V - Z') \cup \{z\}$. Conversely, any cut vertex is obtained in this manner.*

Proof. Let Z and Z' be the two 2-cuts that separate $\{x_1, y'\}$ and $\{x_2, y''\}$, where we assume $Z \cap (X \cup Y) = Z' \cap (X \cup Y) = \{x_1, y'\}$ without loss of generality. Choose Z (resp., Z') as the cut minimizing $|Z|$ (resp., maximizing $|Z'|$) among such 2-cuts. We first show that any other 2-cut Z'' that separates $\{x_1, y'\}$ and $\{x_2, y''\}$ satisfies

$$Z \subset Z'' \subset Z' \text{ (and hence } Z \subset Z'). \tag{3.1}$$

If Z'' crosses Z, we have $|\delta(Z \cap Z'')| \geq 4$ from the choice of Z and $|\delta(Z \cup Z'')| \geq 2$ as $Z \cup Z''$ separates $\{x_1, y'\}$ and $\{x_2, y''\}$. This means

$$|\delta(Z)| + |\delta(Z'')| = 4 < 6 \leq |\delta(Z \cap Z'')| + |\delta(Z \cup Z'')|,$$

which is a contradiction to (2.1). Similarly, we see that Z'' also cannot cross Z', and hence (3.1) holds. Since $|\delta(Z' - Z)| \leq |\delta(Z)| + |\delta(Z')|$ $(= 4)$, $Z' - Z$ is a 2- or 4-cut satisfying $(Z' - Z) \cap (X \cup Y) = \emptyset$. Then, by irreducibility, $Z' - Z$ must be a 4-cut consisting of a single vertex (say, z). This implies that z is a cut vertex. From $|Z' - Z| = 1$ and (3.1), $(G; X, Y)$ has at most two 2-cuts that separate $\{x_1, y'\}$ and $\{x_2, y''\}$.

Conversely, let z be a cut vertex in G. Since G is Eulerian, z cannot have degree 2. By Lemma 3.3(ii), (iii), and (iv), z is a nonterminal vertex and there are four distinct vertices, say, u_1, u_2, v_1, v_2, adjacent to z. Again since G is Eulerian, $G - \{z\}$ has exactly two components. Let $W_1, W_2 \subset V$ be the vertex sets of these components, and assume without loss of generality that $(u_1, z), (u_2, z), (z, v_1), (z, v_2) \in E$, $u_1, v_1 \in W_1$ and $u_2, v_2 \in W_2$ since G is Eulerian and at least one arc is going out (resp., going in) of each W_1 and W_2. Clearly W_1 and $W_2 = V - (W_1 \cup \{z\})$ are both 2-cuts. By irreducibility and Lemma 3.3(i), these satisfy $|W_1 \cap X| = |W_1 \cap Y| = |(W_1 \cup \{z\}) \cap X| = |(W_1 \cup \{z\}) \cap Y| = 1$. This implies that a cut vertex z is obtained in the manner of the lemma statement. □

LEMMA 3.5. *Let $(G = (V, E); X, Y)$ be an irreducible infeasible instance which has an IPR, and let B be the cycle of the outer face in the IPR. If $|V| \geq 5$ and there is no 2-cut Z such that $|Z \cap X| = |Z \cap Y| = 1$ and $\min\{|Z|, |V - Z|\} \geq 3$, then $V - (X \cup Y)$ induces a connected digraph in $G - E(B)$.*

Proof. By $|V| \geq 5$, $V - (X \cup Y) \neq \emptyset$. By Lemma 3.3(ii) and the definition of IPR, each terminal in $X \cup Y$ is an isolated vertex in $G - E(B)$. Assume that $V - (X \cup Y)$ induces two connected components H_1 and H_2 in $G - E(B)$. By the planarity of the IPR, any vertices $u_1, v_1 \in V(H_1) \cap V(B)$ and any vertices $u_2, v_2 \in V(H_2) \cap V(B)$ cannot appear alternately (in the order of u_1, u_2, v_1, v_2) along cycle B. This means that there are two arcs $(u, v), (u', v') \in E(B)$ such that $V(H_1)$ and $V(H_2)$ are contained in two distinct connected components H_1' and H_2' in $G - \{(u, v), (u', v')\}$, respectively. Hence $Z = V(H_1')$ is a 2-cut, and by the irreducibility, both $V(H_1')$ and $V(H_2')$ must contain two terminals, one from X and the other from Y by Lemma 3.3(i). Clearly, each of $V(H_1')$ and $V(H_2')$ contains a nonterminal, and has at least three vertices; i.e., $\min\{|Z|, |V - Z|\} \geq 3$. □

The next lemma can be shown by inspecting all possible irreducible and infeasible instances with $|V| \leq 7$, based on Lemma 3.3.

LEMMA 3.6. *Let $(G; X, Y)$ be an irreducible infeasible instance with $|V| \leq 7$. If $|V| \in \{4, 5, 7\}$, then $(G; X, Y)$ has an IPR. If $|V| = 6$, $(G; X, Y)$ is the instance shown in Fig.* 3(c) *(in this case there is no irreducible infeasible instance with $|V| = 6$ in which some two terminals are adjacent to each other).* □

In this paper, we prove the next result.

THEOREM 3.7. *Let $(G; X, Y)$ be a minimal infeasible instance, and let it satisfy $|V| \neq 6$. Then $(G; X, Y)$ has an IPR.* □

We shall need sections 4–8 to prove Theorem 3.7 for general $|V| \geq 8$.

4. Outline of the proof. This section describes an outline of how to prove Theorem 3.7 in sections 5–8. We first assume that there is a smallest counterexample $(G^*; X, Y)$ to Theorem 3.7; i.e.,

$$(G^*; X, Y) \text{ is a minimal infeasible instance with } n \neq 6 \text{ vertices, but has no IPR,} \tag{4.1}$$

where G^* minimizes the number n^* of vertices among such instances. By Lemma 3.6, $n^* \geq 8$ is assumed. In sections 5 and 6, we characterize cut vertices, 2-cuts, and 6-cuts in G^*. Then in sections 7 and 8, as outlined below, we derive a contradiction from the existence of such G^*, which proves Theorem 3.7.

For the subsequent discussion, we introduce two operations. Let w be a nonterminal vertex with four incident arcs $(s_0, w), (s_1, w), (w, s_2), (w, s_3)$, where s_0, s_1, s_2, and s_3 are all distinct. We say that arcs (s_0, w) and (w, s_2) are *split off* at w when four arcs $(s_0, w), (s_1, w), (w, s_2), (w, s_3)$ are replaced with two new arcs (s_0, s_2) and (s_1, s_3) after eliminating w. Conversely, we say that two arcs $e = (u, v)$ and $e' = (u', v')$ are *hooked up* (with a new vertex w) when we replace these two arcs with the new arcs $(u, w), (w, v), (u', w)$, and (w, v') after introducing a new vertex w.

Now we choose a nonterminal vertex w adjacent to a terminal (say, x_2) by arc (x_2, w) in G^* and split off two arcs at w (recall that $deg(w) = 4$). If the resulting instance

$$(G^*_w; X, Y) \text{ remains connected and irreducible,}$$

then we call such splitting (or two arcs) *admissible*. Based on the properties obtained in sections 5 and 6, we show in section 7 that G^* always has an admissible splitting. Clearly

$$(G^*_w; X, Y) \text{ is infeasible}$$

since $(G^*; X, Y)$ is infeasible. Also if $(G^*_w; X, Y)$ is irreducible, then

$$(G^*_w; X, Y) \text{ has an IPR}$$

by the assumption on G^*.

However, we shall show in section 8 that, for the arc $e = (x_2, v)$ and any other arc e' in an irreducible infeasible instance $(G; X, Y)$ that has an IPR, the instance $(G_{e,e'}; X, Y)$ obtained by hooking up e and e' satisfies one of the following properties:

(i) $(G_{e,e'}; X, Y)$ is reducible,

(ii) $(G_{e,e'}; X, Y)$ has an IPR,

(iii) $(G_{e,e'}; X, Y)$ is feasible.

Notice that G^* is obtained from G^*_w by hooking up two arcs in an IPR of G^*_w. However, this leads to a contradiction because $G^* = (G^*_w)_{e,e'}$ satisfies none of (i)–(iii). Hence no such counterexample $(G^*; X, Y)$ exists.

5. Cut vertex and 2-cuts in G^*.

LEMMA 5.1. *The minimum counterexample* $(G^*; X, Y)$ *in* (4.1) *has the following properties:*

(i) *There is no cut vertex.*

(ii) *There is no* 2*-cut* Z *such that* $|Z \cap X| = |Z \cap Y| = 1$ *and* $\min\{|Z|, |V - Z|\} \geq 3$.

Proof. (i) Assume that $G^* = (V, E)$ has a cut vertex z, since no vertex with degree 2 is a cut vertex in a connected Eulerian digraph and z is nonterminal and has degree 4 by Lemma 3.3(ii), (iii).

Let Z' and Z'' be the vertex sets of the two components in $G^* - \{z\}$. By Lemma 3.4, $|Z' \cap X| = |Z' \cap Y| = 1$ holds and each of $Z' \cup \{z\}$ and $Z'' \cup \{z\}$ is a 2-cut in G^*. Let $(u', z), (u'', z), (z, v'), (z, v'') \in E$ be the four arcs incident to z. Without loss of generality we can assume that $u', v' \in Z'$ and $u'', v'' \in Z''$, $Z' \cap (X \cup Y) = \{x_1, y_2\}$, and G^* has an Eulerian cycle which visits terminals in the order of y_2, x_1, y_1, x_2 (if there is an Eulerian cycle in the order of y_2, x_1, x_2, y_1, then the instance is feasible). We decompose $(G^*; X, Y)$ into $(G'; X', Y')$ and $(G''; X'', Y'')$ as follows. Let $G^*[Z']$ (resp., $G^*[Z'']$) denote the subdigraph of G^* induced by Z' (resp., Z''), and let G' (resp., G'') be the Eulerian digraph obtained by adding new vertices y'_1, x'_2 and new arcs $(u', y'_1), (y'_1, x'_2), (x'_2, v')$ (resp., new vertices y''_2, x''_1 and new arcs $(u'', y''_2), (y''_2, x''_1), (x''_1, v'')$) to $G^*[Z']$ (resp., $G^*[Z'']$). Regard $X' = \{x_1, x'_2\}$, $Y' = \{y'_1, y_2\}$, $X'' = \{x''_1, x_2\}$, and $Y'' = \{y_1, y''_2\}$ as the sets of new terminals. We show that $(G'; X', Y')$ is irreducible. If $(G'; X', Y')$ has a reducible cut W, then W must separate y'_1 and x'_2 (otherwise W would be reducible in $(G^*; X, Y)$). Then W is a 2-cut such that $|W| \geq 2$ and $W \cap (X' \cup Y') = \{y'_1\}$ or $\{x'_2\}$. Since $deg(y'_1) = deg(x'_2) = 2$, $W - \{y'_1\}$ (or $W - \{x'_2\}$) is a 2-cut, which is reducible in $(G^*; X, Y)$, which is a contradiction. Note that $(G^*; X, Y)$ is infeasible only when both new instances $(G';X',Y')$ and $(G'';X'',Y'')$ are infeasible. Therefore, $(G';X',Y')$ must be irreducible and infeasible. Clearly, instance $(G';X',Y')$ is smaller than G^* (since $Z'' \cup \{z\}$ is replaced with two vertices in the new instance), and hence has an IPR by definition of G^* (note that $|V(G')| \neq 6$ by Lemma 3.6 since $(G';X',Y')$ has two adjacent terminals). Analogously, we can show that $(G'';X'',Y'')$ also has an IPR. However, it is easy to see that G^* has an IPR if both instances $(G';X',Y')$ and $(G'';X'',Y'')$ have IPRs, which is a contradiction.

(ii) Let Z be such a 2-cut in $(G^*;X,Y)$, where $\delta(Z) = \{(u',v''),(u'',v')\}$ and $u',v' \in Z$ and $u'',v'' \in V-Z$. Clearly, $\overline{Z} = V - Z$ is also such a 2-cut. Note that u' and v' (resp., u'' and v'') are distinct (otherwise it would be a cut vertex, contradicting the above (i)). Without loss of generality assume that $Z \cap (X \cup Y) = \{x_1, y_2\}$ and G^* has an Eulerian cycle that visits terminals in the order of y_2, x_1, y_1, x_2. We decompose instance $(G^*;X,Y)$ into the two instances $(G';X',Y')$ and $(G'';X'',Y'')$ as follows. Let G' (resp., G'') be the digraph obtained by adding new vertices y_1', x_2' and new arcs $(u',y_1'),(y_1',x_2'),(x_2',v')$ (resp., new vertices y_2'', x_1'' and new arcs $(u'',y_2''),(y_2'',x_1''),(x_1'',v'')$) to $G^*[Z]$ (resp., $G^*[\overline{Z}]$). Regard $X' = \{x_1, x_2'\}$, $Y' = \{y_1', y_2\}$, $X'' = \{x_1'', x_2\}$, and $Y'' = \{y_1, y_2''\}$ as the sets of new terminals. Analogously to (i), we see that each of the new instances is an irreducible infeasible instance. From the assumption of $\min\{|Z|, |V-Z|\} \geq 3$, each of the new instances is smaller than G^* and has an IPR by definition of G^* or by Lemma 3.6. However, it is again clear that G^* has an IPR if these new instances have IPRs, which is a contradiction. □

6. 6-cuts. We first observe a property of a 6-cut Z.

LEMMA 6.1. *Let $(G=(V,E);X,Y)$ be an infeasible instance. If there exists a 6-cut Z with $Z \cap (X \cup Y) = \emptyset$ satisfying the following* (i)–(iv), *then $(G;X,Y)$ is irreducible.*

(i) $|Z| \geq 3$.

(ii) *Any cut W with $W \subseteq Z$ is irreducible.*

(iii) *Any cut W with $W \supseteq Z$ or $W \cap Z = \emptyset$ is irreducible.*

(iv) *$\delta(Z)$ contains no multiple arcs.*

Proof. Let Z be such a 6-cut. From (ii) and (iii), it suffices to show that any cut W which intersects Z is irreducible; i.e., $|\delta(W)| = 2$ or 4. Assume that a cut W intersecting Z is reducible. Since W contains at most one terminal, $V - (W \cup Z)$ contains a terminal, and hence cuts W and Z cross each other. By (2.1),

$$|\delta(W)| + |\delta(Z)| \geq |\delta(W \cap Z)| + |\delta(W \cup Z)|, \tag{6.1}$$

and by (2.2),

$$|\delta(W)| + |\delta(Z)| = |\delta(W - Z)| + |\delta(Z - W)| + 2|\delta(W \cap Z, V - (W \cup Z))|. \tag{6.2}$$

Since Z contains no reducible cut by (ii), we have $|\delta(W \cap Z)| \geq 4$. Also by (iii), we see that $|\delta(W \cup Z)| \geq |\delta(W)| + 2$ holds (otherwise if $|\delta(W \cup Z)| \leq |\delta(W)|$, then $W \cup Z$ would be reducible by $(W \cup Z) \cap (X \cup Y) = W \cap (X \cup Y)$). Therefore, $|\delta(Z)| = 6$, $|\delta(W \cap Z)| \geq 4$, and $|\delta(W \cup Z)| \geq |\delta(W)| + 2$ imply that (6.1) holds by equality. Hence $|\delta(W \cap Z)| = 4$ holds, and this means $|W \cap Z| = 1$ since Z contains no reducible cut. Then $|Z - W| \geq 2$ by (i), and again by (ii), $|\delta(Z - W)| \geq 6$. By (iii), we have $|\delta(W - Z)| \geq |\delta(W)|$ (otherwise, if $|\delta(W - Z)| \leq |\delta(W)| - 2$, then $W - Z$ would be reducible). By $|\delta(Z - W)| \geq 6$, $|\delta(Z)| = 6$, and $|\delta(W - Z)| \geq |\delta(W)|$, (6.2) implies that $|\delta(Z - W)| = |\delta(Z)| = 6$, $|\delta(W - Z)| = |\delta(W)|$, and $|\delta(W \cap Z, V - (W \cup Z))| = 0$. By (iii), $|W - Z| = 1$ must hold, since $|W - Z| \geq 2$ and $|\delta(W - Z)| \doteq |\delta(W)|$ mean

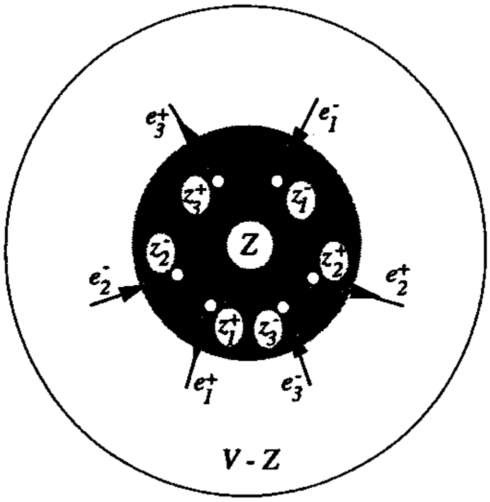


FIG. 5. *A 6-cut Z.*

that $W-Z$ is reducible. Now the vertex $v \in W \cap Z$ has degree 4 and has no adjacent vertex in $V-(W \cup Z)$ by $|\delta(W \cap Z, V-(W \cup Z))| = 0$. From $|\delta(Z-W)| = |\delta(Z)|$ and $|\delta(W-Z)| = |\delta(W)|$, we then have $|\delta(\{v\}, W-Z)| = |\delta(\{v\}, Z-W)| = 2$. However, by $|W-Z| = 1$, the two arcs in $\delta(\{v\}, Z-W)$ are multiple, contradicting (iv). □

6.1. Interchangeability. Let Z be a 6-cut in $(G; X, Y)$ such that

$$Z \cap (X \cup Y) = \emptyset,$$

$$\delta^-(Z) = \{e_1^-, e_2^-, e_3^-\}, \quad \delta^+(Z) = \{e_1^+, e_2^+, e_3^+\}, \tag{6.3}$$

where z_i^- (resp., z_i^+) denote the head vertices of arcs e_i^- (resp., the tail vertices of arcs e_i^+) for $i = 1, 2, 3$ (see Fig. 5). Note that these vertices z_i^- and z_i^+ may not be distinct. We say that Z is $(e_1^-, e_2^-, e_3^-; e_{j_1}^+, e_{j_2}^+, e_{j_3}^+)$-*interchangeable*, where $\{j_1, j_2, j_3\} = \{1, 2, 3\}$, if the subdigraph $G[Z]$ of G induced by Z has three arc-disjoint $z_i^- z_{j_i}^+$-paths P_{i,j_i} ($i = 1, 2, 3$). Z is called *fully interchangeable* if it is $(e_1^-, e_2^-, e_3^-; e_{j_1}^+, e_{j_2}^+, e_{j_3}^+)$-interchangeable for any choice of j_1, j_2, j_3 with $\{j_1, j_2, j_3\} = \{1, 2, 3\}$.

LEMMA 6.2. *An irreducible infeasible instance $(G; X, Y)$ has no fully interchangeable 6-cut Z with $Z \cap (X \cup Y) = \emptyset$.*

Proof. Assume that there is such a 6-cut Z, and let G_Z be the digraph obtained by contracting Z into a nonterminal vertex z. It is easy to see the following:

(i) $(G; X, Y)$ is feasible if and only if $(G_Z; X, Y)$ is feasible, and

(ii) $(G_Z; X, Y)$ is irreducible.

Therefore, $(G_Z; X, Y)$ is also an irreducible infeasible instance, but $deg(z) = 6$ contradicts Lemma 3.3(iii). □

A directed cycle of length 3 is called a *triangle.*

LEMMA 6.3. *Let $(G; X, Y)$ be an irreducible infeasible instance. Then the following hold.*

(i) *If Z is a 6-cut such that $Z \cap (X \cup Y) = \emptyset$, $|Z| = 3$, and the induced subdigraph $G[Z]$ is connected, then $G[Z]$ is a triangle.*

(ii) *If $|Z| = 3$ and the three vertices in Z are mutually adjacent, then the induced subdigraph $G[Z]$ is a triangle.*

Proof. (i) From Lemma 3.3(iv) and $|\delta(Z)| = 6$, it is easy to see that the connected subdigraph $G[Z]$ contains exactly three arcs and these three arcs form an undirected cycle C of length 3 if the orientation is neglected. This C must be a directed cycle in G, because otherwise it is not difficult to see, by checking all possibilities, that Z is fully interchangeable, contradicting Lemma 6.2.

(ii) If all vertices in Z are nonterminal, (ii) follows from (i). Therefore, assume that Z contains a terminal. $Z = \{v_1, v_2, v_3\}$ can contain at most two terminals since any terminal has degree 2 from Lemma 3.3(ii). If Z contains two terminals, then clearly $G[Z]$ forms a triangle. Then assume that Z contains exactly one terminal, say, $Z \cap (X \cup Y) = \{v_2\}$. If $G[Z]$ is not a triangle, then we can assume without loss of generality that $G[Z]$ has arcs $(v_1, v_2), (v_2, v_3), (v_1, v_3)$. Let G_{v_2} be the digraph obtained from G by contracting Z into terminal v_2. It is easy to see that $(G_{v_2}; X, Y)$ is also an irreducible infeasible instance. But v_2 has degree 4 and contradicts Lemma 3.3(ii). □

Let $\delta(Z; G)$ denote $\delta(Z)$ in a digraph G.

LEMMA 6.4. *Let $(G; X, Y)$ be an irreducible infeasible instance, and let Z be a 6-cut in $(G; X, Y)$ as defined in* (6.3). *If Z is not $(e_1^-, e_2^-, e_3^-; e_1^+, e_2^+, e_3^+)$-interchangeable, then properties* (i)–(iv) *hold.*

(i) *The induced subdigraph $G[Z]$ is connected.*

(ii) *If $G[Z]$ has no $z_i^- z_i^+$-path for some $i \in \{1, 2, 3\}$, then $Z = \{z_i^-, z_i^+\}$.*

(iii) *$z_i^- \neq z_i^+$ for all $i \in \{1, 2, 3\}$.*

(iv) *If $|Z| \geq 3$, then $z_i^- \neq z_{i'}^-$ and $z_i^+ \neq z_{i'}^+$ for $1 \leq i < i' \leq 3$.*

Proof. Note that $|\delta^-(\{u\}; G[Z])| = |\delta^+(\{u\}; G[Z])|$ for all $u \in Z - \{z_i^-, z_i^+ \mid i = 1, 2, 3\}$. Hence if $G[Z]$ has a $z_i^- z_j^+$-path P, then $G[Z] - E(P)$ has arc-disjoint $z_{i'}^- z_{j'}^+$- and $z_{i''}^- z_{j''}^+$-paths for some i', i'', j', j'' with $\{i', i''\} = \{1, 2, 3\} - \{i\}$ and $\{j', j''\} = \{1, 2, 3\} - \{j\}$.

(i) If the subdigraph $G[Z]$ of G consists of more than one connected component, then there would be a reducible 2-cut Z' with $Z' \subset Z$.

(ii) Assume without loss of generality that $G[Z]$ has no $z_1^- z_1^+$-path. Then, by Menger's theorem, $G[Z]$ has a cut $W \subset Z$ such that $z_1^- \in W$, $z_1^+ \in Z - W$, and $|\delta^+(W; G[Z])| = 0$. Here $|\delta^-(W; G[Z])| \geq 1$ since $G[Z]$ is connected by (i). Let H denote the Eulerian digraph obtained by adding three new arcs $e_i^* = (z_i^+, z_i^-)$ $(i = 1, 2, 3)$ to $G[Z]$. Now $e_1^* \in \delta^-(W; H)$ and $|\delta^-(W; H)| \geq 2$ hold, and hence $|\delta^+(W; H)| \geq 2$ since H is Eulerian. Since $|\delta^+(W; G[Z])| = 0$, we see that $e_2^*, e_3^* \in \delta^+(W; H)$. Therefore, by $|\delta^+(W; H)| = 2 = |\delta^-(W; H)|$, we have $|\delta^-(W; G[Z])| = 1$. This implies that $z_1^-, z_2^+, z_3^+ \in W$ and $z_1^+, z_2^-, z_3^- \in Z - W$ and that $|\delta(W; G)| = |\delta(Z - W; G)| = 4$ holds. Hence the 4-cut W (resp., $Z - W$) in G consists of a single vertex z_1^- (resp., z_1^+), respectively, from the irreducibility of G.

(iii) If $z_1^- = z_1^+$, then $G[Z]$ has a $z_1^- z_1^+$-path of null length. Since G is Eulerian, $G[Z]$ has two arc-disjoint $z_2^- z_2^+$- and $z_3^- z_3^+$-paths, because even if $G[Z]$ has arc-disjoint $z_2^- z_3^+$- and $z_3^- z_2^+$-paths, the connectivity of $G[Z]$ (which follows from (i)) implies that these paths have a common vertex v from which $z_2^- z_2^+$- and $z_3^- z_3^+$-paths can be constructed. This contradicts that Z is not $(e_1^-, e_2^-, e_3^-; e_1^+, e_2^+, e_3^+)$-interchangeable.

(iv) From (ii), $|Z| \geq 3$ means that $G[Z]$ has paths from z_i^- to z_i^+ for all $i = 1, 2, 3$ (but they may not be arc-disjoint). Assume $z_1^- = z_2^-$ since other cases are analogous, and choose a $z_3^- z_3^+$-path P_3 in $G[Z]$. Note that $G[Z] - E(P_3)$ together with additional arcs (z_1^+, z_1^-) and (z_2^+, z_2^-) becomes Eulerian. This means that $G[Z] - E(P_3)$ has arc-disjoint $z_1^- z_1^+$- and $z_2^- z_2^+$-paths, where $z_1^- = z_2^-$. This contradicts that Z is not $(e_1^-, e_2^-, e_3^-; e_1^+, e_2^+, e_3^+)$-interchangeable. □

6.2. Proper 6-cuts in G^*. We call a 6-cut Z *proper* if

(a) $|Z| \geq 3$,

(b) $Z \cap (X \cup Y) = \emptyset$,

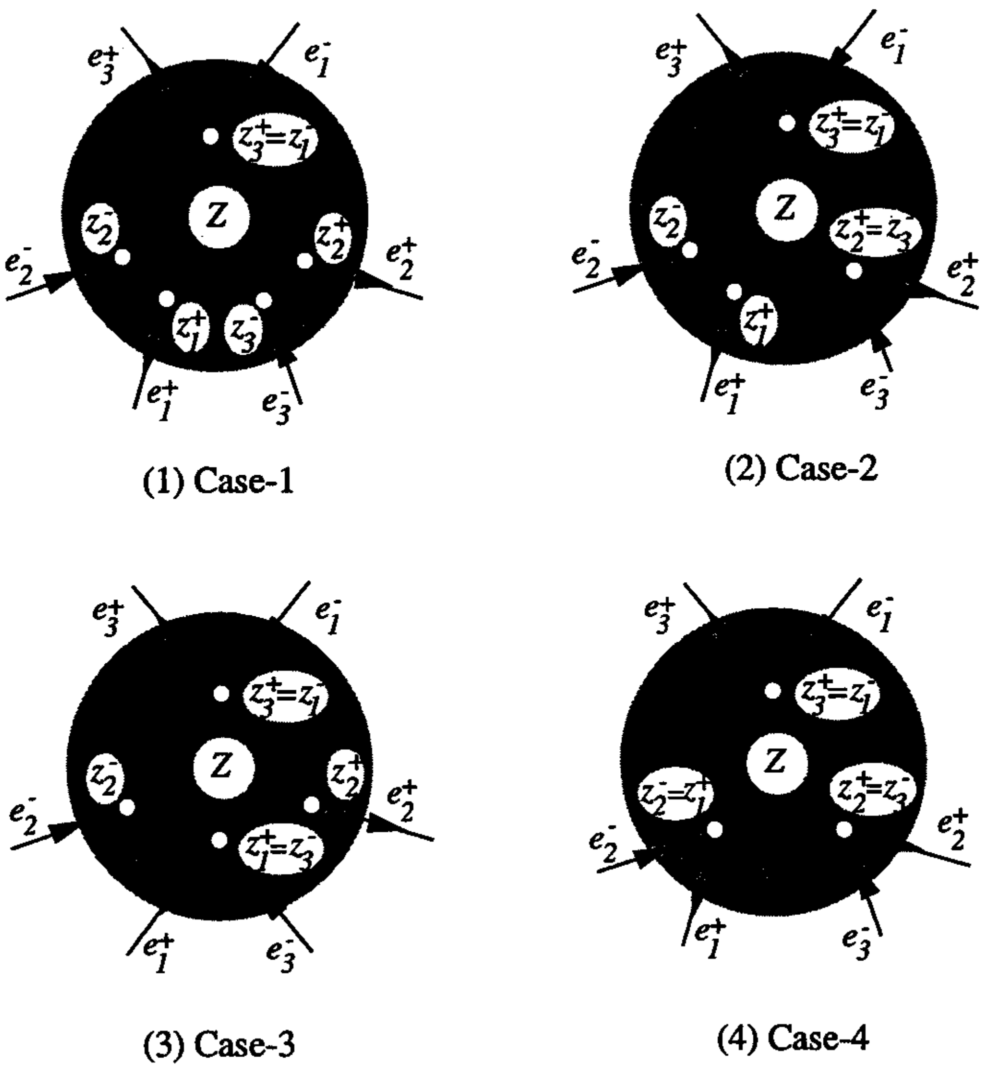


FIG. 6. *Four possible cases for a proper* 6*-cut* Z *in* $(G^*; X, Y)$.

(c) Z contains a vertex z such that $(u, z) \in \delta^-(Z)$ and $(z, v) \in \delta^+(Z)$.

In this subsection, we prove that any proper 6-cut Z induces a triangle in the minimum counterexample $(G^*; X, Y)$.

Let Z be a proper 6-cut in the minimum counterexample $(G^*; X, Y)$ for which $e_i^+, e_i^-, z_i^+, z_i^-$ $(i = 1, 2, 3)$ are defined by (6.3). As Z is not fully interchangeable by Lemma 6.2, assume that Z is not $(e_1^-, e_2^-, e_3^-; e_1^+, e_2^+, e_3^+)$-interchangeable without loss of generality. From condition (c) and Lemma 6.4(iii), $z_i^- = z_j^+$ holds for some $i \neq j$. Here we assume without loss of generality that $z_1^- = z_3^+$ (if necessary, exchange the indices $i = 2, 3$). By Lemma 6.4(iii) and (iv), we have the following four possible cases.

Case 1. $z_2^-, z_3^-, z_2^+, z_3^+$ are all distinct (see Fig. 6(1)).

Case 2. $z_2^+ = z_3^-$ and $z_2^- \neq z_1^+$ (or symmetrically, $z_2^- = z_1^+$ and $z_2^+ \neq z_3^-$) (see Fig. 6(2)).

Case 3. $z_1^+ = z_3^-$ and $z_2^- \neq z_2^+$ (see Fig. 6(3)).

Case 4. $z_2^+ = z_3^-$ and $z_2^- = z_1^+$ (see Fig. 6(4)).

Now let H_Z^* be the Eulerian digraph obtained from $G^*[Z]$ as follows (see Fig. 7):

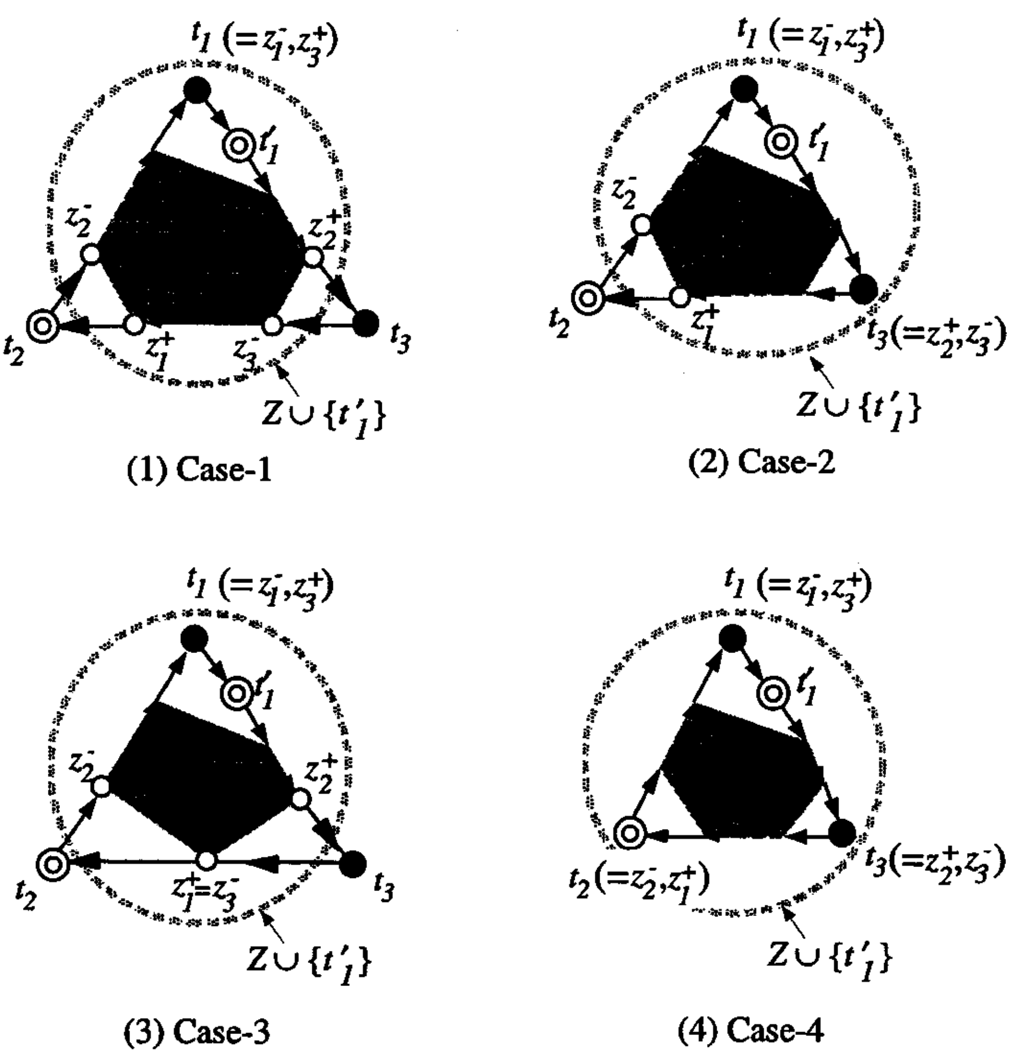


FIG. 7. *The instance* $(H_Z^*; \tilde{X}, \tilde{Y})$ *transformed from* $G^*[Z]$.

1. For $i = 1, 2$, if z_i^+ and z_{i+1}^- are distinct, add a new vertex t_{i+1} together with new arcs (z_i^+, t_{i+1}) and (t_{i+1}, z_{i+1}^-); if $z_i^+ = z_{i+1}^-$, then let $t_{i+1} = z_{i+1}^-(= z_i^+)$.
2. Let $t_1 = z_1^-(= z_3^+)$.
3. Replace the arc (t_1, v) with two arcs (t_1, t_1') and (t_1', v), inserting a new vertex t_1' between t_1 and v.

Define sets of terminals $\tilde{X} = \{t_1, t_3\}$ and $\tilde{Y} = \{t_1', t_2\}$. Obviously, instance $(H_Z^*; \tilde{X}, \tilde{Y})$ is feasible if and only if Z is $(e_1^-, e_2^-, e_3^-; e_1^+, e_2^+, e_3^+)$-interchangeable. Thus, $(H_Z^*; \tilde{X}, \tilde{Y})$ must be infeasible. Note that H_Z^* contains at most $|Z| + 3$ $(< |Z| + |X \cup Y| \leq n^*)$ vertices, where n^* is the number of vertices in G^*. The next lemma summarizes the properties of H_Z^*.

LEMMA 6.5. *Let* $(G^*; X, Y)$ *be the minimum counterexample, and let* Z *be a proper 6-cut in* $(G^*; X, Y)$, *which is not* $(e_1^-, e_2^-, e_3^-; e_1^+, e_2^+, e_3^+)$*-interchangeable, and* $(H_Z^*; \tilde{X}, \tilde{Y})$ *be the instance defined in the above. Then the following properties* (i)–(iv) *hold in all the above cases.*

(i) $(H_Z^*; \tilde{X}, \tilde{Y})$ *is infeasible.*
(ii) $(H_Z^*; \tilde{X}, \tilde{Y})$ *is connected and irreducible and has no 2-cut* W *such that* $|W \cap \tilde{X}| = |W \cap \tilde{Y}| = 1$ *and* $\min\{|W|, |V(H_Z^*) - W|\} \geq 3$.
(iii) $(H_Z^*; \tilde{X}, \tilde{Y})$ *has an IPR.*

(iv) *Z is $(e_1^-, e_2^-, e_3^-; e_{j_1}^+, e_{j_2}^+, e_{j_3}^+)$-interchangeable for any choice of j_1, j_2, j_3 from $\{1,2,3\}$ except $(j_1, j_2, j_3) = (1,2,3)$.*

Proof. In what follows, we consider all four cases simultaneously.

(i) Already proved.

(ii) Since $G^*[Z]$ is connected by Lemma 6.4(i), H_Z^* is connected. Assume that $(H_Z^*; \tilde{X}, \tilde{Y})$ has a reducible cut W. If $W \cap (\tilde{X} \cup \tilde{Y}) = \emptyset$, then W would also be a reducible cut in $(G^*; X, Y)$, which is a contradiction. Therefore, W must be a 2-cut in $(H_Z^*; \tilde{X}, \tilde{Y})$ such that $|W| \geq 2$ and $|W \cap (\tilde{X} \cup \tilde{Y})| = 1$. Since no such W with $|W| = 2$ attains $|\delta(W; H_Z^*)| = 2$ as easily checked, we further assume $|W| \geq 3$. Let $\{t^*\} = W \cap (\tilde{X} \cup \tilde{Y})$. Clearly, $|\delta(W; H_Z^*)| = 2$ implies that $|\delta(W'; G^*)| \leq 4$ holds for $W' = W - \{t^*\} \subseteq V$. This and $|W'| = |W| - 1 \geq 2$ mean that W' is a reducible 2- or 4-cut in $(G^*; X, Y)$, which is a contradiction. Therefore, $(H_Z^*; \tilde{X}, \tilde{Y})$ is irreducible. Assume that $(H_Z^*; \tilde{X}, \tilde{Y})$ has a 2-cut W with $|W \cap \tilde{X}| = |W \cap \tilde{Y}| = 1$ and $\min\{|W|, |V(H_Z^*) - W|\} \geq 3$, and let $t_1' \in W$ without loss of generality. Then $|W| \geq 3$ implies that $(W - (\tilde{X} \cup \tilde{Y})) \cup \{z_3^+\}$ is a reducible 4-cut in G^*, contradicting irreducibility of $(G^*; X, Y)$.

(iii) The instance $(H_Z^*; \tilde{X}, \tilde{Y})$ is infeasible by (i) and is connected and irreducible by (ii). Since H_Z^* contains at most $|Z| + 3 < n^*$ vertices, the instance $(H_Z^*; \tilde{X}, \tilde{Y})$ has an IPR by the minimality assumption on G^* and by Lemma 3.6 (note that terminals $t_1 \in \tilde{X}$ and $t_1' \in \tilde{Y}$ are adjacent).

(iv) Let B be the directed cycle of the outer face in an IPR of $(H_Z^*; \tilde{X}, \tilde{Y})$, where B visits t_1, t_1', t_3, t_2 in this order, and let $B(u,v)$ denote the uv-path on B, where $B(u,u)$ means a path of null length. Note that $(j_1, j_2, j_3) \neq (1,2,3)$ implies (a) $j_1 = 3$, (b) $j_1 = 2$, or (c) $j_1 = 1$ and $j_2 = 3$. If $|V(H_Z^*)| = 4$, then only Case 4 can occur and the IPR is a cycle of length 4 visiting t_1, t_1', t_3, t_2 in this order. In this case, we can easily check by inspection that (iv) holds. We then assume $|V(H_Z^*)| \geq 5$. Since $(H_Z^*; \tilde{X}, \tilde{Y})$ has no 2-cut W stated in the above (ii) and $|V(H_Z^*)| \geq 5$ holds, $V(H_Z^*) - (\tilde{X} \cup \tilde{Y})$ induces a connected component in $H_Z^* - E(B)$ by Lemma 3.5.

(a) $j_1 = 3$. We first take a $z_1^- z_3^+$-path P_A of null length in H_Z^*. We then consider path $B(z_2^+, z_2^-)$, which contains a $z_3^- z_1^+$-path $P_B = B(z_3^- z_1^+)$, and remove the arcs of $E(B(z_2^+, z_2^-))$ from H_Z^*. Now $indeg(u) = outdeg(u)$ holds for all $u \in V(H_Z^*) - \{z_2^+, z_2^-\}$. Then, the set $E(H_Z^*) - E(B(z_2^+, z_2^-))$ of remaining arcs can be regarded as a $z_2^- z_2^+$-path P_C. Therefore, Z is $(e_1^-, e_2^-, e_3^-; e_3^+, e_2^+, e_1^+)$-interchangeable. To show the $(e_1^-, e_2^-, e_3^-; e_3^+, e_1^+, e_2^+)$-interchangeability, it suffices to prove that the above $z_3^- z_1^+$-path $P_B = B(z_3^- z_1^+)$ and $z_2^- z_2^+$-path P_C have a common vertex by which we can reconstruct arc-disjoint $z_3^- z_2^+$-path and $z_2^- z_1^+$-path. Now since $V(H_Z^*) - (\tilde{X} \cup \tilde{Y})$ induces a connected component in $H_Z^* - E(B)$, we obtain $V(P_B) \cap V(P_C) \neq \emptyset$.

(b) $j_1 = 2$. It is easy to see that $z_1^- z_2^+$-path $P_A = B(z_1^-, z_2^+)$ and path $B(z_2^+, z_2^-)$ (which contains a $z_3^- z_1^+$-path $P_B = B(z_3^- z_1^+)$) and $z_2^- z_3^+$-path $P_C = E(H_Z^*) - E(B(z_1^-, z_2^-))$ are arc-disjoint. Therefore, Z is $(e_1^-, e_2^-, e_3^-; e_2^+, e_3^+, e_1^+)$-interchangeable. By the connectedness of $V(H_Z^*) - (\tilde{X} \cup \tilde{Y})$ in $H_Z^* - E(B)$, $V(P_B) \cap V(P_C) \neq \emptyset$ holds and arc-disjoint $z_3^- z_3^+$-path and $z_2^- z_1^+$-path can be reconstructed from P_B and P_C, implying $(e_1^-, e_2^-, e_3^-; e_2^+, e_1^+, e_3^+)$-interchangeability.

(c) $j_1 = 1$ and $j_2 = 3$. Consider a $z_2^- z_3^+$-path $P_A = B(z_2^-, z_3^+)$ path $B(z_2^+, z_2^-)$ (which contains a $z_3^- z_1^+$-path $P_B = B(z_3^- z_1^+)$), and $z_1^- z_2^+$-path $P_C = E(H_Z^*) - E(B(z_2^+, z_1^-))$. These three paths are arc-disjoint. By the connectedness of $V(H_Z^*) - (\tilde{X} \cup \tilde{Y})$ in $H_Z^* - E(B)$, $V(P_B) \cap V(P_C) \neq \emptyset$ holds and arc-disjoint $z_3^- z_2^+$-path and $z_1^- z_1^+$-path can be reconstructed from P_B and P_C, implying $(e_1^-, e_2^-, e_3^-; e_1^+, e_3^+, e_2^+)$-interchangeability. $\square$

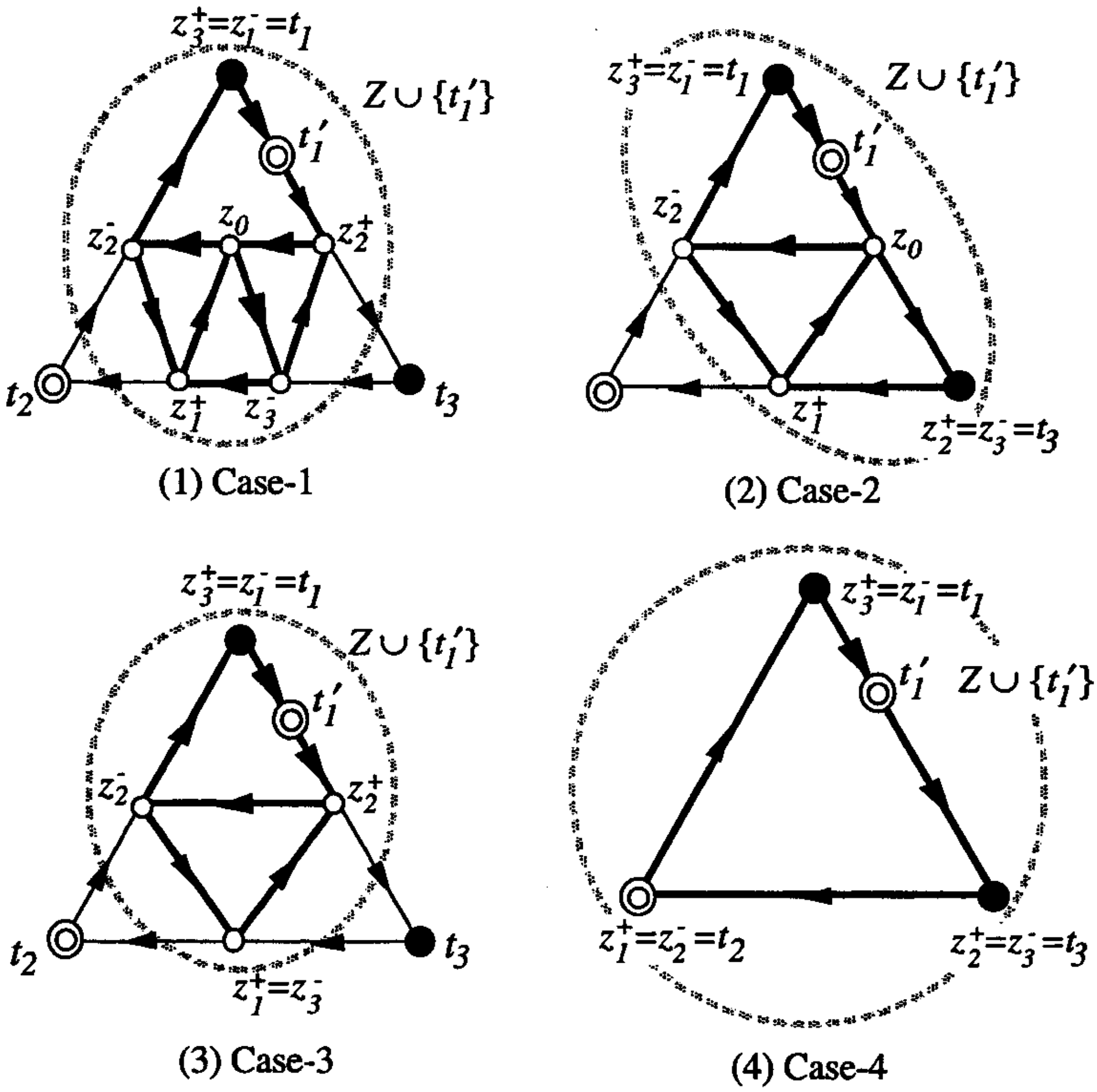


FIG. 8. *The smallest IPRs of H_Z^*.*

Figure 8 shows the smallest IPR of H_Z^* in Cases 1–4. Let $H_Z^*[Z\cup\{t'\}]$ be the subdigraph induced by $Z\cup\{t'\}$ from the smallest IPR of H_Z^* in Fig. 8, and let $H_Z^\#$ be the digraph obtained from $H_Z^*[Z\cup\{t'\}]$ by deleting the vertex t_1' (merging the two arcs $(z_3^+,t_1'),(t_1',v)$ into an arc (z_3^+,v)). Let us consider the digraph $G_{H_Z}^\#$ obtained from the minimum counterexample G^* by replacing $G^*[Z]$ by $H_Z^\#$, as shown in Fig. 9. Let $Z_0=V(H_Z^\#)$. Let Z_0' be the set of vertices $u\in Z_0$ with $|\delta(\{u\};H_Z^\#)|=2$ and $z_0\in Z_0$ be the vertex with $|\delta(\{u\};H_Z^\#)|=4$ in Cases 1 and 2. Note that each vertex in Z_0' is either z_i^+ or z_i^- for some i.

LEMMA 6.6. *For a proper 6-cut Z in the minimum counterexample $(G^*;X,Y)$, which is not $(e_1^-,e_2^-,e_3^-;e_1^+,e_2^+,e_3^+)$-interchangeable, let $G_Z^\#$ be defined as above. Then the following properties* (i)–(iv) *hold in all Cases* 1–4 *(defined in the beginning of this subsection).*

(i) *$(G_Z^\#;X,Y)$ is infeasible.*

(ii) *$(G_Z^\#;X,Y)$ is connected and irreducible.*

(iii) *$(G^*;X,Y)$ has an IPR if $(G_Z^\#;X,Y)$ has an IPR.*

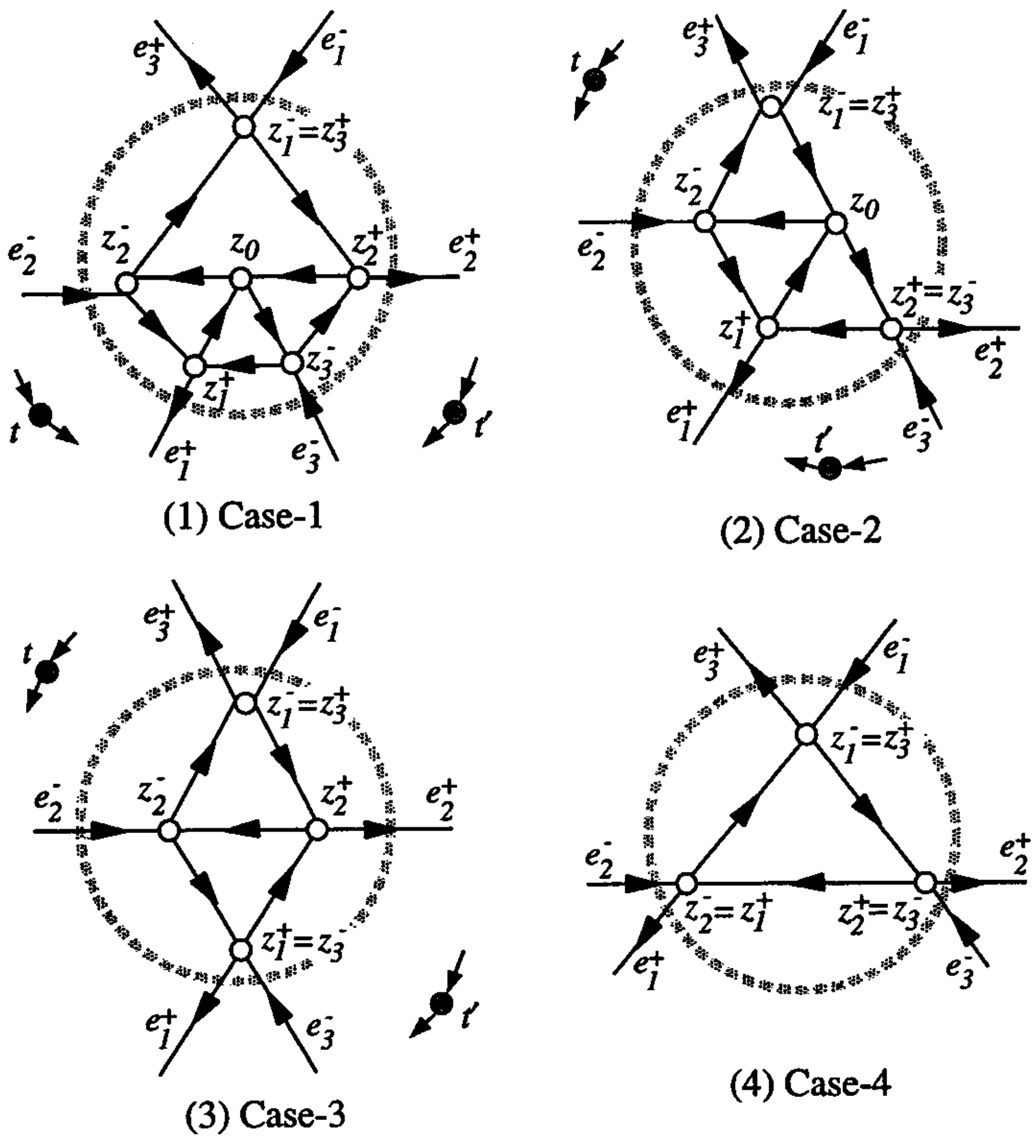


FIG. 9. *The instance $G_Z^{\#}$ obtained from G^* by replacing $G^*[Z]$ with $H_Z^{\#}$.*

(iv) *$G^*[Z] = H_Z^{\#}$ holds.*

Proof. (i) By Lemma 6.5(iv), the 6-cut Z in G^* is $(e_1^-, e_2^-, e_3^-; e_{j_1}^+, e_{j_2}^+, e_{j_3}^+)$-interchangeable for any choice of j_1, j_2, j_3 from $\{1,2,3\}$, except $(j_1, j_2, j_3) = (1,2,3)$. Then it is easy to see that the corresponding 6-cut $Z_0 = V(H_Z^{\#})$ also has the same interchangeability in $G_Z^{\#}$, implying that $(G_Z^{\#}; X, Y)$ is feasible if and only if $(G^*; X, Y)$ is feasible. Since $(G^*; X, Y)$ is infeasible, $(G_Z^{\#}; X, Y)$ is also infeasible.

(ii) Clearly, $(G_Z^{\#}; X, Y)$ is connected since $(G^*; X, Y)$ is. We apply Lemma 6.1 to $(G_Z^{\#}; X, Y)$ and 6-cut $Z_0 = V(H_Z^{\#})$. Clearly, $Z_0 \cap (X \cup Y) = \emptyset$ and $|Z_0| \geq 3$, and $\delta(Z_0)$ contains no multiple arcs, satisfying conditions (i) and (iv) of Lemma 6.1. We see by inspection that there is no reducible cut $W \subseteq Z_0$ and from the irreducibility of $(G^*; X, Y)$ that there is no reducible cut W with $W \supseteq Z_0$ or $W \cap Z_0 = \emptyset$, satisfying conditions (ii) and (iii) of Lemma 6.1. Therefore, $(G_Z^{\#}; X, Y)$ is irreducible by Lemma 6.1.

(iii) Consider Case 1 (other cases can be treated analogously). Assume that $(G_Z^{\#}; X, Y)$ has an IPR in which we assume without loss of generality that arcs

$e_3^+, e_1^-, e_2^+, e_3^-, e_1^+, e_2^-$ appear in this order along the cycle $\{z_3^+ = z_1^-, z_2^+, z_3^-, z_1^+, z_2^-\}$ (recall that, in an IPR, the arcs incident to each vertex are alternately oriented out and in). By Lemma 6.5(iii), $(H_Z^*; \tilde{X}, \tilde{Y})$ has an IPR in which we can assume without loss of generality that all terminals t_1, t_1', t_3, t_2 appear in this order along the cycle of the outer face, and hence the arcs $e_3^+, e_1^-, e_2^+, e_3^-, e_1^+, e_2^-$ appear in the same way as in the IPR of $(G_Z^\#; X, Y)$. This implies that $H_Z^\#$ in $G_Z^\#$ can be replaced with H_Z^* so that the resulting digraph G^* also has an IPR.

(iv) From (i)–(iii) and the assumption on G^*, $(G_Z^\#; X, Y)$ is an irreducible infeasible instance, but has no IPR. Clearly, by $|V(G^\#)| \geq |Z_0| + |X \cup Y| \geq 7$, $(G_Z^\#; X, Y)$ is also a counterexample to Theorem 3.7. Then by the minimality of G^*, $|Z_0| = |Z|$. By inspection, we see that Z with $|Z| = |Z_0|$ can induce no other subdigraph than $G^*[Z] = H_Z^\#$ of Fig. 9 in all Cases 1–4. □

In what follows, we strengthen Lemma 6.6(iv) and show that any proper 6-cut Z in G^* induces a triangle, i.e., none of Cases 1, 2 or 3 occurs. A proper 6-cut Z in $(G^*; X, Y)$ is called *maximal* if there is no proper 6-cut Z' with $Z \subset Z'$.

LEMMA 6.7. *Let Z be a maximal proper 6-cut in the minimum counterexample $(G^*; X, Y)$, defined by* (6.3). *If Z satisfies one of Case* 1, *Case* 2, *or Case* 3 *(i.e., $G^*[Z] = H_Z^*$ of Fig,* 9(1), (2), *and* (3), *respectively), then the following properties* (i)–(v) *hold.*

(i) *In Case* 1, *there is no pair of terminals $t, t' \in X \cup Y$ such that (t, z_2^-), (z_1^+, t), (t', z_3^-), $(z_2^+, t') \in \delta(Z)$. In Case* 2, *there is no pair of terminals $t, t' \in X \cup Y$ such that $(t, z_2^-), (z_3^+, t)$, $(t', z_3^-), (z_1^+, t') \in \delta(Z)$. In Case* 3, *there is no pair of terminals $t, t' \in X \cup Y$ such that $(t, z_2^-), (z_3^+, t)$, $(t', z_3^-), (z_2^+, t') \in \delta(Z)$.*

(ii) *In Case* 1, *assume that there is no terminal t with $(t, z_2^-), (z_1^+, t) \in \delta(Z)$ (without loss of generality by* (i)*). Then the instance $(G'; X, Y)$ obtained from $(G^*; X, Y)$ by splitting off arcs e_2^- and (z_2^-, z_1^+) at z_2^- is infeasible and irreducible.*

(iii) *In Case* 2, *assume that there is no terminal t with $(t, z_2^-), (z_3^+, t) \in \delta(Z)$ (without loss of generality by* (i)*). Then the instance $(G'; X, Y)$ obtained from $(G^*; X, Y)$ by splitting off arcs (z_2^-, z_3^+) and e_3^+ at z_3^+ $(= z_1^-)$ is infeasible and irreducible.*

(iv) *In Case* 3, *assume that there is no terminal t with $(t, z_2^-), (z_3^+, t) \in \delta(Z)$ or $(t, z_1^-), (z_2^+, t) \in \delta(Z)$ (without loss of generality by* (i)*). Then the instance $(G'; X, Y)$ obtained from $(G^*; X, Y)$ by splitting off arcs (z_2^-, z_3^+) and e_3^+ at z_3^+ $(= z_1^-)$ is infeasible and irreducible.*

(v) *Let $(G'; X, Y)$ be the instance of* (ii) *of Case* 1 *(resp.,* (iii) *of Case* 2 *and* (iv) *of Case* 3*). Then $(G^*; X, Y)$ has an IPR if $(G'; X, Y)$ has an IPR.*

Proof. (i) Assume that there are such terminals t and t' in Cases 1, 2, and 3. By Lemma 3.3(ii), terminals t and t' have degree 2 and $Z \cup \{t, t'\}$ is a 2-cut. In Case 1, G^* would have a cut vertex $z_1^- = z_3^+$ (see Fig. 9(1)), which contradicts Lemma 5.1(i). Then consider Cases 2 and 3. By Lemma 3.3(i), $|\{t, t'\} \cap X| = |\{t, t'\} \cap Y| = 1$ holds, and assume $t = x_1$ and $t' = y_1$ without loss of generality. Furthermore, by Lemma 5.1(ii), we obtain $V - (Z \cup \{t, t'\}) = \{x_2, y_2\}$. Now G^* has $|Z| + 4 = 9$ vertices in Case 2 and $|Z| + 4 = 8$ vertices in Case 3. By inspection, we see that $(G^*; X, Y)$ in Case 2 is feasible or has an IPR and $(G^*; X, Y)$ in Case 3 is feasible, which is a contradiction.

(ii) Obviously $(G'; X, Y)$ is infeasible. We show that $(G'; X, Y)$ has no multiple arcs. If there are such multiple arcs, then they must be $(u, z_1^+), (z_1^+, u)$ for some vertex $u \in V - Z$, since G^* has no multiple arcs by Lemma 3.3(iv). This means that u is adjacent to both z_2^- and z_1^+ in G^*. By the assumption that there is no terminal t with $(t, z_2^-), (z_1^+, t) \in \delta(Z)$, u is not a terminal. Then $Z \cup \{u\}$ is a proper 6-cut, contradicting the maximality of $|Z|$. Now we apply Lemma 6.1 to $(G'; X, Y)$ and $Z' = Z - \{z_2^-\}$. Clearly, 6-cut Z' satisfies $Z' \cap (X \cup Y) = \emptyset$ and conditions (i) and (iv) of Lemma 6.1. From the irreducibility of G^*, condition (iii) of Lemma 6.1 holds for Z'. By inspection, we see that Z' satisfies condition of (ii) of Lemma 6.1. Therefore, $(G'; X, Y)$ is irreducible by Lemma 6.1.

(iii) This proof is analogous to (ii).

(iv) The assumption that there is no terminal t with $(t, z_2^-), (z_3^+, t) \in \delta(Z)$ or $(t, z_1^-), (z_2^+, t) \in \delta(Z)$ in Case 3 does not lose generality, because if a terminal t is adjacent to both z_2^- and z_3^+, then no terminal is adjacent to both z_3^- and z_2^+ by (i) and two vertices z_2^-, z_3^- cannot be adjacent to another terminal. This assumption ensures that $(G'; X, Y)$ has no multiple arcs. The rest of the proof is analogous to (ii).

(v) It should be noted that if an instance has an IPR, then any triangle (if any) in the instance gives rise to a face in its IPR. Let $Z' = Z - \{z_2^-\}$ in Case 1 and $Z' = Z - \{z_1^-\}$ in Cases 2 and 3. It is easy to see that $(G'; X, Y)$ still contains a triangle in $G'[Z']$ in each of Cases 1, 2, and 3, and the vertices on these triangles are uniquely embedded in an IPR. Based on this, we can observe that if $(G'; X, Y)$ has an IPR, then $(G^*; X, Y)$ has an IPR. □

From this lemma, we can conclude that none of Cases 1, 2, or 3 can happen in $(G^*; X, Y)$ as follows. If situations (ii), (iii), or (iv) occurs, then the instance $(G'; X, Y)$ is irreducible and infeasible, as shown in the lemma. Since the instance $(G'; X, Y)$ is smaller than $(G^*; X, Y)$ and $|V(G')| = |V(G^*)| - 1 \geq 7$, it has an IPR by the assumption on G^* and Lemma 3.6. Then, $(G^*; X, Y)$ also has an IPR by Lemma 6.7(v). This is a contradiction. Therefore, only Case 4 is possible for a maximal proper 6-cut Z (note that Lemma 6.7(iv) no longer holds for Case 4, since $G^*[Z]$ has no triangle after splitting off arcs, say, (z_2^-, z_3^+) and e_3^+ at z_3^+). This implies that any maximal proper 6-cut (and hence any proper 6-cut, which is not necessarily maximal) always induces a triangle.

LEMMA 6.8. *Any proper* 6-*cut in* $(G^*; X, Y)$ *induces a triangle.* □

7. Admissible splitting. In this section, we derive a condition for splitting two arcs at a nonterminal vertex to be admissible (defined in section 4) and then show that $(G^*; X, Y)$ always has an admissible splitting.

LEMMA 7.1. *Let* $(G^*; X, Y)$ *be the minimum counterexample, and let* w *be a nonterminal vertex in* G^*, *where* $(s_0, w), (s_1, w), (w, s_2), (w, s_3)$ *are the four arcs incident with* w. *If* s_0 *and* s_2 *are not adjacent, and* s_1 *and* s_3 *are not adjacent, then the instance* $(G_w^*; X, Y)$ *obtained by splitting off* (s_0, w) *and* (w, s_2) *at* w *is connected and irreducible* (*i.e., this splitting is admissible*).

Proof. By Lemma 5.1(i), w is not a cut vertex in G^*, and hence $(G_w^*; X, Y)$ is connected. Assume that $(G_w^*; X, Y)$ has a reducible cut $W \subsetneq V - \{w\}$. Since s_0, s_1, s_2, s_3 are distinct by Lemma 3.3(iv), let $S = \{s_0, s_1, s_2, s_3\}$. We see that $W \cap S = \{s_0, s_2\}$ or $W \cap S = \{s_1, s_3\}$ holds (otherwise W would be reducible in $(G^*; X, Y)$). Without loss of generality, assume $W \cap S = \{s_0, s_2\}$ (see Fig. 10). We consider three cases: (a) $|\delta(W; G_w^*)| = 2$ and $W \cap (X \cup Y) = \emptyset$; (b) $|\delta(W; G_w^*)| = 4$, $W \cap (X \cup Y) = \emptyset$, and $|W| \geq 2$; and (c) $|\delta(W; G_w^*)| = 2$, $|W \cap (X \cup Y)| = 1$, and $|W| \geq 2$.

(a) In this case, $W' = W \cup \{w\}$ satisfies $|\delta(W'; G^*)| = |\delta(W; G_w^*)| + 2 = 4$ and $|W'| \geq 2$, implying that W' was reducible in $(G^*; X, Y)$, which is a contradiction.

(b) $W' = W \cup \{w\}$ satisfies $|\delta(W'; G^*)| = |\delta(W; G_w^*)| + 2 = 6$, and $|W'| \geq 3$. Since arcs $(s_1, w), (w, s_3)$ are adjacent to w, W' is a proper 6-cut in G^*, and by Lemma 6.8

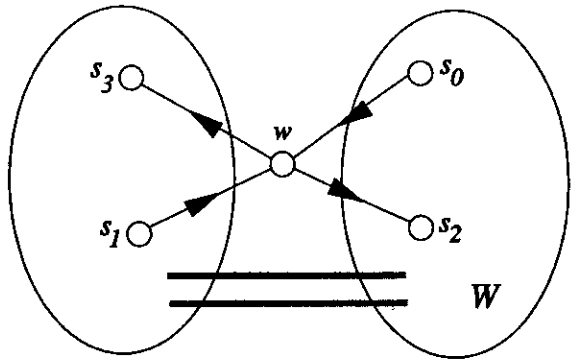


FIG. 10. *Illustration for Lemma* 7.1.

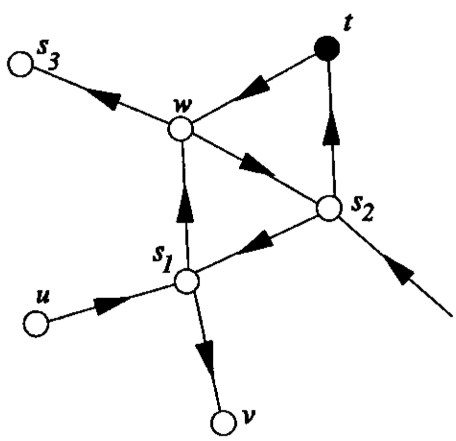


FIG. 11. *Illustration for Lemma* 7.2.

it induces a triangle. However, this contradicts that s_0 and s_2 are not adjacent.

(c) Let $\{t\} = W \cap (X \cup Y)$. We see that $t \neq s_0, s_2$, because otherwise if $t = s_0$ or $t = s_2$, then $W - \{t\}$ is a 4-cut in G^* and $|W - \{t\}| = 1$ must hold by irreducibility of G^*, contradicting that s_0 and s_2 are not adjacent. Then $|W| \geq 3$. This means that $W' = (W - \{t\}) \cup \{w\}$ is a proper 6-cut in G^*, which induces a triangle by Lemma 6.8. However, this again contradicts that s_0 and s_2 are not adjacent. □

LEMMA 7.2. *Let w be a nonterminal vertex adjacent to a terminal t by arc (t, w) in the minimum counterexample $(G^*; X, Y)$, and let $(s_1, w), (w, s_2), (w, s_3)$ be three other arcs incident with w. Then the following property* (i) *or* (ii) *holds.*

(i) *t and s_3 are not adjacent, and s_1 and s_2 are not adjacent (i.e., splitting (t, w) and (w, s_3) at w is admissible by Lemma* 7.1).

(ii) *t and s_2 are not adjacent, and s_1 and s_3 are not adjacent (i.e., splitting (t, w) and (w, s_2) at w is admissible by Lemma* 7.1).

Proof. (a) Consider the case in which t is adjacent to s_2 (i.e., G^* has arc (s_2, t)). Clearly, t cannot be adjacent to s_3. Assume that s_1 and s_2 are adjacent (i.e., G^* has arc (s_2, s_1) by Lemma 6.3(ii)). Let u, v be two other vertices adjacent to s_1, where $(u, s_1), (s_1, v) \in E$ (see Fig. 11). We will show that w (resp., s_2) is not adjacent to u (resp., v). Assume first that w and u are adjacent (i.e., $(w, u) \in E$ by Lemma 6.3(ii), and hence $u = s_3$). If u is a terminal, $W = \{t, u = s_3, w, s_1, s_2\}$ is a 2-cut with $|W \cap (X \cup Y)| = 2$. By Lemma 3.3(i), $|W \cap X| = |W \cap Y| = 1$, and by Lemma 5.1(ii), $|V - W| = 2$, implying $|V - W| + |W| = 7 < n^*$, which is a contradiction. (Recall

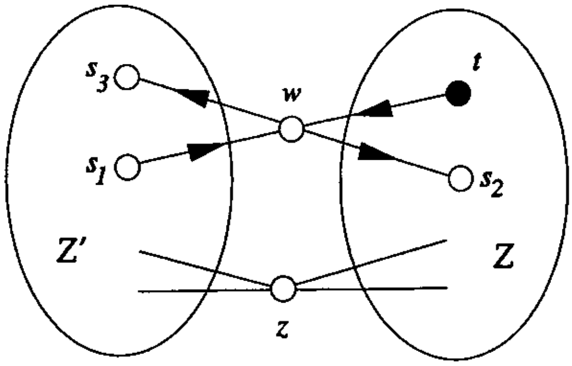


FIG. 12. *Illustration for Lemma* 7.3.

that $n^* \geq 8$ holds by Lemma 3.6, as noted after (4.1).) Then u must be nonterminal. However, in this case $Z = \{u = s_3, w, s_1, s_2\}$ would be a proper 6-cut with four vertices, contradicting Lemma 6.8. Therefore, u is not adjacent to w. Similarly, we see that v is not adjacent to s_2. In other words, splitting off $(u, s_1), (s_1, w)$ at s_1 is admissible by Lemma 7.1 (i.e., the resulting instance $(G^*_{s_1}; X, Y)$ is irreducible by Lemma 7.1). $(G^*_{s_1}; X, Y)$ (containing $n^* - 1 \geq 7$) has an IPR by the assumption on G^* and Lemma 3.6. Since the arcs incident to any terminal lie on the outer face in such IPR, arcs $(s_2, t), (t, w), (w, s_3)$ form part of the boundary of the outer face. Hence, arcs $(u, w), (w, s_2), (s_2, v)$ belong to the boundary of a face in the IPR. This implies that $(G^*; X, Y)$ also has an IPR, which can be obtained by hooking up (u, w) and (s_2, v). This is a contradiction, implying that s_1 and s_2 are not adjacent. Therefore, in this case, we have (i).

(b) If t is adjacent to s_3, we can show that (ii) holds by an analogous argument.

(c) Finally, consider the case in which t is adjacent to neither s_2 or s_3. Assume that s_1 and s_3 are adjacent. We only have to show that s_1 and s_2 are not adjacent. However, if these are adjacent, $Z = \{w, s_1, s_2, s_3\}$ would be a proper 6-cut with four vertices, contradicting Lemma 6.8. $\square$

This lemma says that $(G^*; X, Y)$ always has an admissible splitting at vertex w, which is adjacent to a terminal. We further characterize the digraph obtained by such splitting.

LEMMA 7.3. *Let t be a terminal which is not adjacent to any other terminal, w be a nonterminal vertex adjacent to t by arc (t, w) in the minimum counterexample $(G^*; X, Y)$, and $(s_1, w), (w, s_2), (w, s_3)$ be three other arcs incident with w. Let $G^*_{s_2}$ (resp., $G^*_{s_3}$) denote the instance obtained from G^* by splitting arcs $(t, w), (w, s_2)$ (resp., $(t, w), (w, s_3)$) at w. Then one of these instances is connected and irreducible and has no cut vertex.*

Proof. By Lemma 7.2, one of the instances $G^*_{s_2}$ and $G^*_{s_3}$ is connected and irreducible. Assume without loss of generality that $G^*_{s_2}$ is connected and irreducible, i.e., Lemma 7.2(ii) holds. Then s_1 and s_3 are not adjacent and t and s_2 are not adjacent in G^*. If $G^*_{s_2}$ does not have a cut vertex, then the lemma is shown. Therefore, assume that $G^*_{s_2}$ has a cut vertex z (see Fig. 12).

We first show that w and z are not adjacent in G^* by contradiction. By Lemma 5.1(i), z is not a cut vertex in G^*. Let Z and $Z' = V - \{w, z\} - Z$ be the vertex sets of the two connected components in $G^*_{s_2} - \{z\}$, where $t \in Z$ is assumed. Consider the three possible cases, in which $z = s_1$, $z = s_2$, and $z = s_3$.

First consider the case of $z = s_1$. Then Z' is a 2-cut in G^* and $|Z'| \geq 2$ holds since

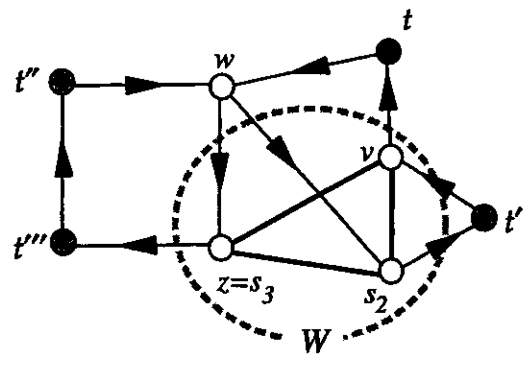


FIG. 13. *The proof of Lemma* 7.3.

s_1 and s_3 are not adjacent. Then $|Z \cap (X \cup Y)| = |Z' \cap (X \cup Y)| = 2$ (otherwise Z' or $V - Z'$ would be a reducible 2-cut in $G^*_{s_2}$). Then, by Lemma 3.3(i), $|Z' \cap X| = |Z' \cap Y| = 1$. If $Z = \{t, s_2\} \subset X \cup Y$, then $|\delta(t; G^*)| = |\delta(s_2; G^*)| = 2$ holds by Lemma 3.3(ii), and in this case we see that $(G^*; X, Y)$ is feasible, which is a contradiction. Then $Z - (X \cup Y) \neq \emptyset$. By this and Lemma 5.1(ii), we have $Z' - (X \cup Y) = \emptyset$ and $|Z'| = 2$. Let t' be the terminal in $Z - \{t\}$. We see that $W = (Z \cup \{w, z\}) - \{t, t'\}$ is a 6-cut in G^* (if W is a 2- or 4-cut, then it would be reducible). By assumption of $n^* \geq 8$, we have $|W| = n^* - 4 \geq 4$, and hence W is a proper 6-cut. However, $|W| = 3$ must hold by Lemma 6.8, which is a contradiction.

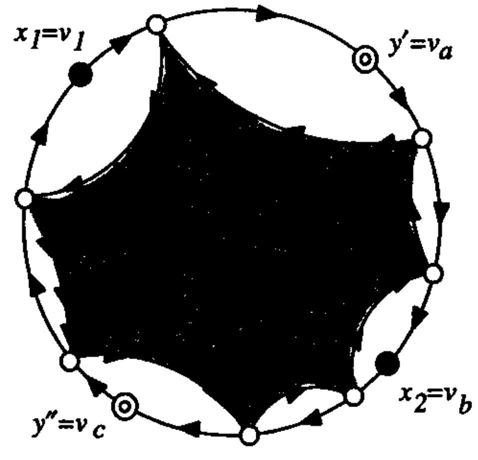
Next, consider the case of $z = s_3$. In this case, we can observe $|Z'| = 2$ and $|Z' \cap X| = |Z' \cap Y| = 1$ in a similar manner as in the case of $z = s_1$. Let t' be the terminal in $Z - \{t\}$ and $Z' = \{t'', t'''\}$. We see that $W = (Z \cup \{z\}) - \{t, t'\}$ is a 6-cut in G^* (if W is a 2- or 4-cut, then it would be reducible). By $n^* \geq 8$, $|W| = n^* - 5 \geq 3$ and W is a proper 6-cut. By Lemma 6.8, W induces a triangle. By considering that t and s_2 are not adjacent and G^* has no multiple arc, G^* is given as the instance shown in Fig. 13, where the triangle $W = \{z, s_2, v\}$ has two possible orientations. For any choice of terminals $\{t, t', t'', t'''\}$ from $X \cup Y$ and orientation of the triangle, we can check that the instance is always feasible, which is a contradiction.

Finally, consider the case of $z = s_2$. In this case, we can obtain $|\delta(Z)| = 2$, $|Z| = 2$, and $|Z \cap X| = |Z \cap Y| = 1$ in a similar manner as in the above cases. This implies that t is adjacent to a terminal $\{t'\} = Z - \{t\}$, contradicting the assumption on t of this lemma.

Therefore, w and z are not adjacent in G^*.

Then, $\{s_1, s_3\}$ and $\{t, s_2\}$ are contained in distinct components in $G^*_{s_2} - \{z\}$ since z is a cut vertex in $G^*_{s_2}$ but not in G^*. That is, s_1 and s_2 (resp., t and s_3) are not adjacent, and hence $G^*_{s_3}$ is connected and irreducible by Lemma 7.2.

We show that $G^*_{s_3}$ has no cut vertex. Let $G^*_{s_3}[Z]$ (resp., $G^*_{s_3}[Z']$) denote the subdigraph of $G^*_{s_3}$ induced by Z (resp., Z'). Note that $G^*_{s_3}[Z] = G^*[Z]$ and $G^*_{s_3}[Z'] = G^*[Z']$. Clearly, all vertices in Z (resp., all vertices in Z') are connected in $G^*_{s_3}[Z]$ (resp., $G^*_{s_3}[Z']$) since otherwise G^* would be reducible. This implies that z is no longer a cut vertex in $G^*_{s_3}$. Assume that $G^*_{s_3}$ has another cut vertex z' ($\neq z$). By a similar argument as above, z' is not equal to any of s_1, s_2, s_3 and $\{s_1, s_2\}$ and $\{t, s_3\}$ are contained in distinct components in $G^*_{s_3} - \{z'\}$. However, this is impossible because if $z' \in Z'$, then t and s_2 are connected in $G^*[Z](= G^*_{s_3}[Z])$ (without using z'), and otherwise if $z' \in Z$, then s_1 and s_3 are connected in $G^*[Z'](= G^*_{s_3}[Z'])$ (without using z'). $\square$



(1)

FIG. 14. *Case 1 in the proof of Lemma 8.1 (the shaded area indicates $\mathcal{R}'$; i.e., B' and its interior).*

8. Hooking up arcs in IPR.

To complete the proof of Theorem 3.7, this section shows that, given an irreducible infeasible instance G that has an IPR, hooking up any two arcs in G cannot yield G^*. More precisely, hooking up two arcs in G makes G satisfy at least one of the conditions in the following lemma, none of which G^* can satisfy.

LEMMA 8.1. *Let $(G=(V,E);X,Y)$ (where $|V|\geq 7$) be an irreducible instance which has an IPR and has no cut vertex. For arc $e=(x_2,u)\in E$ with $x_2\in X$ and any arc $e'=(v,v')\in E$, let $(G_{e,e'};X,Y)$ be the resulting instance obtained by hooking up e and e' with a new vertex w. Then $G_{e,e'}$ is connected and one of the following properties* (i)–(v) *holds.*

(i) *$G_{e,e'}$ has a cut vertex.*

(ii) *$(G_{e,e'};X,Y)$ has a 2-cut Z such that $Z\subseteq V$, $|Z\cap X|=|Z\cap Y|=1$, $|Z|\geq 3$, and $|(V\cup\{w\})-Z|\geq 3$.*

(iii) *$(G_{e,e'};X,Y)$ is reducible.*

(iv) *$(G_{e,e'};X,Y)$ has an IPR.*

(v) *$(G_{e,e'};X,Y)$ is feasible.*

Proof. Assuming that $(G_{e,e'};X,Y)$ satisfy neither (i) nor (ii), we show that $(G_{e,e'};X,Y)$ satisfies one of (iii)–(v). Since G has no cut vertex, we only have to consider IPRs as illustrated in Figs. 14, 15, and 16, which correspond respectively to the following three cases.

Case 1. $(G;X,Y)$ has no 2-cut W such that $|W\cap X|=|W\cap Y|=1$.

Case 2. $(G;X,Y)$ has a 2-cut Z such that $|Z\cap X|=|Z\cap Y|=1$, $Z-(X\cup Y)\neq\emptyset$, and $(V-Z)-(X\cup Y)\neq\emptyset$, where $x_2\in Z$.

Case 3. $(G;X,Y)$ has a 2-cut Z such that $|Z\cap X|=|Z\cap Y|=1$, $Z\subseteq X\cup Y$, or $V-Z\subseteq X\cup Y$, where $x_2\in Z$.

Let $\mathcal{R}$ be the IPR of $(G;X,Y)$, and let B denote the cycle of the outer face of $\mathcal{R}$, which is a simple cycle since G has no cut vertex. Let $v_1,v_2,\ldots,v_p$ $(p=|V(B)|)$ be the vertices that appear along B clockwise, where $v_1=x_1, v_a=y', v_b=x_2$, and $v_c=y''$ $(1<a<b<c)$, and $\{y',y''\}=Y$ are assumed without loss of generality. Let $B(u,v)$ denote the subpath of B from u to v, where $B(u,u)$ means a path of null length. Let $\mathcal{R}'$ denote the planar representation obtained from the IPR of G by eliminating the arcs in $E(B)$ together with $X\cup Y$. We denote components of $\mathcal{R}'$ by

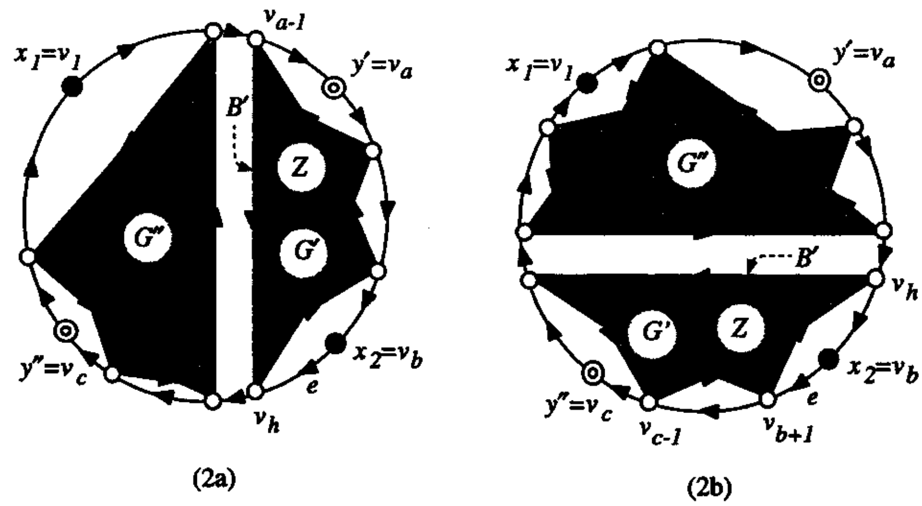


FIG. 15. *Illustration of graph G used in the proof of Lemma* 8.1.

$G', G'', \ldots$ and the directed cycles representing their outer faces by $B', B'', \ldots$. For these cycles, say, B', we denote by $B'(u,v)$ the subpath of B' from u to v. The proof will be given separately for the above three cases.

Case 1 (Fig. 14). In this case, $\mathcal{R}'$ consists of a single component G' by Lemma 3.5. Also, no two terminals in $\mathcal{R}$ are adjacent on B. Note that the directed cycle B' (which may not be simple) visits all vertices in $V(B) - X - Y$ in the order reverse to B (i.e., counterclockwise). Choose $e = (x_2, v_{b+1})$, and partition the arc set $E - e$ into the three subsets

$$\begin{aligned}
E_1 &= \{e'' | e'' \text{ is adjacent to } e\},\\
E_2 &= E(B(v_{b+1}, y'')) \cup E(B'(v_{b+1}, v_{b-1})) - E_1,\\
E_3 &= E - e - E_1 - E_2 \text{ (see Fig. 15).}
\end{aligned}$$

It is easy to see that $(G_{e,e'}; X, Y)$ satisfies (iii) (resp., (iv)) for any $e' \in E_1$ (resp., $e' \in E_2$). We then show that (v) holds for all $e' \in E_3$. Since no two terminals in $\mathcal{R}$ are adjacent on B and $\mathcal{R}$ has no cut vertex, G has at least four nonterminal vertices in $V(B')$.

Case 1a. $e' = (v_{a-1}, y') \in E_3$. Then $G_{e,e'}$ has a $y'y''$-path,

$$P_Y = \langle B(y', v_{a+1}), B'(v_{a+1}, v_{a-1}), (v_{a-1}, w), (w, v_{b+1}), B(v_{b+1}, y'')\rangle.$$

Clearly $G_{e,e'} - E(P_Y)$ has an x_1x_2-path, where e and e' are hooked up with vertex w. $P_X = \langle (x_1, v_2), B'(v_2, v_{b-1}), (v_{b-1}, x_2)\rangle$, which implies that $(G_{e,e'}; X, Y)$ is feasible.

Case 1b. $e' = (v_k, v_{k+1}) \in E(B(y', v_{b-1})) \subseteq E_3$. Then $G_{e,e'}$ has a $y'y''$-path

$$P_Y = \langle B(y', v_k), (v_k, w), (w, v_{b+1}), B(v_{b+1}, y'')\rangle.$$

It is also easy to see that x_1 and x_2 are still connected in $G_{e,e'} - E(P_Y)$, implying that $(G_{e,e'}; X, Y)$ is feasible by Lemma 2.1.

Case 1c. $e' \in E_3 - \{(v_{a-1}, y')\} - E(B(y', v_{b-1}))$. In this case, consider the following $y'y''$-chain in G:

$$Q_Y = \langle B(y', v_{b-1}), B'(v_{b+1}, v_{b-1}), B(v_{b+1}, y'')\rangle.$$

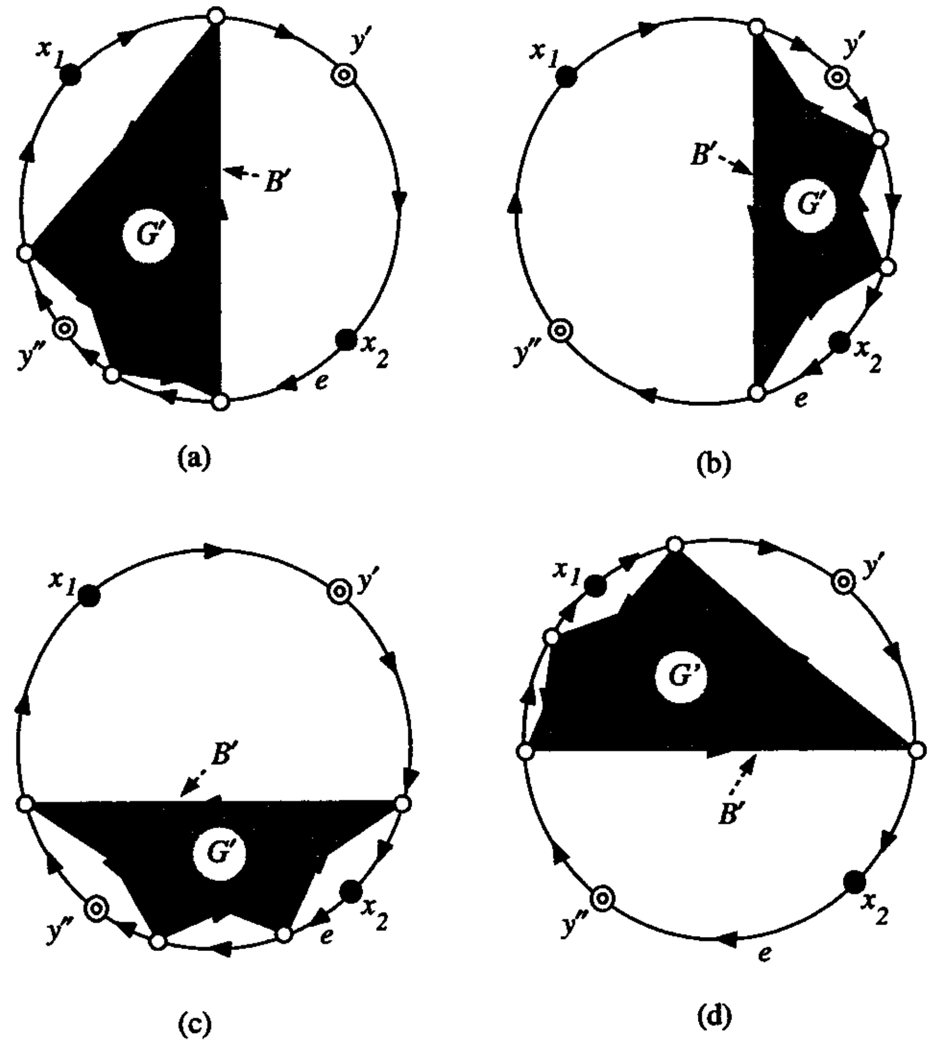


FIG. 16. *Illustration for the proof of Case* 1c *in Lemma* 8.1.

Clearly, $e' \notin E(Q_Y)$ holds, and $G_{e,e'}$ still has $y'y''$-chain Q_Y.

Let $\mathcal{H}$ be the IPR resulting from $\mathcal{R}$ by removing the arcs in $E(B(v_{a-1}, v_{c+1})) \cup E(B'(v_{c+1}, v_{b-1}))$ (see Fig. 16). We now claim that x_1 is reachable in $\mathcal{H}$ from any vertex which is located on the boundary B' or in the area surrounded by B'. By $E(\mathcal{H}) \cap E(Q_Y) = \emptyset$, the claim will mean that, for any $e' = (u', v') \in E_3 - \{(v_{a-1}, y')\} - E(B(y', v_{b-1}))$, $G_{e,e'} - E(Q_Y)$ has a $v'x_1$-path (hence, it has an x_2x_1-path). Then, by Lemma 2.1, this will complete the proof that $(G_{e,e'}; X, Y)$ is feasible. To prove the claim, it is sufficient to show that x_1 is reachable from any vertex on B', since any vertex inside B' is clearly reachable to a vertex on B'.

Partition set $V(B')$ into two subsets $V_1 = V(B'(v_{b-1}, v_{c+1}))$ and $V_2 = V(B') - V_1$. Since two paths $B'(v_{b-1}, v_{c+1})$ and $B(v_{c+1}, v_{a-1})$ remain in $\mathcal{H}$, x_1 is clearly reachable in $\mathcal{H}$ from any vertex $v \in V_1$. We then show that x_1 is reachable from any vertex $v \in V_2$ in $\mathcal{H}$. Let us denote the vertex set $V(B'(v_{c+1}, v_{b-1}))$ $(= V_2 \cup \{v_{c+1}, v_{b-1}\})$ by $\{u_0, u_1, u_2, \ldots, u_q, u_{q+1}\}$, where B' visits these vertices $u_0, \ldots, u_{q+1}$ in this order. Assume that there is a vertex $u_k \in V_2$ which cannot reach any vertex in V_1 and that u_k has the smallest index among such vertices in V_2. We follow the *leftmost* path P^* from u_k in $\mathcal{H}$ until the path returns to u_k (note that P^* must come back to u_k since

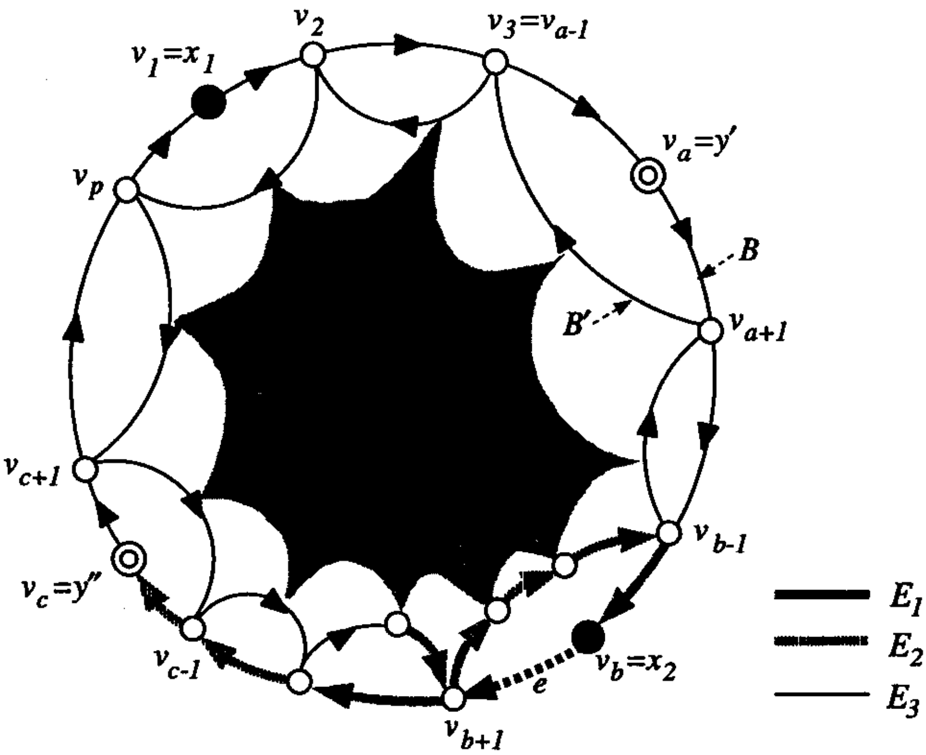


FIG. 17. *Case 2 in the proof of Lemma* 8.1.

$indeg(v) = outdeg(v)$ for all $v \in V - V_1$ in $\mathcal{H}$). Clearly, $|V(P^*)| \geq 2$. Let u_h be the first vertex in V_2 that P^* visits. Note that P^* visits no vertex $u_i \in V_2$ with $i > h$. Consider the set V^* of vertices in $V - V(P^*)$ that are adjacent to a vertex of $V(P^*)$ in $\mathcal{H}$. Since the arcs incident to a nonterminal vertex are alternately oriented in and out from the definition of IPR, all vertices in V^* are located in the inside area surrounded by P^*. In other words, V_1 and $V(P^*) \cup V^*$ are disconnected in $\mathcal{H}$. Thus, removal of the four arcs $\{(u_{k-1}, u_k), e_k, (u_h, u_{h+1}), e_h\}$ from G disconnects V_1 and $V(P^*) \cup V^*$, where e_k (resp., e_h) is the arc in $B(x_2, y'')$ such that e_k and (u_{k-1}, u_k) (resp., e_h and (u_h, u_{h+1})) belong to the same face in $\mathcal{R}$. This contradicts that $(G; X, Y)$ has no reducible 4-cut. Therefore, x_1 is reachable from any vertex in $V_1 \cup V_2$.

Case 2 (Fig. 17). In this case, $\mathcal{R}'$ consists of two components G' and G'', where G' (resp., G'') is induced by $V' = Z - (X \cup Y)$ (resp., $V'' = (V - Z) - (X \cup Y)$). There are two cases. Case 2a: $y', x_2 \in Z$ and $y'', x_1 \in V - Z$ (see Fig. 17(2a)), and Case 2b: $x_2, y'' \in Z$ and $x_1, y' \in V - Z$ (see Fig. 17(2b)). In both Cases 2a and 2b, e' must be chosen from $E(G[V - Z])$ to avoid the case (ii) in the lemma statement.

Case 2a(i). $e' \in E(B(x_2, y'')) \cap E(G[V - Z])$. It is easy to see that $(G_{e,e'}; X, Y)$ has an IPR.

Case 2a(ii). $e' \in E(G[V-Z]) - E(B(x_2, y''))$. Let v_h be the vertex in $B(y', y'') \cap Z$ with the largest index, where $b < h < c$ must hold (otherwise V' would be a reducible cut or a cut vertex). Then $G_{e,e'}$ has a $y'y''$-path

$$P_Y = \langle (y', v_{a+1}), B'(v_{a+1}, v_h), B(v_h, y'') \rangle.$$

Furthermore, it is also easy to see that x_1 and x_2 are connected in $G_{e,e'} - E(P_Y)$. Therefore, $(G_{e,e'}; X, Y)$ is feasible by Lemma 2.1.

Case 2b(i). $e' = (v_k, v_{k+1}) \in E(B(y', x_2)) \cap E(G[V - Z])$. Then $G_{e,e'}$ has a $y'y''$-path $P_Y = \langle B(y', v_k), (v_k, w), (w, v_{b+1}), B(v_{b+1}, y'') \rangle$. Obviously, x_1 and x_2 remain

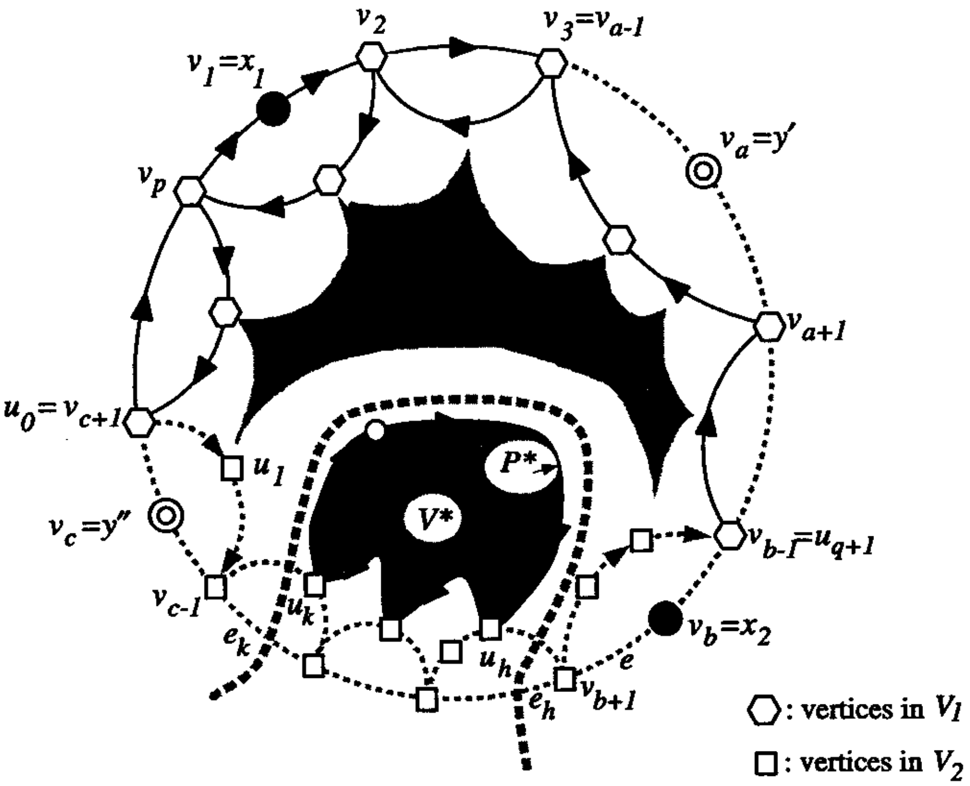


FIG. 18. *Case* 3 *in the proof of Lemma* 8.1.

connected in $G_{e,e'} - E(P_Y)$, implying by Lemma 2.1 that $(G_{e,e'}; X, Y)$ is feasible.

Case 2b(ii). $e' \in E(G[V-Z]) - E(B(y', x_2))$. Let v_h be the vertex in $B(y', y'') \cap Z$ with the smallest index, where $a < h < b$ must hold. Then $G_{e,e'}$ has a $y'y''$-path $P_Y = \langle B(y', v_h), B'(v_h, v_{c-1}), (v_{c-1}, y'')\rangle$. Since it is obvious that x_1 and x_2 are connected in $G_{e,e'} - E(P_Y)$, $(G_{e,e'}; X, Y)$ is feasible by Lemma 2.1.

Case 3 (Fig. 18). In this case, there are two terminals $x^* \in X$ and $y^* \in Y$ which are adjacent on B, and $\mathcal{R}'$ has a single component G'.

Case 3a. $Z = \{y', x_2\}$ (i.e., $(y', x_2) \in E(B)$) (see Fig. 18(3a)). This case can be treated in the same manner as in Case 1, where the corresponding partition of $E - e$ is defined by $E_1 = \{e'' | e'' \text{ is adjacent to } e\}$, $E_2 = \{(v_{a-1}, y')\} \cup E(B'(v_{b+1}, v_{a-1})) \cup E(B(v_{b+1}, y'')) - E_1$, and $E_3 = E - e - E_1 - E_2$ and $y'y''$-chain Q_Y in Case 1c is chosen as $Q_Y = \langle (v_{a-1}, y'), B'(v_{b+1}, v_{a-1}), B(v_{b+1}, y'')\rangle$.

Case 3b. $V - Z = \{y'', x_1\}$ (i.e., $(y'', x_1) \in E(B)$) (see Fig. 18(3b)). This case can be treated in the same manner as in Case 2a.

Case 3c. $V - Z = \{x_1, y'\}$ (i.e., $(x_1, y') \in E(B)$) (see Fig. 18(3c)). This case can be treated in the same manner as in Case 2b.

Case 3d. $Z = \{x_2, y''\}$ (i.e., $(x_2, y'') \in E(B)$) (see Fig. 18(3d)). This case can be treated in the same manner as in Case 1, where the corresponding partition of $E - e$ is defined by $E_1 = \{e'' | e'' \text{ is adjacent to } e\}$, $E_2 = E(B'(v_{c+1}, v_{b-1}))$, and $E_3 = E - e - E_1 - E_2$ and $y'y''$-chain Q_Y in Case 1c is chosen as $Q_Y = \langle B(y', v_{b-1}), B'(v_{c+1}, v_{b-1}), B(y'', v_{c+1})\rangle$.

Now we are ready to prove Theorem 3.7 by deriving a contradiction from the assumption that a minimum counterexample $(G^*; X, Y)$ exists. G^* must have a nonterminal vertex adjacent to a terminal (otherwise, G^* consists of only terminals, contradicting $n^* \geq 8$). We can assume without loss of generality that $(G^*; X, Y)$ has a terminal x_2 which is not adjacent to any other terminal (because if any terminal is adjacent to some other terminal, then $|\delta(V-(X\cup Y))| \le 4$, and hence $|V-(X\cup Y)| \le 1$ would hold by irreducibility of G^*). Then by Lemma 7.3, two arcs (x_2, w) and (w, s_2) in $(G^*; X, Y)$ can be split off at w so that the resulting instance $(G^*_w; X, Y)$ is still connected and irreducible, and has no cut vertex, where (s_1, w) and (w, s_3) are the other arcs incident to w. In other words, G^* can be obtained from G^*_w by hooking up two arcs $e = (x_2, s_2)$ and $e' = (s_1, s_3)$ after introducing w. We then apply Lemma 8.1 to $G = G^*_w$ and $G_{e,e'} = G^*$. By Lemma 5.1, neither (i) nor (ii) of Lemma 8.1 holds for G^*. Furthermore, none of the remaining (iii)–(v) of Lemma 8.1 is possible by the definition of G^*. This is a contradiction and proves the next lemma.

LEMMA 8.2. *Let $(G; X, Y)$ be an infeasible irreducible instance that satisfies $|V| \ne 6$. Then $(G; X, Y)$ has an IPR.* □

Finally, this proves Theorem 3.7, since, by Lemma 3.2, this is a stronger statement than Theorem 3.7.

9. Complexity results. Based on Theorem 3.7 (or Lemma 8.2 to be more precise), we can test if a given instance $(G; X, Y)$ is feasible or not in polynomial time.

LEMMA 9.1. *Given an instance $(G; X, Y)$, one of its irreducible instances $(G'; X, Y)$ can be found in $O(m + n\log n)$ time, where n and m denote the numbers of vertices and arcs in G, respectively.* □

Before describing the algorithm for computing an irreducible instance, let us review a *cactus representation* [1], a compact representation of all minimum cuts in an undirected graph. A connected undirected graph is called a *cactus* if, for each edge, there is exactly one simple cycle that contains it, where the cycle may be of length 2. Then, in a cactus, two cycles (if any) have at most one common vertex, which is a cut vertex. A vertex with degree 2 in a cactus is called a *leaf vertex*. Given an undirected graph $G = (V, E)$, we map it to a cactus $\Gamma = (W, F)$ by a mapping $\varphi : V \to W$, where φ may not be an onto-mapping. The size of a minimum cut in G (resp., in Γ) is defined by $\lambda(G) = \min\{|\delta(Z; G)| \mid \emptyset \ne Z \subset V\}$ (resp., $\lambda(\Gamma) = \min\{|\delta(S; \Gamma)| \mid \emptyset \ne S \subset W\}$), where $\delta(Z; G)$ denotes the set of edges between Z and $V - Z$ in G (similarly for $\delta(S; \Gamma)$). Clearly, in a cactus $\Gamma = (W, F)$ with $|W| \ge 2$, $\lambda(\Gamma) = 2$ holds.

Let $\mathcal{C}(G) = \{Z \mid \emptyset \ne Z \subset V, |\delta(Z; G)| = \lambda(G)\}$ and $\mathcal{C}(\Gamma) = \{S \mid \emptyset \ne S \subset W, |\delta(S; \Gamma)| = \lambda(\Gamma)\}$ denote the sets of all minimum cuts of G and Γ, respectively. Note that S belongs to $\mathcal{C}(\Gamma)$ if and only if two arcs in $\delta(S; \Gamma)$ belong to the same cycle. In the following description, we use the term "vertex" to denote an element in V and the term "node" to denote an element in W. There may be a node $x \in W$ with $\varphi^{-1}(x) = \emptyset$, which is called an empty node. Define

$$\varphi(Z) \equiv \{\varphi(v) \in W \mid v \in Z\} \quad \text{for } Z \subseteq V \text{ and}$$
$$\varphi^{-1}(S) \equiv \{v \in V \mid \varphi(v) \in S\} \quad \text{for } S \subseteq W.$$

A pair (Γ, φ) of a cactus and a mapping φ is called a cactus representation for $\mathcal{C}(G)$ if it satisfies (i) and (ii) below.

(i) For any cut $Z \in \mathcal{C}(G)$, there exists a cut $S \in \mathcal{C}(\Gamma)$ such that $Z = \varphi^{-1}(S)$ and $V - Z = \varphi^{-1}(W - S)$.

(ii) Conversely, for any 2-cut $S \in \mathcal{C}(\Gamma)$, $Z = \varphi^{-1}(S)$ satisfies $Z \in \mathcal{C}(G)$.

It is known [1] that $G = (V, E)$ always has such a cactus representation $(\Gamma = (W, F), \varphi)$ with $|W| = O(|F|) = O(|V|)$, which can be constructed in $O(|E| + \lambda(G)^2|V|\log|V|)$ time [7]. We say that a cut $Z \in \mathcal{C}(G)$ and a cut $S \in \mathcal{C}(\Gamma)$ correspond to each other if $Z = \varphi^{-1}(S)$ and $V - Z = \varphi^{-1}(W - S)$. Note that if Γ has an empty node, a minimum cut in $\mathcal{C}(G)$ may correspond to more than one minimum cut in $\mathcal{C}(\Gamma)$, while any minimum cut in $\mathcal{C}(\Gamma)$ always corresponds to exactly one minimum cut in $\mathcal{C}(G)$. Obviously, any leaf node $w \in W$ corresponds to a minimum cut in $\mathcal{C}(G)$, and there are at least two leaf nodes in Γ.

LEMMA 9.2. *For an undirected graph $G = (V, E)$ and a designated vertex $t^* \in V$, let $\mathcal{Z} = \{Z_1, Z_2, \dots, Z_q\}$ be the set of all cuts Z_i such that*

(i) *$Z_i \in \mathcal{C}(G)$ and $Z_i \subseteq V - \{t^*\}$,*

(ii) *$|Z_i|$ is maximal subject to* (i) *(i.e., no cut $Z' \in \mathcal{C}(G)$ with $Z_i \subset Z' \subseteq V - \{t^*\}$).*

Then any two cuts $Z_i, Z_j \in \mathcal{Z}$ are mutually disjoint, and the set $\mathcal{Z}$ can be computed in $O(|E| + \lambda(G)^2|V| \log |V|)$ time.

Proof. Consider a cactus representation $(\Gamma = (W, F), \varphi)$ for $\mathcal{C}(G)$. Let $w^* = \varphi(t^*) \in W$, and let Γ have p cycles passing through w^*. In other words, removal of w^* from Γ creates p connected components with node sets W_i, $i = 1, 2, \dots, p$. Let $Z_i = \varphi^{-1}(W_i)$, $i = 1, \dots, p$. Since each W_i is a 2-cut in Γ, we have $W_i \in \mathcal{C}(\Gamma)$ and hence $Z_i \in \mathcal{C}(G)$ by definition of a cactus representation. Hence, each Z_i satisfies condition (i). If there is a cut $Z' \in \mathcal{C}(G)$ such that $Z_i \subset Z' \subseteq V - \{t^*\}$, then there is a cut $W' \in \mathcal{C}(\Gamma)$ such that $Z' = \varphi^{-1}(W')$, $W_i \subset W'$, and $w^* \in W - W'$. However, Γ cannot have any such 2-cut W' separating w^* and W_i by the definition of W'. Therefore, each Z_i satisfies condition (ii). Obviously, $Z_i, \dots, Z_p$ are mutually disjoint by disjointness of $W_1, \dots, W_p$. The stated time complexity follows from the fact that a cactus representation $(\Gamma = (W, F), \varphi)$ with $|W| + |F| = O(|V|)$ can be obtained in $O(|E| + \lambda(G)^2|V| \log |V|)$ time [7], and computing connected components in $\Gamma - w^*$ can be done in $O(|W| + |F|) = O(|V|)$ time. □

Proof of Lemma 9.1. Given an instance $(G; X, Y)$, where $n = |V|$ and $m = |E|$, the following algorithm applies all reductions of type (1), (2), and (3), defined in the beginning of section 3.

1. Type (1) reductions (i.e., 2-cuts Z such that $|Z| \geq 1$ and $Z \cap (X \cup Y) = \emptyset$): We contract four terminals x_1, x_2, y_1, y_2 into a single vertex t^* and ignore arc orientation in G. Let $\overline{G}$ denote the resulting undirected graph. Clearly, $\lambda(\overline{G}) \geq 2$, since G is connected and Eulerian. It is easy to see that a cut $Z \subseteq V - \{x_1, x_2, y_1, y_2\}$ is 2-cut in G if and only if $\lambda(\overline{G}) = 2$, $Z \in \mathcal{C}(\overline{G})$, and $Z \subseteq V(\overline{G}) - \{t^*\}$. We can check if $\lambda(\overline{G}) \geq 2$ in $O(m + n \log n)$ time [7]. If $\lambda(\overline{G}) > 2$, then there is no cut of type (1), and we go to 2. If $\lambda(\overline{G}) = 2$, then by Lemma 9.2 the set $\{Z_1, \dots, Z_p\}$ of these cuts Z with maximal $|Z|$ is uniquely determined and obtained in $O(m + n \log n)$ time. Apply reduction (1) to all cuts Z_i in $(G; X, Y)$. This can be done in $O(m + n)$ time (since $Z_i \cap Z_j = \emptyset$ for $1 \leq i < j \leq p$ if $p \geq 2$). Go to 2 after letting $(G; X, Y)$ be the resulting instance.
2. Type (2) reductions (i.e., 2-cuts Z such that $|Z| \geq 2$ and $|Z \cap (X \cup Y)| = 1$): For each terminal $t \in X \cup Y$, let $\overline{G}_t$ denote the undirected graph obtained from G by contracting the other three terminals $X \cup Y - \{t\}$ into a single vertex $\bar{t}$ and ignoring arc orientations. We easily see that if G has a 2-cut Z with $Z \cap (X \cup Y) = \{t\}$ and $|Z| \geq 2$, then $\lambda(\overline{G}_t) = 2$, $Z \in \mathcal{C}(\overline{G}_t)$, and $Z \subseteq V(\overline{G}) - \{\bar{t}\}$ hold. Then such Z contains t (otherwise, Z would be a cut of type (1), which has been eliminated in the above 1) and is unique if it is maximal (since at most one cut can contain t). Furthermore, such Z can be obtained in $O(m + n \log n)$ time; see Lemma 9.2. We apply reduction (2) to the cut Z in $(G; X, Y)$ in $O(m + n)$ time. This procedure for all four terminals can be done in $O(m + n \log n)$ time. Go to 3 after letting $(G; X, Y)$ be the resulting instance.

3. Type (3) reductions (i.e., 4-cuts Z such that $G[Z]$ is connected, $|Z| \geq 2$, and $Z \cap (X \cup Y) = \emptyset$): Since the current $(G; X, Y)$ has no cut of type (1), any 4-cut $Z \subseteq V - (X \cup Y)$ induces a connected subdigraph $G[Z]$. Let $\overline{G}$ be the undirected graph obtained from $(G; X, Y)$ by contracting four terminals x_1, x_2, y_1, y_2 into a single vertex t^* and ignoring arc orientations. Clearly, $\lambda(\overline{G}) \geq 4$ (otherwise, $(G; X, Y)$ would have a reducible 2-cut). We easily see that a cut $Z \subseteq V - (X \cup Y)$ is 4-cut in G if and only if $\lambda(\overline{G}) = 4$, $Z \in \mathcal{C}(\overline{G})$, and $Z \subseteq V(\overline{G}) - \{t^*\}$. We can check if $\lambda(\overline{G}) \geq 4$ in $O(m + n \log n)$ time. If $\lambda(\overline{G}) > 4$, then there is no cut of type (3) and the current instance $(G; X, Y)$ is irreducible. If $\lambda(\overline{G}) = 4$, then by Lemma 9.2 the set $\{Z_1, \ldots, Z_p\}$ of these cuts Z with maximal $|Z|$ is uniquely determined and is obtained in $O(m + n \log n)$ time. Apply reduction (3) to all these cuts Z_i in $(G; X, Y)$ to obtain an irreducible instance. This can be done in $O(m + n)$ time. □

Given an irreducible instance $(G'; X, Y)$, we can check if it is feasible or not in linear time as follows. If G' has less than 7 vertices, its feasibility can be easily checked in $O(1)$ time (since any irreducible infeasible digraph G' with $|V| < 7$ has $O(1)$ arcs). Otherwise, test if the resulting irreducible instance $(G'; X, Y)$ has an IPR, which can be done in $O(m + n)$ time by using a fast planar drawing algorithm [8]. If it has an IPR, then it is infeasible; otherwise it is feasible. Therefore, we have established the next theorem.

THEOREM 9.3. *Given an instance $(G; X, Y)$, where n and m are the numbers of vertices and arcs, respectively, testing if it is feasible or not can be done in $O(m + n \log n)$ time.* □

We now show that, if a given instance $(G; X, Y)$ is feasible, a solution (i.e., a pair of arc-disjoint $x'x''$- and $y'y''$-paths in G, where $\{x', x''\} = X$ and $\{y', y''\} = Y$) can be found in $O(m(m + n \log n))$ time.

Let $(G = (V, E); X, Y)$ be an irreducible feasible instance. If V consists of only four terminals, then a solution is easily found in $O(1)$ time. Otherwise, one of the following four cases A–D occurs, and we can find a pair of arcs such that the instance remains feasible after splitting them off.

A. There is a nonterminal vertex v with $deg(v) \leq 6$ or a terminal v with $deg(v) = 4$ in instance $(G; X, Y)$: Choose such a vertex v, and find two arcs $(v', v) \in \delta^-(v)$ and $(v, v'') \in \delta^+(v)$ such that the instance obtained by splitting (v', v) and (v, v'') at v remains feasible. Note that such a pair of arcs exists (since the instance is feasible), and by Theorem 9.3 it is found in $O(m + n \log n)$ time by checking feasibility among all (at most 9) possibilities. Split off such (v', v) and (v, v'') at v, and recompute an irreducible instance $(G'; X, Y)$ from the resulting instance in $O(m + n \log n)$ time (Lemma 9.1).

B. $deg(v) \geq 8$ for all $v \in V - (X \cup Y)$, $deg(t) \neq 4$ for all $t \in X \cup Y$, and there is a 6-cut $Z \subseteq V - (X \cup Y)$: Then choose a 6-cut Z with minimal $|Z|$ among such 6-cuts, and let v be a nonterminal vertex in Z. Note that any nonempty cut

$$Z^* \subset Z \text{ satisfies } |\delta(Z^*; G)| \geq 8 \text{ from the assumption on } Z.$$

Since $|\delta^-(v)| = |\delta^+(v)| \geq 4$ by $deg(v) \geq 8$ and $|\delta^-(Z)| = |\delta^+(Z)| = 3$, there are arcs $(v', v) \in \delta^-(v) - \delta^-(Z)$ and $(v, v'') \in \delta^+(v) - \delta^+(Z)$, where $v', v'' \in Z$ (possibly $v' = v''$). Let $(G'; X, Y)$ be the instance obtained from $(G; X, Y)$ by splitting off these arcs (v', v) and (v, v'') at v (in the case of $v' = v''$, splitting simply means removal of those two arcs). We show that $(G'; X, Y)$

remains irreducible (hence feasible, because any irreducible instance having a nonterminal vertex v with $deg(v) \geq 6$ is feasible by Lemma 3.3(iii)). Assume that $(G'; X, Y)$ has a reducible cut Z'. Since Z' was not reducible in $(G; X, Y)$, Z' must separate $\{v\}$ and $\{v', v''\}$. Since $Z' \subset Z$ would imply $|\delta(Z'; G)| \geq 8$ from the above and $|\delta(Z'; G')| \geq 6$ (hence, such Z' is not reducible in G'), Z' must intersect Z (and hence Z' and Z cross each other because $(V-(Z' \cup Z))$ contains a terminal). From the above, we have $|\delta(Z \cap Z'; G)| \geq 8$ and

$$|\delta(Z \cap Z'; G')| \geq 6 \ (= |\delta(Z; G')|).$$

Also, we obtain

$$|\delta(Z \cup Z'; G')| \geq |\delta(Z'; G')| + 2$$

(otherwise $|\delta(Z \cup Z'; G')| \leq |\delta(Z'; G')|$ implies that $Z \cup Z'$ is a reducible cut in G). However, these two inequalities contradict (2.1) (i.e., $|\delta(Z; G')| + |\delta(Z'; G')| \geq |\delta(Z \cap Z'; G')| + |\delta(Z \cup Z'; G')|$). This shows that $(G'; X, Y)$ is irreducible, and hence feasible.

A minimal 6-cut Z in the above can be found in $O(m + n \log n)$ time as follows. Since such Z never intersects $X \cup Y$, we contract the four terminals into a single vertex t^* and ignore the arc orientation. Let $\overline{G_{t^*}}$ be the resulting undirected graph. Clearly, $\lambda(\overline{G_{t^*}}) = 6$ by the irreducibility of G and the assumption of case B. Find a cactus representation (Γ, φ) for $\mathcal{C}(\overline{G_{t^*}})$ in $O(m + n \log n)$ time [7]. Recall that Γ has at least two leaf nodes, and one of them, say, z, satisfies $t^* \notin \varphi^{-1}(z)$. By definition of a cactus representation, $Z = \varphi^{-1}(z)$ is a minimal 6-cut in G.

C. $deg(v) \geq 8$ for all $v \in V-(X \cup Y)$, $deg(t) \neq 4$ for all $t \in X \cup Y$, and $|\delta(Z^*)| \geq 8$ for all $Z^* \subseteq V-(X \cup Y)$, but there is a 4-cut Z with $Z \cap (X \cup Y) = \{t\}$ for some terminal t: Then take a minimal Z among them. Since $deg(t) \neq 4$, we see that $Z - \{t\} \neq \emptyset$ and $deg(t) \geq 6$ (if $deg(t) = 2$, then $Z - \{t\}$ is a 6-cut with $Z - \{t\} \subseteq V - (X \cup Y)$, contradicting the assumption of case C). Since $|\delta^-(t)| = |\delta^+(t)| \geq 3$ by $deg(t) \geq 6$ and $|\delta^-(Z)| = |\delta^+(Z)| = 2$, there are arcs $(v', t) \in \delta^-(\{t\}) - \delta^-(Z)$ and $(t, v'') \in \delta^+(\{t\}) - \delta^+(Z)$, where $v', v'' \in Z$ (possibly $v' = v''$). Let $(G'; X, Y)$ be the instance obtained from $(G; X, Y)$ by splitting off these arcs (v', t) and (t, v'') at t (in the case of $v' = v''$, splitting means removal of those two arcs). We show that $(G'; X, Y)$ remains irreducible (hence feasible, because any irreducible instance having a terminal vertex t with $deg(t) \geq 4$ is feasible by Lemma 3.3(ii)). Assume that $(G'; X, Y)$ has a reducible cut Z'. Since Z' is not reducible in $(G; X, Y)$, Z' must separate $\{t\}$ and $\{v', v''\}$. Since $Z' \subset Z$ implies that $|\delta(Z')| \geq 6$ (if $t \in Z'$) by the minimality of $|Z|$ and $|\delta(Z')| \geq 8$ (if $t \notin Z'$) by the assumption of case C (hence, such Z' is not reducible in G'), then Z' intersects Z (and hence Z' and Z cross each other since $(V - (Z' \cup Z))$ contains a terminal). We see that Z' is not a cut of type (1) because otherwise Z' would be a reducible 4-cut in $(G; X, Y)$. If Z' is a cut of type (3) in $(G'; X, Y)$, then Z' is a 6-cut $Z' \subseteq V - (X \cup Y)$ in $(G; X, Y)$, contradicting the assumption of case C. Therefore, Z' must be a reducible cut of type (2) in $(G'; X, Y)$. We first consider the case of $t \in Z \cap Z'$. We have $|\delta(Z \cap Z'; G)| \geq 6$ (by the minimality of $|Z|$) and $|\delta(Z \cap Z'; G')| \geq 4 (= |\delta(Z; G')|)$. Since $Z - Z'$ contains no terminal, we obtain $|\delta(Z \cup Z'; G')| \geq |\delta(Z'; G')| + 2$ (otherwise $|\delta(Z \cup Z'; G)| = |\delta(Z \cup Z'; G')| \leq |\delta(Z'; G')|$ implies that $Z \cup Z'$ is a reducible cut in G). However, these two inequalities contradict (2.1), as in case B. Then assume $t \in Z-Z'$, implying that there is another terminal t' in $Z'-Z$. Clearly, $|\delta(Z-Z';G)| \geq 6$ $(= |\delta(Z;G)|+2)$ (from minimality of Z). From $|\delta(Z';G')| = 2$, we have $|\delta(Z';G)| = 4$ and $deg(t') \geq 6$ (otherwise, if $deg(t') = 2$, then $Z' - \{t'\}$ would be a 6-cut, contradicting the assumption of case C). From this and the irreducibility of $(G;X,Y)$, $|\delta(Z' - Z;G)| \geq 4$ $(= |\delta(Z';G)|)$ holds. These inequalities contradict (2.2). Consequently, $(G';X,Y)$ is irreducible and hence is feasible.

The above minimal 4-cut Z can be found in $O(m+n \log n)$ time as follows. Since such Z always separates $\{t\}$ and $(X \cup Y) - \{t\}$, contract the three other terminals of $(X \cup Y) - \{t\}$ into a single vertex $\overline{t}$ and ignore the arc orientation. Let $\overline{G_{\overline{t}}}$ be the resulting undirected graph. Clearly, $\lambda(\overline{G_{\overline{t}}}) = 4$ (since case C does not occur for this t if $\lambda(\overline{G_{\overline{t}}}) \geq 6$). Find a cactus representation (Γ, φ) for $\mathcal{C}(\overline{G_{\overline{t}}})$ in $O(m+n\log n)$ time [7]. By definition of a cactus representation, Γ has a leaf node z with $t \in \varphi^{-1}(z)$ and $Z = \varphi^{-1}(z)$ is a desired minimal 4-cut in G.

D. $deg(v) \geq 8$ for all $v \in V-(X \cup Y)$, $deg(t) \neq 4$ for all $t \in X \cup Y$, and $|\delta(Z^*)| \geq 8$ for all $Z^* \subseteq V - (X \cup Y)$ and $|\delta(Z)| \geq 6$ for all Z with $Z \cap (X \cup Y) = \{t\}$ and $t \in X \cup Y$: Then choose an arbitrary nonterminal vertex v and two arcs (v', v) and (v, v''). It is easy to see that the instance $(G';X,Y)$ obtained from $(G;X,Y)$ by splitting off these arcs (v',v) and (v,v'') at v remains irreducible (hence feasible, because any irreducible instance having a nonterminal vertex v with $deg(v) \geq 6$ is feasible by Lemma 3.3(iii)).

Recall that none of cases A–D can be applied to an instance only when it has four terminals with degree 2 but no nonterminal vertex. Given an irreducible feasible instance, we continue to split off a pair of arcs to obtain smaller feasible instances by following the above cases A–D until an instance consisting of four terminals with degree 2 is obtained, in which we can easily find a solution. The entire running time of this procedure is $O(m(m + n \log n))$, since the number of arcs decreases at least by 2 after splitting off a pair of arcs. It is easy to see that a solution of the original instance $(G;X,Y)$ can be recovered in the same time complexity from the sequence of such splittings. This establishes the next theorem.

THEOREM 9.4. *Given a feasible instance $(G;X,Y)$, where n and m are the numbers of vertices and edges, respectively, a solution of $(G;X,Y)$ can be computed in $O(m(m + n\log n))$ time.* □

10. Discussion. For the arc-disjoint path problems

$$(G; X_i = \{s_i, t_i\}, i = 1, 2, \ldots, k)$$

associated with Eulerian digraphs, different problem settings are conceivable depending upon the restrictions on G and the directions of the required paths: (i) either $G + H$ is Eulerian, where H is the demand digraph, or G itself is Eulerian, and (ii) either s_it_i-paths are required for all i, or one of the s_it_i- and t_is_i-paths is required for each i. The result in [9] shows that ($G + H$ Eulerian, s_it_i-path, $k = 3$) can be solved in polynomial time, while our result here shows that (G Eulerian, one of the s_it_i- and t_is_i-paths, $k = 2$) can also be solved in polynomial time. By generalizing the proof in [4, 9], it is possible to prove that all types become NP-hard if k is considered as a part of input. Therefore, an interesting theoretical challenge, for each problem type, will be to find out the maximum constant k that permits a polynomial time algorithm, or to show that any constant k permits a polynomial time algorithm.

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