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On the Orientation of Graphs

ANDRÁS FRANK

Research Institute for Telecommunication, H-1525 P.O.B. 15, Budapest, Hungary

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Let $G(V, E)$ be a finite, undirected graph, and let $l(X)$ be a set function on 2^V. When can the edges of G be oriented so that the indegree of every subset X is at least $l(X)$? A necessary and sufficient condition is given for the existence of such an orientation when $l(X)$ is "convex."

INTRODUCTION

Let $G(V, E)$ be a finite undirected graph with vertex set V and edge set E. Multiple edges are allowed but loops are excluded. Let $l(X)$ be a nonnegative integer function on the subsets of V, for which $l(\varnothing) = l(V) = 0$. The indegree $\rho(X)$ of a vertex subset X of a digraph is defined in the natural way: $\rho(X)$ is the number of edges, the head of which is in X but the tail is not.

Under what condition can we orient the edges of G so that the indegree $\rho(X)$ of any subset X of V is at least $l(X)$?

The first result of this type is due to Hakimi [4]. He has solved the problem when $l(X)$ is an arbitrary nonnegative integer for $|X| = 1$ and $l(X) = 0$ otherwise. In [3], generalizing Hakimi's result, the problem was solved for some other special functions, for example, if $l(X)$ is an arbitrary nonnegative integer for $|X| = 1$ and for $|X| = |V| - 1$ and $l(X) \equiv 0$ otherwise; or $l(X)$ is an arbitrary positive integer for $|X| = 1$ and for $|X| = |V| - 1$ and $l(X) \equiv 1$ otherwise. (The latter equality describes the strong connectivity of the obtained digraph.) Nash-Williams [6] settled the question for the function $l(X) \equiv k$.

The purpose of this paper is to investigate the problem in some more general cases. In the main theorem a necessary and sufficient condition is given for the existence of a required orientation when $l(X)$ is "convex." As a consequence of this theorem we obtain a common generalization of Hakimi's and Nash-Williams' results (Theorem 5).

The expression "orientation" has two meanings, without causing any confusion. One may orient an edge (and speak about its orientation), in the sense that the undirected edge is replaced by a directed one. On the other

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hand, if we assign an orientation to all the edges of an undirected graph G, then G is said to possess an orientation.

1. The Main Theorem

All graphs we consider will have the underlying set V. For $X \subseteq V$, we put $\bar{X} = V \setminus X$. If X consists of a single element x we shall write $\rho(x)$ instead of $\rho(\{x\})$. In a directed or undirected graph, $d(X, Y)$ will denote the number of such edges one end vertex of which is in $X \setminus Y$, and the other one in $Y \setminus X$. We use the abbreviation $d(X) = d(X, \bar{X})$. Throughout, the shorter-terms graph and digraph will be used instead of undirected and directed graphs, respectively.

Definition. A pair of subsets X, Y of V is called crossing if $X \cap Y \neq \varnothing$, $X \cup Y \neq V$, $X \not\subseteq Y$, and $Y \not\subseteq X$.

Definition. The nonnegative integer function $l(X)$ defined on the subsets of V is called *convex* with respect to G (briefly convex) when $l(\varnothing) = l(V) = 0$ and the following inequality holds for every crossing pair X, Y:

$$l(X) + l(Y) - d(X, Y) \leqslant l(X \cap Y) + l(X \cup Y). \tag{1}$$

If (1) holds for every pair X, Y then $l(X)$ is called *strongly convex*.

A set function $l(X)$ is called *supermodular* if for any crossing pair X, Y the relation

$$l(X) + l(Y) \leqslant l(X \cap Y) + l(X \cup Y)$$

holds.

Fundamental results on functions of this type can be found in [1].

Obviously a supermodular function is convex with respect to any graph and the reader may easily check that $l(X)$ is convex if and only if the set function $\bar{l}(X) = l(X) - \frac{1}{2}d(X)$ is supermodular. Also note that if $l(X)$ is convex with respect to G, then it is convex with respect to any supergraph of G.

Theorem 1. *For a convex set function $l(X)$ there exists an orientation of the edges of $G(V, E)$ such that*

$$\rho(X) \geqslant l(X) \qquad \text{for} \quad \varnothing \subset X \subset V \tag{2}$$

if and only if for every partition $V = V_1 \cup V_2 \cup \cdots \cup V_t$ of the vertices, the number e_p of edges connecting different V_i's satisfies

$$e_p \geqslant \sum_{i=1}^{t} l(V_i) \tag{3}$$

and

$$e_p \geqslant \sum_{i=1}^{t} l(\bar{V}_i). \tag{4}$$

Proof. *Necessity.* In a digraph the number of edges connecting different V_i's is $\sum_{i=1}^{t} \rho(V_i)$. If we have a good orientation (i.e., if (2) holds) then $\rho(V_i) \geqslant l(V_i)$ and (3) holds.

If there exists a good orientation with respect to $l(X)$, then reversing the orientation of all the edges gives a good orientation with respect to $l'(X) = l(\bar{X})$. Therefore (3) is true for $l'(X)$ as well, which is exactly condition (4).

Sufficiency. If there exists an orientation of the edges and a partition $V = V_1 \cup \cdots \cup V_t$ of the vertices for which

$$\rho(V_1) = l(V_1) - 1, \qquad \rho(V_i) = l(V_i), \qquad i = 2, 3, \ldots, t, \tag{5}$$

then this partition obviously violates (3).

If there exist an orientation of the edges and a partition $V = V_1 \cup \cdots \cup V_t$ of the vertices for which

$$\rho(\bar{V}_1) = l(\bar{V}_1) - 1, \qquad \rho(\bar{V}_i) = l(\bar{V}_i), \qquad i = 2, 3, \ldots, t, \tag{6}$$

then this partition violates (4).

We shall show that if graph G_1' has a good orientation and was obtained from G by duplicating some edges, then after deleting one of the new edges from G_1', the resulting graph G_1 still has a good orientation. This statement proves our theorem, since one may duplicate all the edges of G at first and obtain a good orientation by orienting every old edge and its copy oppositely ($d(X) \geqslant l(X)$ follows from (3) and from the fact that $l(R)$ is nonnegative).

At this point we remark that a graph G_1 obtained from G by duplicating some edges satisfies (1), (3), and (4) provided G does.

Now let G_1' be a graph obtained from G by duplicating some edges of G and assume that G_1' has a good orientation. Let $e(a, b)$ be an edge in G_1' which was not in G. Let G_1 denote the graph, obtained from G_1' by deleting e. If the good orientation of G_1' is still good in G_1, we have finished with the proof. Otherwise we have a "wrong" orientation $\mathbf{G}_1$ of G_1, i.e., in $\mathbf{G}_1$ there are some sets for which $\rho(X) < l(X)$. Such sets will be called *wrong* (ρ and d in the sequel concern the graph $\mathbf{G}_1$).

We try to improve this orientation. Let the orientation of $e(a, b)$ be $e(\mathbf{a}, \mathbf{b})$ in the good orientation of G_1'. Although the orientation $\mathbf{G}_1$ is wrong, it is not too wrong; i.e.,

$$\text{If } X \subset V \text{ is wrong then } \rho(X) = l(X) - 1. \tag{7}$$

Since b is contained in every wrong set, we have

$$\text{The intersection of wrong sets is nonempty.} \tag{8}$$

Since a is not contained in any wrong set, we have

$$\text{The union of all the wrong sets is not } V. \tag{9}$$

Henceforth let $\mathbf{G}_1$ denote such an orientation of G_1 for which (7)–(9) hold and the number of wrong sets is as small as possible. Suppose, indirectly, that this number is not 0. This will lead to a contradiction.

We need some lemmas.

LEMMA 1. *For any pair X, Y of subsets of V,*

$$\rho(X) + \rho(Y) = \rho(X \cap Y) + \rho(X \cup Y) + d(X, Y).$$

Proof. A simple enumeration of edges verifies the statement. ▮

LEMMA 2. *Let W_1, W_2 be two wrong sets in $\mathbf{G}_1$; then both $W_1 \cup W_2$ and $W_1 \cap W_2$ are wrong as well.*

Proof. If $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$, we have nothing to prove. Otherwise W_1, W_2 form a crossing pair by (8) and (9), thus

$$l(W_1) + l(W_2) \leqslant l(W_1 \cap W_2) + l(W_1 \cup W_2) + d(W_1, W_2). \tag{10}$$

On the other hand, by (7), (8), and (9) we get

$$\begin{aligned} \rho(W_1 \cap W_2) &\geqslant l(W_1 \cap W_2) - 1, \\ \rho(W_1 \cup W_2) &\geqslant l(W_1 \cup W_2) - 1. \end{aligned} \tag{11}$$

Hence by Lemma 1:

$$\begin{aligned} l(W_1) - 1 + l(W_2) - 1 &= \rho(W_1) + \rho(W_2) \\ &= \rho(W_1 \cap W_2) + \rho(W_1 \cup W_2) + d(W_1, W_2) \\ &\geqslant l(W_1 \cap W_2) - 1 + l(W_1 \cup W_2) - 1 + d(W_1, W_2). \end{aligned} \tag{12}$$

Equations (10) and (12) imply that every inequality in (10), (11), and (12) must hold as equality. ▮

LEMMA 3. *In $\mathbf{G}_1$ the union of any number of wrong sets is wrong again.*

Proof. The proof is straightforward by Lemma 2. ▮

The union of all the wrong sets is denoted by R. By Lemma 3, R is wrong.

DEFINITION. A nonempty subset S of V is called *strict* (in $\mathbf{G}_1$), if $\rho(S) = l(S)$.

Remark. If, in $\mathbf{G}_1$, $\rho(S) = 0$, the set S is trivially either wrong or strict.

LEMMA 4. *If S_1 and S_2 are strict subsets, $S_1 \cup S_2 \neq V$ and $S_1 \cap S_2 \not\subseteq R$, then $S_1 \cup S_2$ and $S_1 \cap S_2$ are strict.*

Proof. The proof is similar to that of Lemma 2: If $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$, there is nothing to prove. Otherwise S_1, S_2 form a crossing pair, and $S_1 \cup S_2$, $S_1 \cap S_2$ are not wrong by the definition of R. Thus

$$\begin{aligned} l(S_1) + l(S_2) &= \rho(S_1) + \rho(S_2) \\ &= \rho(S_1 \cap S_2) + \rho(S_1 \cup S_2) + d(S_1, S_2) \\ &\geqslant l(S_1 \cap S_2) + l(S_1 \cup S_2) + d(S_1, S_2) \\ &\geqslant l(S_1) + l(S_2), \end{aligned}$$

from which the lemma follows. ▮

LEMMA 5. *If S_1, S_2 are strict, $S_1 \cup S_2$ is not strict, and $(S_1 \cup S_2) \cap R = \varnothing$ then $S_1 \cap S_2 = \varnothing$.*

Proof. This is obvious by Lemma 4. ▮

LEMMA 6. *If S is strict in $\mathbf{G}_1$, R_1 is wrong, S and R_1 are crossing, and $S \not\subseteq R$, then $S \cap R_1$ is wrong.*

Proof. By the same argument used in the proof of Lemma 2 we get that at least one of the sets $S \cap R_1$ and $S \cup R_1$ is wrong. But $S \not\subseteq R$, hence $S \cup R_1$ may not be wrong. ▮

We try to alter the orientation of G_1 so that

$$\begin{aligned} &\text{(i)} \quad R \text{ should become good,} \\ &\text{(ii)} \quad \text{no new wrong set should arise,} \\ &\text{(iii)} \quad \text{statements (7)–(9) should still hold.} \end{aligned} \tag{13}$$

If we find such alteration, then this fact and the minimal property of $\mathbf{G}_1$ are in contradiction, which proves the theorem.

We distinguish some cases.

Case 1.

There exists a vertex $x \notin R$, not contained in any strict set.

Then there exists a directed path P from a vertex y of R to x (a good subset of indegree 0 is strict). Reverse the orientation of the edges of P. It can be easily checked that the resulting orientation of G_1 satisfies (13).

Case 2.

Every vertex $x \notin R$ is contained in a strict set.

Subcase 2.1. Every vertex $x \notin R$ is contained in a strict set which is disjoint from R.

In this case $\bar{R}$ is the union of some strict sets. The maximal strict subsets $V_2, V_3, \ldots, V_t$ of $\bar{R}$ are pairwise disjoint by Lemma 5, hence $V_1 = R, V_2, \ldots, V_t$ is a partition of V. This partition and $\mathbf{G}_1$ satisfy (5); therefore (3) does not hold: a contradiction. This case is impossible.

Subcase 2.2. There exists a vertex $x \notin R$ which is contained in some strict sets, but all these sets intersect R.

Let the minimal strict sets containing x be $M_1, M_2, \ldots, M_r$.

2.2.1. There exists a vertex $y \in M_1 \cap M_2 \cap \cdots \cap M_r \cap R$.

Then no strict set M exists for which $x \in M$ and $y \notin M$. Therefore there exists a directed path from y to x and altering the orientations of all the edges along this path, we obtain a new orientation of G_1 which satisfies (13).

2.2.2. $M_1 \cap M_2 \cap \cdots \cap M_r \cap R = \varnothing$.

This is equivalent to

$$\bar{M}_1 \cup \bar{M}_2 \cup \cdots \cup \bar{M}_r \supseteq R. \tag{14}$$

By the minimality of the M_i's and by Lemma 4 we get

$$M_i \cup M_j = V;$$

that is,

$$\bar{M}_i \cap \bar{M}_j = \varnothing \qquad \text{for} \quad 1 \leqslant i < j \leqslant r. \tag{15}$$

By (8) there exists a vertex b contained in all the wrong sets. Assume that $b \in \bar{M}_i$. In this case $\bar{M}_i \subset R$. Otherwise apply Lemma 6 with the choice $S = M_i$, $R_1 = R$. Then $R \cap M_i$ is wrong and does not contain b: a contradiction.

Assume $\bar{M}_1, \bar{M}_2, \ldots, \bar{M}_k \subset R$ and $\bar{M}_{k+1}, \bar{M}_{k+2}, \ldots, \bar{M}_r \not\subset R$, where $1 \leqslant k \leqslant r$. One can easily check by a simple induction that all the subsets $R \cap M_{k+1}$, $R \cap M_{k+1} \cap M_{k+2}, \ldots, R \cap M_{k+1} \cap M_{k+2} \cap \cdots \cap M_r$ are wrong. Namely, in the j'th step apply Lemma 6 with the choice $R_1 = R \cap M_{k+1} \cap \cdots \cap M_{k+j}$, $S = M_{k+j+1}$ (if S and R_1 are not crossing, it is easy to see that $R_1 \subset S$, hence $R_1 \cap S = R_1$ is wrong).

Now $R \cap M_{k+1} \cap \cdots \cap M_r = \bar{M}_1 \cup \bar{M}_2 \cup \cdots \cup \bar{M}_k$ by (14) and (15). Let $R_1 = \bar{M}_1 \cup \bar{M}_2 \cup \cdots \cup \bar{M}_k$ and let $V_1 = \bar{R}_1$, $V_2 = \bar{M}_1$, $V_3 = \bar{M}_2, \ldots, V_t = \bar{M}_k$ (where $t = k + 1$). $V_1, V_2, \ldots, V_t$ form a partition of V by (15) and by the construction of V_1. This partition satisfies (6) with respect to $\mathbf{G}_1$, hence (4) does not hold. Case 2.2.2 is impossible. This completes the proof. ∎

EXAMPLE. Theorem 1 is not necessarily true for nonconvex functions. Let $V = \{1, 2, 3, 4\}$, $E = \{(1, 2), (3, 4)\}$ and let $l(X)$ be defined on 2^V as follows: $l(1, 3) = l(2, 3, 4) = l(4) = 1$ and $l(X) = 0$ otherwise.

Although conditions (3) and (4) are satisfied, no good orientation exists. The function $l(X)$ is not convex: $X = \{1, 2\}$, $Y = \{1, 3\}$ form a crossing pair violating (1) (every other crossing pair satisfies (1)).

2. Corollaries

Remark 1. My original proof for Theorem 1 followed the same line but originally the theorem was formulated for supermodular set functions only. L. Lovász noticed that my lemmas and the theorem remain true for convex set functions as well.

Remark 2. When $l(X)$ is nonincreasing, that is,

$$X \subset Y \qquad \text{implies} \qquad l(X) \geqslant l(Y),$$

then obviously (3) implies (4); hence (3) is already sufficient for the existence of a good orientation (similarly, if $l(X)$ is nondecreasing then (4) implies (3)).

The following remark and theorem are also due to Lovász.

Remark 3. If $l(X)$ is convex and condition (1) holds not only for crossing pairs but disjoint pairs as well, then (3) is automatically satisfied.

This is true for $t = 2$. If $t > 2$, let $V'_{t-1} = V_t \cup V_{t-1}$ and $V'_i = V_i$ for $i = 1, 2, \ldots, t-2$. Now we get by induction

$$\sum_{i=1}^{t} l(V_i) \leqslant \sum_{i=1}^{t-2} l(V'_i) + l(V_{t-1} \cup V_t) + d(V_{t-1}, V_t)$$

$$\leqslant e'_p + d(V_{t-1}, V_t) = e_p.$$

Similarly if $l(X)$ is convex and (1) holds for all the pairs X, Y, where $X \cup Y = V$, then (4) is satisfied. Hence we get

THEOREM 2. *If $l(X)$ is strongly convex with respect to G then there exists an orientation of G which satisfies* (2) *(we can obtain a simple proof of this theorem using the strong convexity only).*

THEOREM 3. *Let $u(X)$ be a nonnegative integer set function on the subsets of V. Suppose $u(X)$ is concave with respect to $G(V, E)$; that is,*

$$u(X) + u(Y) - d(X, Y) \geqslant u(X \cap Y) + u(X \cup Y) \tag{16}$$

for any crossing pair X, Y. Assume further that

$$u(X) \leqslant d(X). \tag{17}$$

Then there exists an orientation of G such that

$$\rho(X) \leqslant u(X) \quad \text{for} \quad X \subset V, \tag{18}$$

if and only if for every partition $V = V_1 \cup V_2 \cup \cdots \cup V_t$ of the vertices, the number e_p of edges connecting different V_i's satisfies

$$e_p \leqslant \sum_{i=1}^{t} u(V_i) \tag{19}$$

and

$$e_p \leqslant \sum_{i=1}^{t} u(\overline{V}_i). \tag{20}$$

Proof. An orientation satisfies (18) if and only if

$$\rho(X) \geqslant d(X) - u(X) \quad \text{for} \quad X \subset V.$$

Let $l(X) = d(X) - u(X)$. $l(X)$ is nonnegative and a simple calculation shows that it is convex with respect to G. The condition of the existence of the required orientation, by Theorem 1, is that for every partition $V = V_1 \cup V_2 \cup \cdots \cup V_t$,

$$e_p \geqslant \sum_{i=1}^{t} (d(V_i) - u(V_i))$$

and

$$e_p \geqslant \sum_{i=1}^{t} (d(\overline{V}_i) - u(\overline{V}_i)).$$

But $\sum_{i=1}^{t} d(V_i) = 2e_p$, hence the theorem follows. ∎

THEOREM 4. *Let $s(X)$ be an arbitrary nonnegative integer set function on the subsets of V. There exists an orientation of G for which*

$$\rho(X) = s(X) \quad \text{for} \quad X \subset V \tag{21}$$

if and only if

$$s(X) \text{ is strongly convex,} \tag{22}$$

$$s(X) + s(\overline{X}) = d(X) \quad \text{for} \quad X \subset V. \tag{23}$$

Proof. Necessity. Condition (22) follows from Lemma 1 and (23) is trivial.

Sufficiency. By Theorem 2 we have an orientation for which $\rho(X) \geqslant s(X)$ for $X \subset V$. (23) implies that this inequality must hold as an equality. ∎

(Of course there exists an immediate proof for this theorem.)

3. SPECIAL CASES

We shall present some special functions for which Theorem 1 is applicable.

1. Let $l(X)$ be a convex set function with respect to $G(V, E)$ and let $a(x)$ and $b(x)$ be two nonnegative integer functions on V. We seek an orientation of G such that

$$\begin{aligned} \rho(X) &\geqslant l(X) && \text{for} \quad X \subset V, \\ a(x) \leqslant \rho(x) &\leqslant b(x) && \text{for} \quad x \in V. \end{aligned} \tag{24}$$

For this aim we construct the following set function $l'(X)$:

$$\begin{aligned} &= l(X) && \text{if} \quad 1 < |X| < |V| - 1 \\ l'(X) &= \max(l(X), a(x)) && \text{if} \quad X = \{x\} \\ &= \max(l(X), d(X) - b(x)) && \text{if} \quad X = V \setminus \{x\}. \end{aligned} \tag{25}$$

Obviously an orientation satisfies (24) if and only if it satisfies (2) for $l'(X)$. Moreover it is clear that $l'(X)$ is also convex because when X, Y is a crossing pair then

$$1 < |X| < |V| - 1,$$
$$1 < |Y| < |V| - 1;$$

hence (1) remains true for l'.

Applying this method in any of the special cases below we obtain some new results which appear to be interesting for their own sake as well.

2. Let $p(x)$ be a nonnegative integer function on V. Let us define $l(X) = \min_{y \in X} p(y)$ ($\varnothing \subset X \subset V$), $l(\varnothing) = l(V) = 0$.

Now $l(X)$ is supermodular and nonincreasing; therefore it is enough to assume (3) to apply Theorem 1.

3. Consider the last example if $p(x) \equiv k$. In this case (3) is evidently equivalent to

$$d(X) \geqslant 2k \quad \text{for} \quad \varnothing \subset X \subset V, \tag{26}$$

hence Theorem 1 yields a weak form of a famous theorem of Nash–Williams [6].

We formulate Theorem 1 for the situation described in Examples 1 and 3.

THEOREM 5. *Let $G(V, E)$ be a graph and let $a(x)$, $b(x)$ $(a(x) \leqslant b(x))$ be two nonnegative integer functions on V. There exists a k-strongly-edge-connected orientation of G in which*

$$\rho(x) \geqslant a(x) \quad \textit{for} \quad x \in V$$

if and only if for every partition $V = V_0 \cup V_1 \cup \cdots \cup V_t$ of the vertices (now V_0 may be empty) the number e_p of edges connecting different V_i's $(i \geqslant 0)$ and the number e_p' of edges lying in V_0 satisfy:

$$e_p + e_p' \geqslant kt + \sum_{x \in V_0} a(x). \tag{27}$$

There exists a k-strongly-edge-connected orientation of G for which

$$\rho(x) \leqslant b(x), \quad \textit{when} \quad x \in V,$$

if and only if for every partition $V = V_0 \cup V_1 \cup \cdots \cup V_t$ of the vertices, and for the number e_p'' of edges connecting different V_i's $(i \geqslant 1)$:

$$-e_p' + e_p'' \geqslant kt - \sum_{x \in V_0} b(x). \tag{28}$$

There exists a k-strongly-edge-connected orientation of G for which

$$a(x) \leqslant \rho(x) \leqslant b(x) \quad \textit{for} \quad x \in V$$

if and only if (27) *and* (28) *hold.*

4. Let $G(V, E)$ be a graph and let r be a vertex of G called its root. Let us define $l(X)$ as follows:

$$\begin{aligned} l(X) &= 0 \quad \text{if} \quad r \in X, \\ &= k \quad \text{otherwise.} \end{aligned}$$

This function is supermodular and nonincreasing; thus by a direct application of Theorem 1 we get

THEOREM 6. *There exists an orientation of G such that*

$$\rho(X) \geqslant k \quad \textit{whenever} \quad r \notin X \subset V \tag{29}$$

if and only if

for every partition $V_1 \cup V_2 \cup \cdots \cup V_t$ of the vertices and for the number e_p of the edges connecting different V_i's: (30)

$$e_p \geqslant (t-1) \cdot k.$$

Of course we can again apply the method used in Theorem 1. In this case, however, the exact formulation of Theorem 1 for this special case is left to the reader.

A digraph satisfying (29), by a well-known theorem of Edmonds [2, 5], has k edge-disjoint arborescences of root r. In this way we get a celebrated theorem of Tutte [7]: The graph G has k edge-disjoint spanning trees if and only if (30) holds. (Of course the theorem of Tutte immediately implies Theorem 6).

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