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Directed vertex-connectivity augmentation

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Abstract. The problem of finding a smallest set of new edges to be added to a given directed graph to make it k-vertex-connected was shown to be polynomially solvable recently in [6] for any target connectivity $k \geq 1$. However, the algorithm given there relied on the ellipsoid method. Here we refine some results of [6] which makes it possible to construct a combinatorial algorithm which is polynomial for any fixed k. Short proofs for (extensions of) some earlier results related to this problem will also be given.

Key words. connectivity of directed graphs – augmentation – polynomial algorithm

1. Introduction

In this paper we investigate the directed vertex-connectivity augmentation problem: given a digraph $G = (V, E)$, find a smallest set F of new edges for which $G = (V, E \cup F)$ is k-vertex-connected. (A digraph $G = (V, E)$ is *k-vertex-connected* if $|V| \geq k + 1$ and $G - X$ is strongly connected for any subset $X \subset V$ with $|X| \leq k - 1$.) In what follows "k-connected" refers to k-vertex-connected and "digraph" refers to a simple directed graph.

This problem was shown to be solvable in linear time in the special case of $k = 1$ by Eswaran and Tarjan [2]. The general case, where $k \geq 1$ is arbitrary, is also polynomially solvable, as it was shown recently by the present authors [6]. The proof, however, relied on the ellipsoid method. The existence of a purely combinatorial polynomial algorithm was left as an open problem, even for a strongly connected starting graph G and for $k = 2$.

In this paper we give a partial solution to this open problem. We describe a combinatorial algorithm which solves the directed connectivity augmentation problem optimally. Its running time is polynomial for every *fixed* $k \geq 1$, that is, it can be bounded by a product of a polynomial of $|V|$ and an exponential function of k. The algorithm is based on refinements of some results of [6]. These structural results lead to new and simpler proofs of (extensions of) some earlier results on the augmentation problem, as well.

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We say that an ordered pair (X, Y), $\emptyset \neq X, Y \subset V$, $X \cap Y = \emptyset$ is a *one-way pair* in a digraph $G = (V, E)$ if there is no edge in G with tail in X and head in Y. We call X and Y the *tail* and the *head* of the pair, respectively. The *deficiency* of a one-way pair – with respect to k-connectivity – is $p_{def}(X, Y) := (k - |V - (X \cup Y)|)^+$, where $(x)^+ := \max\{x, 0\}$ for some real number x. Two pairs are *independent* if their tails or their heads are disjoint.

The following good characterization was proved in [6].

Theorem 1 [6]. *A digraph $G = (V, E)$ can be made k-connected by adding at most γ new edges if and only if*

$$\sum (p_{def}(X, Y) : (X, Y) \in \mathcal{F}) \leq \gamma \tag{1}$$

holds for every family $\mathcal{F}$ of pairwise independent one-way pairs. □

In the case of the directed edge-connectivity augmentation problem the corresponding dual families in the good characterization are *subpartitions*, that is, collections of pairwise disjoint subsets of V, see [3, Theorem 3.1]. However, it is important to observe that if $k \geq 2$ then in Theorem 1 it is not sufficient to consider the deficiencies of *subpartition-type* families of one-way pairs only, that is, families whose members are either pairwise tail-disjoint or pairwise head-disjoint.

For example, we need 3 edges to make the graph on Figure 1 two-connected, but the maximum sum of deficiencies over subpartition-type families is just 2. A family of three independent one-way pairs with total deficiency 3 is $\mathcal{F} = \{(a, cde), (be, ca), (cde, b)\}$.

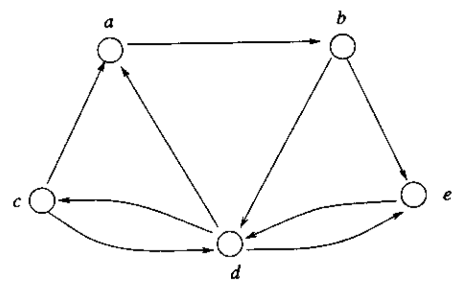

Fig. 1.

On the other hand, we shall prove in Section 3 that if the maximum of the sum of deficiencies in (1) is large enough (at least $2k^2 - 1$), then there exists a subpartition-type dual family $\mathcal{F}$ whose deficiency attains this maximum value. Our algorithm is based on the fact that a subpartition-type optimal dual family, if there is any, is easy to detect.

Before presenting this result and the algorithm, the special case where the starting digraph is already $(k-1)$-connected will be treated separately in Section 2 in order to obtain a simplified min-max equality for the 2-connectivity augmentation problem and to give a different proof for the following (strenghtening of an) earlier result from [11]: if the starting graph is $(k-1)$-connected then the difference between the size of a smallest augmenting set and the maximum sum of deficiencies over the subpartition-type families is at most $k - 1$.

A special case of the augmentation problem which was solved earlier is a theorem due to Masuzawa et al. [12], stating that if the starting digraph G is an arborescence, then the size of a smallest augmenting set equals the sum of out-degree deficiencies. We give a new and short proof for this result in Section 3 using Theorem 1 and its refinement.

Section 4 contains the description of the algorithm. In Section 5 we show a reduction of our augmentation problem to a generalization of the directed k-edge-connectivity augmentation problem.

Note, that the corresponding undirected k-connectivity augmentation problem has not been solved yet, even if the starting graph is $(k-1)$-connected. Polynomial solvability was verified only for $k \leq 3$, see [2,9,10,15]. In the case of edge-connectivity the general problem – directed and undirected – can be solved in polynomial time, see [3],[13] and [14] for efficient combinatorial algorithms. A survey on connectivity augmentations can be found in [4].

2. k-separated independent families

In this section we restrict our attention to the special case where the graph to be made k-connected is already $(k-1)$-connected. Although some results will be stated in more general form in Section 3, it is worth dealing with this special case first since the proofs are simpler and give sharp results in this special case.

Given a finite ground-set S, by a *pair* we always mean an ordered pair of disjoint non-empty subsets of S. We will use the notation (T, H) for a pair with *tail* T and *head* H. Two pairs (T_1, H_1) and (T_2, H_2) are said to be *independent* if $T_1 \cap T_2 = \emptyset$ or $H_1 \cap H_2 = \emptyset$. A family $\mathcal{F}$ of pairs is *independent* if the members of $\mathcal{F}$ are pairwise independent.

Let us consider families of pairs of a special type. Let $n := |S|$. We say that a family $\mathcal{F}$ of pairs on S is *k-separated* for some $0 \leq k \leq n-2$ if the following three conditions hold:

$$|H \cup T| = n - k \text{ for every } (T, H) \in \mathcal{F}, \tag{2}$$

and for every $(T, H), (T', H') \in \mathcal{F}$ we have

$$|T \cup T'| \leq n - k - |H \cap H'|, \text{ if } H \cap H' \neq \emptyset \text{ and} \tag{3}$$

$$|H \cup H'| \leq n - k - |T \cap T'|, \text{ if } T \cap T' \neq \emptyset. \tag{4}$$

The following easy observation motivates the investigation of these families in the context of the connectivity augmentation problem of digraphs. For a one-way pair (X, Y) in a digraph $G = (V, E)$ let $h(X, Y) := |V - (X \cup Y)|$.

Lemma 1. *Let $G = (V, E)$ be a k-connected directed graph. Let $\mathcal{F}$ consist of the one-way pairs (X, Y) of subsets of V for which $h(X, Y) = k$. Then $\mathcal{F}$ is a k-separated family of pairs.* □

Clearly, only the one-way pairs (X, Y) with $h(X, Y) = k-1$ have positive deficiency if the goal is to increase the connectivity of a $(k-1)$-connected digraph to k. Thus, in this special case, a family which attains the maximum in (1) is an independent $(k-1)$-separated family.

In the following lemmas let $\mathcal{F}$ be a k-separated independent family of pairs on the ground-set S. For convenience, we use the notation $C_i := S - (T_i \cup H_i)$ for a pair (T_i, H_i).

Lemma 2. *If $|H_1| \geq |T_2|$ for two pairs $(T_1, H_1), (T_2, H_2) \in \mathcal{F}$, then $T_1 \cap T_2 = \emptyset$. Analogously, if $|T_1| \geq |H_2|$, then $H_1 \cap H_2 = \emptyset$.*

Proof. By symmetry, it is enough to prove the first assertion. For suppose $|H_1| \geq |T_2|$, but the pairs are not tail-disjoint. Thus $H_1 \cap H_2 = \emptyset$, since $\mathcal{F}$ is independent. From the definition of a k-separated family we get $k = |T_1 \cap C_2| + |C_1 \cap C_2| + |H_1 \cap C_2|$ and $|T_1 \cap C_2| + |C_1 \cap C_2| + |T_2 \cap C_1| \geq k$. This gives $|T_2 \cap C_1| \geq |H_1 \cap C_2|$, which yields $|T_2| > |H_1|$, a contradiction.

□

We say that the tail T of some pair (T, H) in a k-separated family is *small* if $|T| \leq \lfloor (n-k)/2 \rfloor$. Similarly, a head H is *small* if $|H| \leq \lfloor (n-k)/2 \rfloor$. The following lemma is straightforward from Lemma 2.

Lemma 3. *The small heads in $\mathcal{F}$ are pairwise disjoint, and the small tails in $\mathcal{F}$ are pairwise disjoint.*

□

Lemma 4. *If $H_1 \cap H_2 \neq \emptyset$ for two pairs $(T_1, H_1), (T_2, H_2) \in \mathcal{F}$, then $|S - (H_1 \cup H_2)| \leq k$.*

Proof. From the definition of a k-separated family we get $2k = |C_1| + |C_2| = |C_1 \cap H_2| + |C_2 \cap H_1| + |C_1 \cap C_2| + |S - (H_1 \cup H_2)| \geq k + |S - (H_1 \cup H_2)|$, as required.

□

A *homogeneous bipartition* $\{\mathcal{F}_t, \mathcal{F}_h\}$ of $\mathcal{F}$ is a partition $\mathcal{F} = \mathcal{F}_t \cup \mathcal{F}_h$ of $\mathcal{F}$ into two disjoint parts, such that the pairs in $\mathcal{F}_t$ are pairwise tail-disjoint and the pairs in $\mathcal{F}_h$ are pairwise head-disjoint. We define the *canonical bipartition* $\{\mathcal{S}_t, \mathcal{S}_h\}$ of $\mathcal{F}$ as follows. Let $\mathcal{S}_t$ contain the pairs of $\mathcal{F}$ with small tails and let the remaining pairs belong to $\mathcal{S}_h$. By Lemma 3, this is a homogeneous bipartition of the family. We say that a pair $(T, H) \in \mathcal{F}$ is *free* subject to a given homogeneous bipartition $\{\mathcal{A}_t, \mathcal{A}_h\}$ of $\mathcal{F}$ if $(T, H) \in \mathcal{A}_t$ ($(T, H) \in \mathcal{A}_h$) implies that $\{\mathcal{A}_t - (T, H), \mathcal{A}_h + (T, H)\}$ ($\{\mathcal{A}_t + (T, H), \mathcal{A}_h - (T, H)\}$, resp.) is also a homogeneous bipartition.

Lemma 5. *Every homogeneous bipartition $\{\mathcal{A}_t, \mathcal{A}_h\}$ has a free pair.*

Proof. Let us consider the tails in $\mathcal{A}_t$ and the heads in $\mathcal{A}_h$. Choose a set of maximum size from these sets. By symmetry we can assume it is a tail T'. We claim that the pair $(T', H') \in \mathcal{A}_t$ is free. Indeed, $|T'| \geq |H|$ holds for every pair $(T, H) \in \mathcal{A}_h$ by the choice of T', and $H' \cap H = \emptyset$ by Lemma 2.

□

The following theorem simplifies the structure of a k-separated independent family $\mathcal{F}$ provided that $|\mathcal{F}|$ is high enough.

Theorem 2. *Suppose that $\mathcal{F}$ is a k-separated independent family of one-way pairs with $|\mathcal{F}| \geq 2k+2$. Then the heads are pairwise disjoint or the tails are pairwise disjoint in $\mathcal{F}$.*

Proof. First we show that there are no two heads (and similarly there are no two tails) such that one is included in the other. To see this suppose that $H_1 \subseteq H_2$. In this case $C_1 = T_2 \cup (S - (T_1 \cup T_2) - (H_1 \cap H_2))$. By (3) and $|T_2| \geq 1$ this implies $|C_1| \geq k+1$, a contradiction.

The next claim is that there exist $k+2$ pairs in $\mathcal{F}$ which are pairwise head-disjoint or pairwise tail-disjoint. Indeed, the bigger part of the canonical bipartition will do, except when $|\mathcal{S}_t| = |\mathcal{S}_h| = k+1$. However, there exists a free pair by Lemma 5, which proves that we can find $k+2$ pairwise head- or tail-disjoint pairs.

Thus we have $k+2$ pairs $(T_1, H_1), \ldots, (T_{k+2}, H_{k+2})$ with pairwise disjoint – say – heads. Add further pairs to this sub-family as long as it is possible to maintain the disjointness of the heads. Suppose that there is a pair (T, H) which cannot be added. Let – without loss of generality – $H \cap H_1 \neq \emptyset$. By Lemma 4 this implies $|S - (H \cup H_1)| \leq k$. But this yields that the head of some pair of the current subfamily must be a subset of H, a contradiction.

□

Let $\Gamma^+(X)$ and $\Gamma^-(X)$ denote the set of out-neighbours and the set of in-neighbours of a set X of vertices in a digraph $G = (V, E)$, respectively. (That is, $\Gamma^+(X) := \{v \in V - X :$ there exists $u \in X$ with $uv \in E\}$.) The *connectivity* $\kappa(G)$ of a digraph G is the biggest k for which G is k-connected. Let G be a digraph with $\kappa := \kappa(G)$. We say that a set $X \subset V$ with $|V - X| \geq \kappa + 1$ is *out-tight* in G if $|\Gamma^+(X)| = \kappa$. Similarly, X is *in-tight* if $|\Gamma^-(X)| = \kappa$ and $|V - X| \geq \kappa + 1$ hold. The maximum number of pairwise disjoint in-tight and out-tight sets in G is denoted by $b(G)$ and $t(G)$, respectively. (Observe, that if G is $(k-1)$-connected, then the maximum value of (1) restricted to subpartition-type families is precisely the maximum of $b(G)$ and $t(G)$.) Let $m_k(G)$ denote the minimum number of new edges to be added to make G k-connected. Clearly, $\max\{b(G); t(G)\} \leq m_k(G)$.

Theorems 1, 2 and Lemmas 1, 3 imply the following two theorems.

Theorem 3. *Let G be a digraph with $\kappa(G) = k-1$ and for which* $\max\{b(G); t(G)\} \geq 2k-1$. *Then $m_k(G) = \max\{b(G); t(G)\}$.*

□

Theorem 4. *Let G be a digraph with $\kappa(G) = k-1$. Then $m_k(G) \leq \max\{b(G); t(G)\} + k - 1$.*

□

Theorem 4 is a strenghtening of a similar approximation result of [11]. Let $b'(G)$ and $t'(G)$ denote the number of minimal in-tight and minimal out-tight sets in G with $\kappa(G) = k-1$, respectively. It is not hard to check (see [11, Lemma 2.4]) that these parameters are lower bounds for $m_k(G)$. Clearly, $b(G) \leq b'(G)$ and $t(G) \leq t'(G)$.

Corollary 1 [11]. *Let G be a digraph with $\kappa(G) = k-1$. Then $m_k(G) \le \max\{b'(G); t'(G)\} + k - 1$ holds.*

□

Note, that Theorem 4 (and Corollary 1) are sharp in the sense that there exist $(k-1)$-connected digraphs G for all $k \ge 1$ such that $b(G) = b'(G) = t(G) = t'(G) = k$ and $m_k(G) = 2k-1$. (Examples are given in [11]. For $k = 2$ see Figure 1.)

In the rest of this section we further simplify Theorem 1 (and 3) in the special case of $k = 2$. As we mentioned, the subpartition-type lower bound does not provide a good characterization. However, the few "pathological" cases can be classified as follows. (We use the notation $d^+(v)$ and $d^-(v)$ for the out-degree and the in-degree of a vertex v.)

Theorem 5. *Let $G = (V, E)$ be a strongly connected digraph and let γ be a non-negative integer for which $\gamma \ge \max\{b(G); t(G)\}$ holds. Then either $\gamma \ge m_2(G)$ or else $\gamma = 2$, $m_2(G) = 3$, and one of the following two cases holds:*

(a) There exist two disjoint out-tight sets X_1 and X_2 and an edge uv such that $u \in X_1$, $v \in X_2$ and $d^-(v) = 1$.
(b) There exist two disjoint in-tight sets Y_1 and Y_2 and an edge uv such that $u \in Y_1$, $v \in Y_2$ and $d^+(v) = 1$.

Proof. Suppose that $\gamma \ge \max\{b(G); t(G)\}$ but $m := m_2(G) > \gamma$ holds. By Theorem 1 there exists a 1-separated family $\mathcal{F}$ of pairwise independent one-way pairs in G with $|\mathcal{F}| = m$. If $m \ge 4$, then by Theorem 2 the tails (or the heads) in $\mathcal{F}$ are pairwise disjoint. This implies $\gamma \ge \max\{b(G), t(G)\} \ge m > \gamma$, a contradiction. If $\gamma = 1$, then $m \ge 2$, hence there are at least two independent one-way pairs in G. This obviously implies the existence of two disjoint out-tight (or in-tight) sets in G, contradicting $\max\{b(G); t(G)\} \le 1$.

Thus we can assume that $\gamma = 2$ and $m = 3$. Let $\mathcal{F}$ be a 1-separated family of three pairwise independent one-way pairs (T_i, H_i), $i = 1, 2, 3$. The three tails (heads) in the family cannot be pairwise disjoint, since $\max\{b(G), t(G)\} > 2 = \gamma$ would follow. Therefore two pairs have intersecting tails and two pairs have intersecting heads in $\mathcal{F}$. Since $\mathcal{F}$ is independent, these two pairs of pairs are different.

Without loss of generality we can assume that $T_1 \cap T_2 \ne \emptyset$ and $H_2 \cap H_3 \ne \emptyset$, and by symmetry we can assume $H_1 \cap H_3 = \emptyset$. Hence by Lemma 4 we obtain that $|H_1| = 1$. Let $H_1 = \{v\}$. This vertex v cannot be in $C_2 \cap C_3$, since otherwise $|\Gamma^-(\{v\})| \ge 2$ would follow. Thus v is in $T_2 - H_3$ or $T_3 - H_2$. By symmetry we can assume $v \in T_2 - H_3$, which implies that the unique in-neighbour u of v must be in T_3. This is precisely the configuration described by case (a). Case (b) follows similarly in the symmetric situation.

□

Theorem 5 shows that the independent families in Theorem 1 can be described in a well-structured form in this special case. Based on this form, a combinatorial polynomial algorithm can be obtained which makes a strongly connected digraph 2-connected optimally. These details are omitted. Instead, the algorithmic aspects of the general case are discussed in Section 4. Note, that a direct algorithmic proof of Theorem 5 was given in [5] which does not rely on the results of [6].

3. Independent families and subpartitions

Let $G = (V, E)$ be the digraph to be augmented up to a given target-connectivity k. We shall always assume $|V| - 1 \ge k \ge 2$.

In the following four lemmas let $\mathcal{F} = \{(T_1, H_1), \ldots, (T_r, H_r)\}$ be a family of pairwise independent one-way pairs of G for which $p(\mathcal{F}) := \sum(p_{def}(T_i, H_i) : 1 \le i \le r)$ is maximum, and subject to this, $|\mathcal{F}|$ is minimum, and subject to these $\sum(|T_i||H_i| : 1 \le i \le r)$ is as small as possible. Observe that the minimality of $|\mathcal{F}|$ implies that $p_{def}(T_i, H_i) = k - |C_i| > 0$ for each $(T_i, H_i) \in \mathcal{F}$.

For convenience, we introduce the following notation. Given two head-disjoint pairs (T_1, H_1) and (T_2, H_2), the six sets $T_1, H_1, T_2, H_2, C_1 := V - (T_1 \cup H_1)$, $C_2 := V - (T_2 \cup H_2)$ partition V into eight parts, whose cardinality will be denoted as follows: $a = |C_2 \cap H_1|$, $b = |C_1 \cap H_2|$, $c = |C_1 \cap C_2|$, $d = |T_1 \cap T_2|$, $e = |H_1 \cap T_2|$, $f = |C_1 \cap T_2|$, $g = |T_1 \cap H_2|$, $h = |C_2 \cap T_1|$.

Lemma 6. *Let $P_1 = (T_1, H_1)$ and $P_2 = (T_2, H_2)$ be pairs in $\mathcal{F}$ with $T_1 \cap T_2 \ne \emptyset$. Then $|V - (T_1 \cup T_2)| \le k - 1$.*

Proof. The deficiencies of the corresponding pairs P_1 and P_2 are $k - b - c - f$ and $k - a - c - h$, respectively. Let $\mathcal{F}' = \mathcal{F} - P_1 - P_2 + (T_1 \cap T_2, H_1 \cup H_2)$. It is easy to see that $\mathcal{F}'$ is independent. By the choice of $\mathcal{F}$, the deficiency of the new family $\mathcal{F}'$ is strictly less than $p(\mathcal{F})$.

This gives $2k - a - b - 2c - f - h - (k - c - f - h) \ge 1$, thus $k - 1 \ge a + b + c = |V - (T_1 \cup T_2)|$ follows.

□

We say that a tail T_i of some pair (T_i, H_i) is *small*, if $|T_i| \le |H_i|$. Similarly, some head H_j is *small*, if $|H_j| \le |T_j|$ holds.

Lemma 7. *The small tails in $\mathcal{F}$ are pairwise disjoint and the small heads in $\mathcal{F}$ are pairwise disjoint.*

Proof. Let T_1 and T_2 be tails in $\mathcal{F}$ with $T_1 \cap T_2 \ne \emptyset$, and assume that T_2 is small. By Lemma 6 $a + b + c \le k - 1$ holds.

Claim 1. $e + f + g + h \ge 1$.

Proof. Suppose that $e = f = g = h = 0$ holds. In this case the deficiencies of the corresponding pairs P_1 and P_2 are $k - b - c$ and $k - a - c$, respectively.

If $a = b = 1$, then $d = 1$ must hold, since T_2 is small. Thus $|V| = a + b + c + d \le k - 1 + d = k$, a contradiction. Let us assume that $a \ge 2$ and let $u \in H_1$. Consider the family $\mathcal{F}' = \mathcal{F} - P_1 + (T_1, \{u\}) + (T_1, H_1 - u)$. This family is clearly independent. For the new deficiency, Lemma 6 gives that

$$p(\mathcal{F}') \ge p(\mathcal{F}) + k - b - c - (a-1) + k - b - c - 1 - (k - b - c) = p(\mathcal{F}) + k - a - b - c \ge p(\mathcal{F}) + 1,$$

a contradiction. (If $b \ge 2$, the argument is similar.) This proves the claim.

□

Claim 2. $g + h \geq 1$.

Proof. If $g + h = 0$ holds, then Claim 1 implies that $e + f \geq 1$, whence $d + e + f = |T_2| \leq |H_2| = b$.

Let $\mathcal{F}' = \mathcal{F} - P_1 - P_2 + (T_1 \cap T_2, H_1 \cup H_2) + (T_2 - T_1, H_2)$ be a new independent family. Comparing the deficiencies, we get

$$p(\mathcal{F}') - p(\mathcal{F}) \geq k - c - f + k - a - c - d - (k - b - c - f) - (k - a - c) = b - d.$$

By the previous observation $b \geq d + 1$, which is impossible by the choice of $\mathcal{F}$. □

In the rest of the proof assume that T_1 is small, too. Claim 2 implies that $g + h \geq 1$ and $e + f \geq 1$ hold. The next observation is that $|H_1| \geq |T_2|$ or $|H_2| \geq |T_1|$ holds, since otherwise $|H_1| < |T_2| \leq |H_2| < |T_1| \leq |H_1|$ would follow, a contradiction. Now we can assume, without loss of generality, that $|H_1| \geq |T_2|$, or using the other notation, $a \geq f + d$. Let us form a new independent family $\mathcal{F}'$ by deleting P_1 and P_2 from $\mathcal{F}$ and adding the pairs $(T_1 - T_2, H_1)$ and $(T_1 \cap T_2, H_1 \cup H_2)$. As before, we can estimate the difference of their deficiencies:

$$p(\mathcal{F}') - p(\mathcal{F}) \geq k - b - c - f - d + k - h - c - f - (k - b - c - f) - (k - a - c - h) = a - f - d \geq 0.$$

Furthermore, $\sum_{\mathcal{F}} |T_i||H_i| - \sum_{\mathcal{F}'} |T_i||H_i| = (e + f)(b + g) > 0$, contradicting the choice of $\mathcal{F}$. Thus the lemma follows. □

Corollary 2. *For a digraph $G = (V, E)$ and some given target-connectivity k there exists an independent family $\mathcal{F}$ of one-way pairs which has maximum deficiency (with respect to k) and which is either subpartition-type or can be divided into two families of subpartition-type. Thus the maximum deficiency of a subpartition-type family is at least the half of the maximum deficiency taken over the independent families.* □

Like the lemmas in Section 2 led to Theorem 2, the previous results imply that if the maximum deficiency of families of independent one-way pairs is high enough – depending on k – then there exists a subpartition-type optimal dual family. Before proving this we need one more lemma.

Lemma 8. *If $\mathcal{F}$ contains at least $k + 1$ pairwise tail-disjoint pairs then all the pairs in $\mathcal{F}$ are tail-disjoint.*

Proof. Let $\mathcal{F}' \subset \mathcal{F}$ be a maximal tail-disjoint subfamily of $\mathcal{F}$ with at least $k + 1$ members. By our assumption, such a subfamily exists and we can assume that there exists a pair $(T, H) \in \mathcal{F} - \mathcal{F}'$. Let $\mathcal{F}' = \{(T_1, H_1), ..., (T_s, H_s)\}$.

The maximality of $\mathcal{F}'$ implies that T has a non-empty intersection with the tail of at least one member of $\mathcal{F}'$. Without loss of generality, we can assume that $T \cap T_1 \neq \emptyset$.

By Lemma 6 we have $|T \cup T_1| \geq n - k + 1$, whence $T_i \subset T$ holds for some $2 \leq i \leq s$. Applying Lemma 6 this implies $|T| \geq n - k + 1$. Let $\alpha := |\{T_i : T_i \subseteq T, 1 \leq i \leq s\}|$. The number of tails in $\mathcal{F}'$ not included in T is at most $n - |T|$, thus $\alpha \geq s - n + |T| \geq k + 1 - n + |T|$. Observe, that if T includes T_i, then $H \cap H_i = \emptyset$. It is possible to form a new independent family $\mathcal{F}^*$ by deleting (T, H) from $\mathcal{F}$ and replacing (T_i, H_i) by $(T_i, H_i \cup H)$ whenever $T_i \subseteq T$ holds. The number of pairs in the new family is one less than the number of pairs in $\mathcal{F}$ and the difference of deficiencies is $p(\mathcal{F}^*) - p(\mathcal{F}) = \alpha|H| - (k - (n - |T \cup H|))$. However, the previous inequalities give $\alpha|H| - (k - (n - |T \cup H|)) \geq (k + 1 - n + |T|)|H| - (k - n + |T| + |H|) = |T||H| + |H| + k|H| - n|H| - (k - n + |T| + |H|) = |T|(|H| - 1) + k(|H| - 1) - n(|H| - 1) = (|T| + k - n)(|H| - 1) \geq 0$, contradicting the choice of $\mathcal{F}$. □

Lemma 9. *If $p(\mathcal{F}) \geq 2k^2 - 1$, then the tails are pairwise disjoint or the heads are pairwise disjoint in $\mathcal{F}$.*

Proof. Since the maximum possible deficiency of a pair – with respect to k – is k, the family $\mathcal{F}$ has at least $2k$ members. Let $\mathcal{F}_t$ (and $\mathcal{F}_h$) be a maximal subfamily of $\mathcal{F}$, which contains all the small tails (small heads), and which has only pairwise tail-disjoint (pairwise head-disjoint, respectively) members. By Lemma 7 we know that $\mathcal{F}_t \cup \mathcal{F}_h = \mathcal{F}$ and by symmetry we can assume that $|\mathcal{F}_t| \geq |\mathcal{F}_h|$ holds. If $|\mathcal{F}_t| \geq k + 1$, then Lemma 8 implies that $\mathcal{F}$ is subpartition-type and we are done.

Thus we may focus on the case where $|\mathcal{F}_t| = k$. In this case $|\mathcal{F}_h| = k$ must hold and by symmetry we can assume that $p(\mathcal{F}_t) \geq p(\mathcal{F}_h)$ also holds. Furthermore, the deficiency of $\mathcal{F}$ can be attained only if every member of $\mathcal{F}_t$ has deficiency k (in other words, $T = V - H$ for every $(T, H) \in \mathcal{F}_t$) and there is at least one pair $(T', H') \notin \mathcal{F}_t$ for which $T' = V - H'$ holds, too. It is clear that if $T = V - H$ holds for a pair (T, H) in an independent family, then T cannot be included by any other tail of the family.

The maximality of $\mathcal{F}_t$ implies that T' (properly) intersects some tail T_i with $(T_i, H_i) \in \mathcal{F}_t$. They can be independent only if $V - T_i \subseteq T'$. Since $|\mathcal{F}_t| = k \geq 2$, this implies that T' includes some member of $\mathcal{F}_t$, a contradiction. □

Note, that the bound $2k^2 - 1$ in Lemma 9 is usually not sharp. For instance one can prove that for $k = 2$ it can be replaced by 6 (but not by 5).

In the rest of this section we show that if the goal is to increase the connectivity of a *branching* then the size of a smallest augmenting set can be expressed by a very simple formula. Namely, the smallest size equals the sum of the out-degree deficiencies. (A branching is a directed forest where each in-degree is at most 1.) This result was obtained – in the special case of a directed tree – by Masuzawa et al. [12] by a long and sophisticated algorithmic proof. Here we deduce this result from Theorem 1 (using its refinement Lemma 9).

Lemma 10. *Let $G = (V, E)$ be an acyclic digraph for which the maximum deficiency γ (with respect to k-connectivity) of an independent family of one-way pairs can be attained by a family $\mathcal{F}$ whose tails $\{X_1, ..., X_t\}$ are pairwise disjoint. Then it can be* *attained by the out-degree deficiencies, that is,*

$$p(\mathcal{F}) = \sum_{v \in V} (k - d^+(v))^+.$$

Proof. Let X_i be the tail of some member (X_i, Y_i) of $\mathcal{F}$. Since G is acyclic, the subgraph induced by X_i contains a vertex v_i with out-degree zero for every $1 \le i \le t$. Thus $\Gamma^+(v_i) \subseteq \Gamma^+(X_i)$, from which $k - d^+(v_i) \ge k - |\Gamma^+(X_i)| \ge p_{def}(X_i, Y_i)$ follows. This yields the inequality $p(\mathcal{F}) \ge \sum_{v \in V}(k - d^+(v))^+ \ge \sum_{i=1}^t (k - d^+(v_i)) \ge \gamma = p(\mathcal{F})$. □

Lemma 11. *Let $B = (V, E)$ be a branching on $n \ge k+1$ vertices. Then $\sum_{v \in V}(k - d^+(v))^+ \ge n(k-1)+1$.*

Proof. $\sum_{v \in V}(k - d^+(v))^+ \ge nk - \sum_{v \in V} d^+(v) = nk - |E| \ge nk - (n-1) = n(k-1)+1$. □

Note, that the out-degree deficiency of a branching is always greater or equal to its in-degree deficiency.

Lemma 12. *Let $B = (V, E)$ be a branching on $k+1 \le n \le 2k+1$ vertices with maximum out-degree k. Then $m_k(B) = \sum_{v \in V}(k - d^+(v))^+$.*

Proof. It is enough to prove that there exists a k-connected and k-regular digraph $G = (V, E')$ with $E \subseteq E'$. To see this first we prove that the vertices of B have an acyclic ordering $(v_0, ..., v_{n-1})$ for which $j - i \le k$ holds for every edge $v_i v_j \in E$. We can assume that B is an arborescence (that is, a directed tree). Let $w \in V$ be a vertex with $d^+(w) = k$ and let us denote the out-neighbours of w by $s_1, ..., s_k$. Let B_i denote the subarborescence of B rooted at s_i $(i = 1, ..., k)$ and let B_w denote the subarborescence rooted at w. We can also assume that the indices of the out-neighbours of w are choosen in such a way that $|B_i| \le |B_j|$ for $1 \le i < j \le k$. Now we define the required acyclic ordering as follows: first come the vertices not included in B_w, in arbitrary order. Then comes w, followed by the out-neighbours $s_1, ..., s_k$ of w, in this order. Then come the vertices of $B_1 - s_1$, then the vertices of $B_2 - s_2$, and so on, following the ordering of the out-neighbours of w. The ordering within a group of vertices of $B_i - s_i$ is arbitrary for every $1 \le i \le k$. It can be checked easily that this ordering has the required property. The details are left to the reader.

Finally, based on this ordering, one can easily construct a k-regular and k-connected digraph G which contains B as a spanning subgraph. Consider the indices modulo n and let $E' = \{v_i v_{i+l} : i = 0, ..., n-1; l = 1, ..., k\}$. This digraph $G = (V, E')$ is k-regular and it is easy to see that it is k-connected, as well. By the special choice of the acyclic ordering, B is a spanning subgraph of G. □

The next simple statement follows from [6, Lemma 4.6].

Lemma 13 [6]. *Let $G = (V, E)$ be a digraph and let $\mathcal{F}$ be an independent family of one-way pairs of G which has maximum deficiency with respect to some $k < |V| - 1$. Then $|\mathcal{F}| \le |V|$.* □

Corollary 3 [12]. *Let $B = (V, E)$ be a branching. Then $m_k(B) = \sum_{v \in V}(k - d^+(v))^+$.*

Proof. We use induction on k. For $k = 1$, the result is easy to see. Now let $B = (V, E)$ be a branching and $k \ge 2$. The case $k = |V| - 1$ is obvious, hence we assume $k < |V| - 1$. If $d < k$ for the maximum out-degree d of B, then by induction B has a d-regular and d-connected supergraph B'. The digraph B' has out-degree deficiency $(k - d)|V|$ (with respect to k), which is the maximum possible deficiency of an independent family, as well, by Lemma 13. This implies – using Theorem 1 – that B' has a k-regular and k-connected supergraph, and we are done. If $d > k$, we can delete edges until $d = k$ holds without increasing the out-degree deficiency, therefore in the rest of the proof we can assume that $d = k$. By Lemma 12 we can also assume $n \ge 2k+2$, and hence by Lemma 11 we obtain that the out-degree deficiency of B (and hence the maximum deficiency of the independent families of one-way pairs) is at least $(2k+2)(k-1)+1 = 2k^2+1$. Thus by Lemma 9 the maximum deficiency can be attained by a subpartition-type family. Hence by Lemma 10 it can be attained by the sum of in- or out-degree deficiencies. Since the sum of out-degree deficiencies is at least as big as the sum of in-degree deficiencies in B, the statement of the corollary follows by Theorem 1. □

Recall that in [4] it was conjectured that $m_k(G)$ is equal to the sum of the in- or out-degree deficiencies for every acyclic digraph G.

4. The algorithm

In this section we give a combinatorial algorithm which solves the directed k-connectivity augmentation problem optimally in time $O(n^6)$ for every fixed $k \ge 1$.

Let $G = (V, E)$ be the starting digraph and let $k \ge 2$ be the target connectivity. Let us introduce the *out-deficiency* $p_k^o(X)$ and the *in-deficiency* $p_k^i(X)$ (with respect to k) of a set $\emptyset \ne X \subset V$ as follows:

$$p_k^o(X) := \begin{cases} (k - |\Gamma^+(X)|)^+ & \text{if } X \cup \Gamma^+(X) \ne V \\ 0 & \text{otherwise,} \end{cases}$$

$$p_k^i(X) := \begin{cases} (k - |\Gamma^-(X)|)^+ & \text{if } X \cup \Gamma^-(X) \ne V \\ 0 & \text{otherwise.} \end{cases}$$

The *out-deficiency (in-deficiency) of a subpartition* of V is the sum of out-deficiencies (in-deficiencies, resp.) of its members. Recall, that $m_k(G)$ denotes the minimum size of a set of new edges which makes G k-connected. This number is equal to the maximum deficiency of the independent families of one-way pairs of G (with respect to k) by

Theorem 1. Let $m_k^i(G)$ ($m_k^o(G)$) denote the maximum in-deficiency (out-deficiency) of the subpartitions of V, respectively. Clearly, $\max\{m_k^o(G), m_k^i(G)\} \leq m_k(G)$. Here, as we remarked earlier, equality does not hold in general, even if G is $(k-1)$-connected.

On the other hand, if $m_k(G) \geq 2k^2-1$, equality holds by Lemma 9. Our algorithm is based on this fact. First we give a sketch of the algorithm. Using max-flow computations it is easy to decide whether $\max\{m_k^o(G), m_k^i(G)\}$ (and hence $m_k(G)$) is at least $2k^2-1$, as we shall show, and if yes, $m_k(G)$ can also be computed. This way, if $m_k(G) \geq 2k^2-1$, it is possible to decide whether some new edge e is contained in an optimal augmentation by checking whether $m_k(G+e) = m_k(G) - 1$ holds. Hence in this case we can find a set F of edges for which $m_k(G+F) = m_k(G) - |F|$ and $m_k(G+F) = 2k^2-2$ hold. Then, in the last part, we simply try every possibe augmenting set of $G+F$ of size $2k^2-2$ and choose one which makes $G+F$ k-connected. This set of edges together with the previously chosen set F of edges is an optimal augmenting set of G. In this last phase we shall be allowed to search for possible solutions on a "small" subset of V, thus the number of connectivity-tests will depend only on k. If $m_k(G) \leq 2k^2-2$ then we find an optimal augmenting set by examining all the possible augmenting sets of size at most $2k^2-2$.

Before providing a more detailed description and analysis we recall the well-known submodular property of the functions $|\Gamma^-|$ and $|\Gamma^+|$. Namely, for each pair of subsets $X, Y \subseteq V$ we have

$$|\Gamma^+(X)| + |\Gamma^+(Y)| \geq |\Gamma^+(X \cap Y)| + |\Gamma^+(X \cup Y)|, \tag{5}$$

$$|\Gamma^-(X)| + |\Gamma^-(Y)| \geq |\Gamma^-(X \cap Y)| + |\Gamma^-(X \cup Y)|. \tag{6}$$

The correctness of the algorithm is based on the next two lemmas. The first one deals with a digraph G^o obtained from G by the following construction. Let us add a new vertex s to V and a new set F^o of edges consisting of at most k parallel edges from each vertex $v \in V$ to s such that in the new graph $G^o := (V+s, E \cup F^o)$ we have

$$g^o(X) \geq k \quad \text{for every} \quad \emptyset \neq X \subset V, \ \ V - (X \cup \Gamma_G^+(X)) \neq \emptyset, \tag{7}$$

where $g^o(X) := |\Gamma_G^+(X)| + d(X, s)$. (Here $d(X, s)$ denotes the number of edges in G^o with tails in X and head s.) Observe that g^o is submodular. Such F^o and G^o obviously exist and F^o satisfies $m_k^o(G) \leq |F^o|$. Let $T_{F^o} := \{v \in V : d(v, s) \geq 1\}$ denote the set of tails of the edges of F^o.

Lemma 14. *Let $G^o = (V+s, E \cup F^o)$ be a digraph obtained as above from a digraph $G = (V, E)$, and assume that F^o is (for inclusion) minimal subject to (7). Then $|F^o| \geq k^2 - k + 2$ implies $m_k^o(G) = |F^o|$.*

Proof. Let us call a subset $\emptyset \neq X \subset V$ with $V - (X \cup \Gamma_G^+(X)) \neq \emptyset$ *out-critical* in G^o if $g^o(X) = k$. The minimality of F^o implies that for every edge $e = xs \in F^o$ there exists an out-critical set X which contains x. Thus there exists a family of out-critical sets whose union covers T_{F^o}. Let us choose such a family $\mathcal{F} = \{X_1, ..., X_t\}$ for which t is as small as possible. If $\mathcal{F}$ consists of pairwise disjoint sets, we have $kt = \sum_i g^o(X_i) = \sum_i |\Gamma_G^+(X_i)| + d(X_i, s)$, from which $|F^o| = \sum_i (k - |\Gamma_G^+(X_i)|) \leq m_k^o(G)$, as required.

Suppose that there are two intersecting members X_i, X_j of $\mathcal{F}$. Let $l = |\Gamma_G^+(X_i \cup X_j)|$. Using the submodularity of g^o and (7) we obtain $k+k = g^o(X_i) + g^o(X_j) \geq g^o(X_i \cap X_j) + g^o(X_i \cup X_j) \geq k + g^o(X_i \cup X_j)$. This gives $d(X_i \cup X_j, s) \leq k - l$. The minimality of t implies that $d(X_i \cup X_j, s) \geq 1$ and hence $l \leq k-1$ follows. From these inequalities we obtain $d(X_i \cup X_j \cup \Gamma_G^+(X_i \cup X_j), s) \leq kl + (k-l) \leq k(k-1) + 1 = k^2 - k + 1$, which implies that $V - (X_i \cup X_j \cup \Gamma_G^+(X_i \cup X_j)) \neq \emptyset$. Thus by the submodularity of g^o we obtain

$$k + k = g^o(X_i) + g^o(X_j) \geq g^o(X_i \cap X_j) + g^o(X_i \cup X_j) \geq k + k,$$

therefore $X_i \cup X_j$ is also out-critical. This contradicts the minimality of t, since replacing X_i and X_j by $X_i \cup X_j$ in $\mathcal{F}$ would yield a smaller family of out-critical sets covering T_{F^o}.

□

Similarly, we can construct a new graph $G^i := (V+s, E \cup F^i)$ by adding a new vertex s and a new set F^i of edges with tail s and heads in V for which

$$g^i(X) \geq k \quad \text{for every} \quad \emptyset \neq X \subset V, \ \ V - (X \cup \Gamma_G^-(X)) \neq \emptyset \tag{8}$$

holds, where $g^i(X) := |\Gamma_G^-(X)| + d(s, X)$. Let H_{F^i} denote the set of heads of the edges of such an F^i. A similar proof shows that if F^i is minimal and $|F^i| \geq k^2 - k + 2$, then $m_k^i(G) = |F^i|$.

Let F^o and F^i be two sets of edges satisfying (7) and (8), respectively, as above.

Lemma 15. *There exists an optimal augmenting set of G with tails in T_{F^o} and heads in H_{F^i}.*

Proof. Let J be an optimal augmenting set of G for which the number of those edges in J whose tail is not in T_{F^o} or whose head is not in H_{F^i} is minimal and suppose that this number is positive. Then, say, there exists an edge $e = xy \in J$ with $x \notin T_{F^o}$. Clearly, $G' := (V, E \cup J - e)$ has connectivity $k-1$ and $m_k(G') = 1$. It is easy to see that there exists a unique minimal out-tight set X and a unique minimal in-tight set Y in G'. (This follows from Theorem 1 by observing that different minimal out-tight sets X_1, X_2 determine independent one-way pairs $(X_1, V - X_1 - \Gamma^+(X_1)), (X_2, V - X_2 - \Gamma^+(X_2))$. A direct proof is also easy to find.) Hence for every edge $e' = x'y'$ with $x' \in X$ and $y' \in Y$ the graph $G' + e'$ is k-connected. We know that there exists a vertex $x' \in X \cap T_{F^o}$ and a vertex $y' \in Y \cap H_{F^i}$ (otherwise X or Y would violate (7) or (8) in G^o or G^i). Thus $J - xy + x'y'$ is an optimal augmenting set, too, which contradicts the choice of J.

□

ALGORITHM TO FIND A MINIMUM SIZE k-VERTEX-CONNECTED AUGMENTATION OF A DIGRAPH

PART 1. Construct an auxiliary digraph $G^o = (V+s, E \cup F^o)$ which satisfies (7) and for which F^o is minimal subject to this property. (This can be done by adding a new vertex s and k parallel edges from each vertex of V to s and then greedily deleting new edges as long as (7) holds. It can be decided by max flow computations if some new edge can be deleted without destroying (7).) Let $\gamma_o = |F^o|$. Then repeat the above procedure with the corresponding G^i, F^i and (8). Let $\gamma_i = |F^i|$.

PART 2. If $\max\{\gamma_o, \gamma_i\} = 2k^2 - 2$ then for every possible set H of new edges with tails in T_{F^o} and heads in H_{F^i} and with size $2k^2 - 2$ check whether $G = (V, E \cup H)$ is k-connected. If it is k-connected for some H, stop, H (together with the possible set F of new edges added previously) is an optimal solution. If $\max\{\gamma_o, \gamma_i\} < 2k^2 - 2$ then for every possible set H of new edges with tails in T_{F^o} and heads in H_{F^i} and with size at most $2k^2 - 2$ (starting with the sets of size 1 and then increasing the size step by step) check whether $G = (V, E \cup H)$ is k-connected. If it is k-connected for some H, stop, H is an optimal solution. Lemma 15 guarantees that the required set H exists in both cases.

PART 3. If $\max\{\gamma_o, \gamma_i\} \geq 2k^2 - 1$ then for every possible new edge $e = xy$ with $x \in T_{F^o}$, $y \in H_{F^i}$ check whether $m_k(G + e) = m_k(G) - 1$ holds. (This can be done by computing the corresponding values γ_o and γ_i of $G+e$. Observe that $2k^2 - 1 \geq k^2 - k + 2$. Hence by Lemma 9 and Lemma 14 it is clear that $m_k(G + e) = m_k(G) - 1$ if and only if $\max_{(G+e)^o}\{\gamma_o, \gamma_i\} = \max_{G^o}\{\gamma_o, \gamma_i\} - 1$.) If this holds, add e to the current G (and to the current set F of augmenting edges) and return to PART 1.

Now let us briefly discuss the efficiency of the algorithm. It is easy to see that in the construction of the starting G^o (and G^i) testing whether an edge e can be deleted without destroying (7) or (8) requires at most $n - 1$ max flow computations for each of the (at most kn) possible edges of F^o (and F^i). There are max flow algorithms with running time $O(n^3)$ (see e.g. [7]), thus PART 1 needs $O(kn^5)$ time. In PART 2 one performs a series of k-connectivity tests, where the number $f(k)$ of tests depends only on k, since by Lemma 15 the set of tails and heads of the possible new edges can be chosen from T_{F^o} and H_{F^i}, respectively, and $\max\{|T_{F^o}|, |H_{F^i}|\} \leq 2k^2 - 2$. Using the algorithm of [8] for checking k-connectedness, this can be done in time $O(f(k)kn^3)$.

To decide whether an edge xy can be added to F in PART 3 first we compute the (unique) maximal out-tight set X_m of G^o containing x and the (unique) maximal in-tight set Y_m of G^i containing y. This requires $2n$ max flow computations. Let us assume, by symmetry, that $\gamma^o \geq \gamma^i$. Then it can be observed that xy cannot be added to F if and only if $G^o + xy - x's$ violates (7) for every $x's$ with $x' \in X_m$. Since $d(X_m, s) \leq k$, this can be decided by kn max flow computations. It is also easy to see, using Lemma 15, that one of the edges with tail in $X_m \cap T_{F^o}$ will do. Hence at most kn edges are necessary to examine. This gives a bound $O(k^2n^5)$ for PART 3. Since $m_k(G) \leq kn$ by Lemma 13, PART 1 and PART 3 are executed at most kn times. Therefore the running time of the algorithm is $O(f'(k)n^6)$ for some (exponential) function f' of k.

The results of this section are summarized in the following theorem.

Theorem 6. *The graph constructed by the above algorithm is an optimal k-connected augmentation of G. The running time of the algorithm is $O(n^6)$ for every fixed k.*

□

5. The ST-edge-connectivity version

In this section we show that a generalization of the k-edge-connectivity augmentation problem given in [6] in fact generalizes the k-vertex-connectivity augmentation problem, as well. Thus, the (algorithmic) investigations related to the topic of the present paper can also be done using a certain edge-connectivity framework.

Let $G = (V, E)$ be a digraph with two specified non-empty subsets S, T of vertices (which may or may not be disjoint). We say that G is *k-edge-connected from S to T* (or *k-ST-edge-connected*) if there are k edge-disjoint paths from every vertex of S to every vertex of T. (When $S = T = V$ we are back at k-edge-connectivity.) A subset $X \subseteq V$ is called *essential* if $T \cap X \neq \emptyset$ and $S - X \neq \emptyset$ hold. We say that a family of subsets of V is *(S, T)-independent* if it contains at most one $t\bar{s}$-set for every pair $s \in S$, $t \in T$. (A $t\bar{s}$-set is one which contains t but disjoint from s.) The out-degree function of a digraph G is denoted by ρ_G. The *deficiency* of an (S, T)-independent family $\{X_1, ..., X_r\}$ of essential sets is defined to be $\sum_1^r (k - \rho_G(X_j))$.

Theorem 7 [6]. *Given a digraph $G = (V, E)$ and two non-empty subsets S, T of vertices, G can be made k-edge-connected from S to T by adding at most γ new edges with tails in S and heads in T if and only if*

$$\sum_j (k - \rho_G(Y_j)) \leq \gamma \tag{9}$$

holds for every choice of an (S, T)-independent family $\{Y_1, ..., Y_r\}$ of essential sets of G.

□

Proof of Theorem 1. Let $G = (V, E)$ be a digraph to be made k-vertex-connected. Construct a new digraph G' by using the well-known vertex-splitting procedure, i.e. substitute each vertex v of V by two new vertices v_S and v_T, and add a new edge v_Tv_S at the same time. Then for every edge $vw \in E$ add an edge v_Sw_T to the new graph. Let $S := \{v_S : v \in V\}$ and $T := \{v_T : v \in V\}$. Let $m_k^{ST}(G')$ denote the size of a smallest set of edges which makes G' k-edge-connected from S to T and let $p_k(G)$ (and $p_k^{ST}(G')$) denote the maximum deficiency of independent families of one-way pairs ((S, T)-independent families of essential sets) in G (in G', respectively), with respect to k.

It is easy to see that the digraph G' obtained from G by this construction is l-edge connected from S to T for some $l \geq 1$ if and only if G is l-vertex-connected. It is also easy to see that every set F of new edges with tails in S and heads in T, which makes G' k-edge-connected from S to T, determines a set of new edges of size $|F|$ on V which makes G k-vertex-connected, thus $m_k(G) \leq m_k^{ST}(G')$.

Let $\mathcal{F}' = \{Y_1, ..., Y_p\}$ be an (S, T)-independent family of essential sets of G' with deficiency $p_k^{ST}(G')$ for which $|\mathcal{F}'|$ is minimal and subject to this $\sum_1^p |T - Y_j| + |S \cap Y_j|$ is maximal. We claim that there are no edges from $S - Y_j$ to $T \cap Y_j$ for $1 \leq j \leq p$. Suppose that this property does not hold and let st be an edge in G' with $s \in S - Y_j$ and $t \in T \cap Y_j$ for some $Y_j \in \mathcal{F}'$. There are two cases to consider. First assume that $|T \cap Y_j| \geq 2$. In this case let us replace Y_j by $Y_j - t$ in $\mathcal{F}'$. It is easy to see that $Y_j - t$

is essential, the family obtained is (S, T)-independent and $\rho_{G'}(Y_j - t) \le \rho_{G'}(Y_j)$ by the special structure of G'. This contradicts the choice of $\mathcal{F}'$. Now consider the second case, where $|T \cap Y_j| = \{t\}$. Observe, that now $|S - Y_j| \ge 2$ holds, since otherwise $\rho_{G'}(Y_j) \ge |T| - 1 \ge k$ would follow, and we could delete Y_j from $\mathcal{F}'$ without decreasing its deficiency. Let us replace Y_j by $Y_j + s$ in $\mathcal{F}'$. It is easy to see that $Y_j + s$ is essential, the family obtained is (S, T)-independent and $\rho_{G'}(Y_j + s) \le \rho_{G'}(Y_j)$. This contradicts the choice of $\mathcal{F}'$.

Let us define a family of ordered pairs in G as follows: $\mathcal{F} := \{(\{v \in V : v_S \notin Y_j\}, \{v \in V : v_T \in Y_j \text{ and } v_S \in Y_j\}) : Y_j \in \mathcal{F}'\}$. The tail of a pair in $\mathcal{F}$ is clearly non-empty, since the sets in $\mathcal{F}'$ are essential. To see that the head of a pair is also non-empty, suppose that $v_S \notin Y_j$ for each $v_T \in Y_j$ holds for some $Y_j \in \mathcal{F}'$. Choose a vertex $v_S \notin Y_j$ for which $v_T \in Y_j$. If $S - (Y_j + v_S) \ne \emptyset$ then replacing Y_j by $Y_j + v_S$ in $\mathcal{F}'$ leads to a contradiction by the choice of $\mathcal{F}'$. If $S \subset Y_j + v_S$ then $Y_j \cap T = \{v_T\}$ and $\rho_{G'}(Y_j) \ge |T - Y_j| \ge k$ follows, a contradiction.

Thus, since there are no edges from $S - Y_j$ to $T \cap Y_j$ $(1 \le j \le p)$, the family $\mathcal{F}$ consists of one-way pairs and each pair in $\mathcal{F}$ has deficiency equal to $k - \rho_{G'}(Y_j)$ for the corresponding member $Y_j \in \mathcal{F}'$. Furthermore, $\mathcal{F}$ is an independent family, since $\mathcal{F}'$ is (S, T)-independent. From these observations it follows that $p_k(G) \ge p_k^{ST}(G')$, which gives $p_k^{ST}(G') \le p_k(G) \le m_k(G) \le m_k^{ST}(G')$. By Theorem 7 equality holds everywhere, in particular $p_k(G) = m_k(G)$ holds, as required.

□

Note, that Enni [1] developed a combinatorial polynomial algorithm to make a digraph optimally 1-ST-edge-connected. For $k \ge 2$ no combinatorial algorithm is known for the k-ST-edge-connectivity augmentation problem.

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