COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA **22** (1) (2002) 47–70

GRAPH ORIENTATIONS WITH EDGE-CONNECTION AND PARITY CONSTRAINTS

ANDRÁS FRANK*, ZOLTÁN KIRÁLY†

Received December 14, 1998
Revised January 12, 2001

Parity (matching theory) and connectivity (network flows) are two main branches of combinatorial optimization. In an attempt to understand better their interrelation, we study a problem where both parity and connectivity requirements are imposed. The main result is a characterization of undirected graphs $G=(V,E)$ having a k-edge-connected T-odd orientation for every subset $T\subseteq V$ with $|E|+|T|$ even. (T-odd orientation: the in-degree of v is odd precisely if v is in T.) As a corollary, we obtain that every $(2k)$-edge-connected graph with $|V|+|E|$ even has a $(k-1)$-edge-connected orientation in which the in-degree of every node is odd. Along the way, a structural characterization will be given for digraphs with a root-node s having k edge-disjoint paths from s to every node and $k-1$ edge-disjoint paths from every node to s.

1. Introduction

The notion of parity plays an important role in describing combinatorial structures. The prime example is W.T. Tutte's theorem [20] on the existence of a perfect matching of a graph. Later, the notion of "odd components" has been extended and used by W. Mader [12] in his disjoint A-paths theorem, by R. Giles [8] in describing matching-forests, by L. Nebeský [16] in determining the maximum genus, by W. Cunningham and J. Geelen [2]

Mathematics Subject Classification (2000): 05C75, 05C40

* Supported by the Hungarian National Foundation for Scientific Research, OTKA T029772. Part of research was done while this author was visiting EPFL, Lausanne, June, 1998.

† Supported by the Hungarian National Foundation for Scientific Research, OTKA T029772 and OTKA T030059.

$v \in T$. It is easy to prove that a connected graph has a T-odd orientation if and only if T is G-even. (Namely, if an orientation is not yet T-odd, then there are at least two bad nodes. Let P be a path in the undirected sense that connects two bad nodes. By reversing the orientation of all of the edges of P, we obtain an orientation having two fewer bad nodes.) Therefore we will throughout assume that each undirected graph G occuring in the paper is connected.

Note that if we subdivide each edge of G by a new node and let T' denote the union of T and the set of subdividing nodes, then there is a one-to-one correspondence between T-odd orientations of G and T'-joins of the subdivided graph G'. To recall: a **T'-join** is a subgraph of G' in which a node v is of odd degree precisely if v belongs to T'.

Given a matroid M on a groundset S and a partition of S into pairs, the matroid parity problem consists of deciding if M has a basis intersecting each pair in an even number of elements (that is, in 0 or 2.) There are several equivalent formulations of this, and one of them, which may be called the **polymatroid parity** problem, is as follows. Given an integer-valued submodular function $b: 2^V \to \mathbf{Z}$ (that is, $b(X)+b(Y) \geq b(X \cap Y)+b(X \cup Y)$ for every pair of subsets $X, Y \subseteq S$), decide if the **base-polyhedron** $B(b) := \{x \in \mathbf{R}^V : x(Z) \leq b(Z)$ for every $Z \subseteq V$ and $x(V) = b(V)\}$ contains an integer point whose components are of prescribed parity (say, all even).

As mentioned above, L. Nebeský [16] characterized graphs having a rooted out-connected and T-odd orientation. It is known that, given a connected undirected graph $G=(V,E)$ with a specified root-node s and a positive integer k, those integer-valued vectors $m: V \to \mathbf{Z}_+$ for which G has a rooted k-out-connected orientation with in-degree $\varrho(v) = m(v)$ $(v \in V)$ span a base-polyhedron. Therefore Nebesky's problem may be considered as a special polymatroid parity problem. For general matroids, matroid-parity is known to be intractable, but there are several special cases where deep characterizations are available, and Nebeský's is one of them.

In this view it is quite natural to change the edge-connection property in Nebeský's theorem. For example, the rooted l-out-connected and T-odd orientation problem was recently solved in [5]. In another natural variation, one may be interested in finding a characterization of graphs and G-even subsets T having a strongly-connected (or, more generally, l-ec) T-odd orientation. Note that the notion of strong-connectivity in some respects is more complex than that of rooted out-connectivity. For example, finding a minimum cost rooted out-connected subgraph of a digraph (that is, finding a cheapest spanning arborescence) is polynomially solvable while the minimum cost strongly connected subgraph problem is NP-complete.

Strongly connected and T-odd orientability of graphs can also be shown to be a polymatroid parity problem and the initial goal of our research was to decide whether this problem is still tractable (like Nebeský's) or perhaps it is already NP-complete. Though this problem remains unsolved, it served as the main motivation behind the present work. We consider the following property of undirected graphs G.

(P1): *G has an l-edge-connected T-odd orientation for every G-even subset T of nodes.*

The main result of this paper (Theorem 4.1) is an NP and a co-NP characterization of graphs admitting (P1). As a co-NP characterization, we prove that (P1) holds if and only if G is $(2l+2)^-$-edge-connected. With the help of this characterization, a G-even subset T can be found along with a polynomially checkable certificate for the non-existence of an l-ec T-odd orientation. As an NP characterization, we prove that (P1) holds if and only if G can be constructed from a node by a sequence of two kinds of simple operations. With the help of this building procedure, an l-ec T-odd orientation of G can be found for any concrete G-even set T.

As a corollary, it will be shown that every 4-edge-connected graph $G = (V,E)$ with a G-even subset T has a strongly connected T-odd orientation, or more generally, that every $(2l+2)$-edge-connected graph, has an l-edge-connected T-odd orientation. Here $(2l+1)$-edge-connectivity is not sufficient: take $l=1$, G the complete graph on four nodes and $T=V$.

The proof of the main result has two ingredients. The first one (Theorem 2.3) is a characterization of graphs having k^--edge-connected orientation, a result which may be considered as a counterpart of C.St.J.A. Nash-Williams' classical theorem on the existence of k-edge-connected orientations [15]. The second one (Theorem 2.6) is a splitting-off result concerning k^--edge-connectivity of digraphs which may be considered as a counterpart of W. Mader's directed splitting off theorem [13]. Mader used his theorem to describe a constructive characterization of all k-edge-connected digraphs. We describe an analogous characterization of k^--ec digraphs which will be used for constructing all $(2k)^-$-ec undirected graphs.

An interesting feature of this approach is that it relies on two earlier, rather general results on supermodular functions (Theorems 2.2 and 2.5) which may be viewed as abstract forms of the above-mentioned theorems of Nash-Williams and Mader, respectively. This demonstrates that those general results can quite nicely be applied in concrete situations.

The following analogy may serve as a further motivation to the problem of characterizing graphs admitting property (P1). For any connected graph G and subset T of nodes of even cardinality, the maximum number ν_T of in characterizing optimal path-matchings. L. Lovász' [10] general framework on matroid parity (as its name already suggests) also relies on odd components. Sometimes parity comes in already with the problem formulation. Lovász [9] for example considered the existence of subgraphs with parity prescription on the degree of nodes. The theory of T-joins describes several problems of this type.

Another large class of combinatorial optimization problems concerns connectivity properties of graphs, in particular, the role of cuts, partitions, trees, paths, and flows are especially well studied.

In some cases the two areas overlap. For example, Seymour's theorem [18] on minimum T-joins implies a result on the edge-disjoint paths problem in planar graphs. In [4] some informal analogy was pointed out between results on parity and on connectivity, but in order to understand better the relationship of these two big aspects of combinatorial optimization, it is desirable to explore further problems where both parity and connectivity requirements are imposed. For example, Nebeský provided a characterization of graphs having an orientation in which every node is reachable from a given node by a directed path and the in-degree of every node is odd.

One goal of the present paper is to provide a new result on orientations of undirected graphs simultaneously satisfying connectivity and parity requirements. The following concepts of connectivity will be used.

Let k be a positive integer. A digraph $D=(V,A)$ is **k-edge-connected** (**k-ec**, for short) if the **in-degree** $\varrho(X) = \varrho_D(X)$ of X (:the number of edges entering X) is at least k for every non-empty proper subset X of V. By Menger's theorem, this is equivalent to requiring that there are k edge-disjoint paths from each node to every other. The **out-degree** $\delta(X)=\delta_D(X)$ is the number of edges leaving X, that is, $\delta(X)=\varrho(V-X)$. The 1-ec digraphs are called **strongly connected**. We call D **rooted out-connected** if it has a node s so that $\varrho(X)\geq 1$ for every non-empty subset $X\subseteq V-s$, or equivalently, if every node is reachable by a directed path from s. More generally, D is **rooted k-out-connected** (from s) if $\varrho(X)\geq k$ for every non-empty subset $X\subseteq V-s$. By Menger's theorem, this is equivalent to requiring that there are k-edge-disjoint paths from s to every other node. When the root s is specified we speak of **s-rooted** connectivity.

A digraph D is said to be **k^--edge-connected** (**k^--ec**) if it has a node s, called **root node**, so that $\varrho(X)\geq k$ for every subset X with $\emptyset\subset X\subseteq V-s$, and $\varrho(X)\geq k-1$ for every subset X with $s\in X\subset V$. The name k^--edge-connectivity is motivated by the observation that a k-ec digraph is clearly k^--ec and a k^--ec digraph is $(k-1)$-ec, that is, k^- edge-connectivity of digraphs is somewhere between $(k-1)$- and k-edge-connectivity. When the role of the root is emphasized, we say that D is k^--ec with respect to s. Throughout the root-node will be denoted by s. Note that by reorienting the edges of a directed path from s to another node s' of a k^--ec digraph one obtains a k^--ec digraph with respect to root s'.

Define a set-function p_k as follows. Let $p_k(\emptyset):=p_k(V):=0$ and

$$p_k(X) := \begin{cases} k & \text{if } \emptyset\subset X\subseteq V-s \\ k-1 & \text{if } s\in X\subset V. \end{cases} \tag{1}$$

By the definition, D is k^--ec if and only if $\varrho(X)\geq p_k(X)$ holds for every $X\subseteq V$. By Menger's theorem the k^--edge-connectivity of D is equivalent to requiring that D has k edge-disjoint paths from s to every node and $k-1$ edge-disjoint paths from every node to s.

An undirected graph $G=(V,E)$ is **k-edge-connected** (**k-ec**) if the number $d(X)$ of edges connecting any non-empty proper subset X of V and its complement $V-X$ is at least k. We are going to introduce a refinement of this notion and to this end it is useful to recall W.T. Tutte's classical disjoint tree theorem [19].

Theorem 1.1 (Tutte). *An undirected graph $G=(V,E)$ contains k edge-disjoint spanning trees if and only if*

$$e_G(\mathcal{F})\geq k(t-1) \tag{2}$$

holds for every partition $\mathcal{F}:=\{V_1,V_2,\dots,V_t\}$ of V into non-empty subsets where $e_G(\mathcal{F})$ denotes the number of edges connecting distinct parts of $\mathcal{F}$.

We call an undirected graph $G=(V,E)$ **$(2k)^-$-edge-connected** (**$(2k)^-$-ec**) if

$$e_G(\mathcal{F})\geq kt-1 \tag{3}$$

for every partition $\mathcal{F}:=\{V_1,V_2,\dots,V_t\}$ of V. Throughout we will assume on partitions to admit at least two non-empty classes and no empty ones.

The name $(2k)^-$-edge-connectivity is motivated by the observation that a $(2k)$-ec graph, on one hand, is always $(2k)^-$-ec since $e_G(\mathcal{F})=\sum_i d(V_i)/2\geq 2kt/2=kt$, that is, (3) is satisfied, and, on the other hand, a $(2k)^-$-ec graph G is $(2k-1)$-ec since (3), when specialized to $|\mathcal{F}|=t=2$, requires for every cut of G to have at least $2k-1$ edges. In other words, $(2k)^-$-edge-connectivity of undirected graphs is somewhere between $(2k-1)$- and $(2k)$-edge-connectivity. Note that, by Tutte's disjoint tree theorem, a graph is $(2k)^-$-ec if and only if it contains k edge-disjoint spanning trees even after deleting any subset of $k-1$ edges.

Let T be a subset of V. We call T **G-even** if $|T|+|E|$ is even. An orientation of G is called **T-odd** if the in-degree of a node v is odd precisely when disjoint T-cuts is always at most the minimum cardinality τ_T of a T-join and there are interesting cases when equality holds. For example, this is so if G is planar and T belongs to one face. Also the min-max equality holds if G is bipartite or series-parallel and T is arbitrary (theorems of P.D. Seymour [18], [17]). Here two general problems had been considered. First, characterize those pairs (G,T) for which $\tau_T = \nu_T$ and second, characterize those graphs G for which $\tau_T = \nu_T$ holds for every (!) even subset T of nodes. The first problem is apparently more natural since its property clearly belongs to NP while it is not clear at all whether the property in the second problem belongs to either of NP or co-NP. Moreover, a good characterization for pairs (G,T) in the first problem would likely give rise to a good characterization for the second problem. However, it turned out that the first problem is NP-complete (M. Middendorf and F. Pfeiffer [14]) while the second one behaves nicer as it belongs to co-NP, a theorem of A. Ageev, A. Kostochka, and Z. Szigeti [1].

The organization of the rest of the paper is as follows. The present section is completed by listing definitions and notation. In Section 2 we recall some known theorems concerning orientations and splitting, and derive two ingredients (Theorems 2.3 and 2.6) of the main proof. Section 3 exhibits a constructive characterization of k^--ec digraphs while Section 4 contains the main theorem and its proof. In Section 6 new proofs are included for the two older theorems from Section 2.

A one-element set is called **singleton.** We often will not distinguish between a singleton and its element. In particular, the in-degree of a singleton $\{x\}$ will be denoted by $\varrho(x)$ rather than $\varrho(\{x\})$. For a set X and an element r, we denote $X \cup \{r\}$ by $X+r$.

For a directed or undirected graph G, let $i_G(X) = i(X)$ denote the number of edges having both end-nodes in X. Let $d(X,Y)$ (respectively, $\bar{d}(X,Y)$) denote the number of edges connecting a node of $X-Y$ and a node of $Y-X$ (a node of $X \cap Y$ and a node of $V-(X \cup Y)$). Simple calculation yields the following identities for the in-degree function ϱ of a digraph G:

$$\varrho(X \cap Y) + \varrho(X \cup Y) = \varrho(X) + \varrho(Y) - d(X,Y), \tag{4}$$

$$\varrho(X-Y) + \varrho(Y-X) = \varrho(X) + \varrho(Y) - \bar{d}(X,Y) - [\varrho(X \cap Y) - \delta(X \cap Y)]. \tag{5}$$

Let f be an edge and r a node of G. Then $G-f$ and $G-r$ denote, respectively, the (di-)graphs arising from G by deleting edge f or node r. By **splitting off** a pair of edges $e = ur$ and $f = rv$ of a (di)-graph, we mean the operation that replaces e and f by a new (directed) edge uv. In a digraph with $\varrho(r) = \delta(r)$, by a **complete splitting** at r we mean an operation consisting of pairing first the edges entering and leaving r, splitting then all these pairs off, and finally leaving r out. We may define analogously a complete splitting at a node r of an undirected graph provided the degree of r is even.

Both in the directed and in the undirected case the inverse operation of complete splitting is as follows. Add a new node r, subdivide j existing edges by new nodes and identify the j subdividing node with r. This will be called **pinching j edges (with r)**. When $j=0$ this means adding a single new node r, while in case $j=1$ we subdivide one edge with a node r.

By the operation of adding a new edge to a (di)graph we always mean that the new edge connects existing nodes. Unless otherwise stated, the newly added edge may be a loop or may be parallel to existing edges.

Two subsets X and Y of node-set V are called **intersecting** if none of sets $X-Y$, $Y-X$, $X \cap Y$ is empty. If, in addition, $V-(X \cup Y)$ is non-empty, then X and Y are **crossing.** A family of subsets containing no two crossing (respectively, intersecting) sets is called **cross-free (laminar)**. A family consisting of the complements of the members of a partition of V is called a **co-partition** of V.

Let p be a non-negative, integer-valued set-function on V for which $p(\emptyset) = p(V) = 0$. Function p is called **crossing supermodular** if $p(X)+p(Y) \leq p(X \cap Y) + p(X \cup Y)$ holds for every pair of crossing subsets X,Y of V. When this inequality is required only for crossing sets X,Y with $p(X) > 0$ and $p(Y) > 0$, we speak of **positively crossing supermodular** functions.

For a number x, let $x^+ := \max(0,x)$. For a function $m: V \to \mathbf{R}$ and subset $X \subseteq V$ we will use the notation $m(X) := \sum(m(v) : v \in X)$.

2. Orientation and splitting off

In this section we recall C. St. J. A. Nash-Williams' orientation theorem [15] and W. Mader's directed splitting off theorem [13], exhibit their extensions into an abstract form concerning crossing supermodular functions, and finally derive a special case of each which will in turn be used in Section 4 for the proof of the main theorem.

Theorem 2.1 (Nash-Williams). *An undirected graph G has a k-edge-connected orientation if and only if G is $(2k)$-edge-connected.*

This has been extended in [3] to the following result. A relatively short, new proof will be included in Section 6.

Theorem 2.2. *Let $G=(V,E)$ be an undirected graph. Suppose that p is a non-negative integer-valued crossing supermodular set-function on V for which $p(\emptyset)=p(V)=0$. Then there exists an orientation of G for which $\varrho(X)\geq p(X)$ holds for every $X\subseteq V$ if and only if both*

$$e_G(\mathcal{F})\geq\sum_{i=1}^{t}p(V_i) \tag{6}$$

and

$$e_G(\mathcal{F})\geq\sum_{i=1}^{t}p(V-V_i) \tag{7}$$

hold for every partition $\mathcal{F}=\{V_1,\dots,V_t\}$ of V. Furthermore, if p is monotone decreasing (that is, $p(X)\geq p(Y)$ whenever $\emptyset\subset X\subset Y\subset V$), then already (6) is sufficient. If p is symmetric (that is, $p(X)=p(V-X)$ whenever $X\subseteq V$), then already the special case $t=2$ of (6) is sufficient (which is equivalent to requiring that $d_G(X)\geq 2p(X)$ for every $\emptyset\subset X\subset V$).

Note that the special case $p(X)\equiv k$ of the last part gives back Nash-Williams' theorem. We will need the following corollary of the second part.

Theorem 2.3. *An undirected graph G has a k^--edge-connected orientation if and only if G is $(2k)^-$-edge-connected.*

Proof. Recall the definition of function p_k in (1) and observe that $p:=p_k$ is a monotone decreasing, crossing supermodular function. Since (6) for this function is just equivalent to the $(2k)^-$-edge-connectivity of G, the theorem follows from the second half of Theorem 2.2. ∎

The second important ingredient of our approach is motivated by the following result of W. Mader [13]. We say that a digraph $D=(U+r,A)$ is **k-edge-connected in U** if

$$\varrho(X)\geq k,\quad \delta(X)\geq k \text{ for every subset } \emptyset\subset X\subseteq U. \tag{8}$$

By Menger's theorem this is equivalent to requiring that there are k edge-disjoint paths in D from u to v for every ordered pair of nodes $u,v\in U$.

Theorem 2.4 (Mader). *Let $D=(U+r,A)$ be a directed graph with a special node r for which $\varrho(r)=\delta(r)$. Suppose that D is k-edge-connected in U. Then there is a complete splitting at r resulting in a k-edge-connected digraph on node set U.*

The following result of [6] may be considered as an extension of Mader's theorem. A new proof will be included in Section 6.

Theorem 2.5. *Let U be a ground-set, p a non-negative, integer-valued positively crossing supermodular set-function on U for which $p(\emptyset)=p(U)=0$. Let m_i,m_o be two non-negative integer-valued functions on U for which $m_i(U)=m_o(U)$. There exists a digraph $H=(U,F)$ for which*

$$\varrho_H(X)\geq p(X) \text{ for every } X\subseteq U \tag{9}$$

and

$$\varrho_H(v)=m_i(v),\quad \delta_H(v)=m_o(v) \text{ for every } v\in U \tag{10}$$

if and only if

$$m_i(X)\geq p(X) \text{ for every } X\subseteq U \tag{11}$$

and

$$m_o(U-X)\geq p(X) \text{ for every } X\subseteq U. \tag{12}$$

In [6] it was shown how this implies Mader's theorem: define $p(X):=(k-\varrho_1(X))^+$ $(\emptyset\subset X\subset U)$ and $p(\emptyset):=p(U):=0$ where ϱ_1 denotes the in-degree function of the digraph arising from D by deleting r. Since ϱ_1 is submodular, p is positively crossing supermodular. Furthermore let $m_i(v)$ $(v\in U)$ denote the number of parallel edges of D from r to v and let $m_o(v)$ denote the number of parallel edges from v to r. Clearly, a digraph $H=(V,F)$ satisfies (10) if and only if F is the set of new edges arising from a complete splitting at r. It is also immediate to see that (9) is equivalent to the requirement that the split off digraph is k-edge-connected, while (11) and (12) are equivalent to the property that D is k-edge-connected in U.

The same approach may be applied to k^--ec digraphs. We say that a digraph $D=(U+r,A)$ with root node $s\in U$ is **k^--edge-connected in U** if

$$\varrho(X)\geq k,\delta(X)\geq k-1 \text{ for every subset } \emptyset\subset X\subset U-s. \tag{13}$$

By Menger's theorem this is equivalent to requiring that there are k edge-disjoint paths in D from s to v and there are $k-1$ edge-disjoint paths in D from v to s for every node $v\in U$.

By applying Theorem 2.5 to another suitable function p, we obtain a splitting off theorem concerning k^--edge-connectivity.

Theorem 2.6. *Let $D'=(U+r,A')$ be a directed graph with a root node $s\in U$ and a special node r for which $\varrho'(r)=\delta'(r)$. Suppose that D' is k^--edge-connected in U. Then there is a complete splitting at r resulting in a k^--edge-connected digraph on node set U.*

Proof. Let $D_1=(U,A_1)$ be the digraph arising from D' by deleting r and let ϱ_1 denote the in-degree function of D_1. Let $m_i(u)$ (respectively, $m_o(u)$) denote the number of parallel edges in D' from r to u (from u to r). Since $\varrho'(r)=\delta'(r)$ we have $m_o(U)=m_i(U)$. Let $p(X):=(p_k(X)-\varrho_1(X))^+$ $(X\subset U)$ and $p(\emptyset):=p(U):=0$. Since both p_k and $-\varrho_1$ are crossing supermodular, p is positively crossing supermodular.

We claim that (11) holds. Indeed, for every $\emptyset\subset X\subset U$ one has $p_k(X)\leq \varrho'(X)=\varrho_1(X)+m_i(X)$ from which $p(X)=(p_k(X)-\varrho_1(X))^+\leq m_i(X)$, which is (11).

We claim that (12) holds, as well. Indeed, for every $\emptyset\subset X\subset U$ we have $p_k(X)=p_k(X+r)\leq \varrho'(X+r)=\varrho_1(X)+m_o(U-X)$ from which $p(X)=(p_k(X)-\varrho_1(X))^+\leq m_o(U-X)$, which is (12).

By Theorem 2.5, there exists a digraph $H=(U,F)$ satisfying (9) and (10). It follows from (9) and from the definition of p that the digraph $D_1+H:=(U,A_1\cup F)$ is k^--ec. By (10), D_1+H is a digraph arising from D' by a complete splitting and hence the proof is complete. ∎

3. Constructing all k^--edge-connected digraphs

In [13] Mader proved the following elegant description of all k-edge-connected digraphs, and the goal of this section is to derive an analogous constructive characterization for k^--ec digraphs.

Theorem 3.1 (Mader). *A directed graph $D=(V,A)$ is k-edge-connected if and only if D can be obtained from a single node by the following two operations: (i) add a new edge, (ii) pinch k existing edges.*

This result immediately follows by combining Theorem 2.4 with another impressive result of Mader:

Theorem 3.2 (Mader). *A minimally (with respect to edge-deletion) k-edge-connected directed graph (with at least two nodes) always contains a node r of in-degree k and out-degree k.*

The following result may be considered as a counterpart of this theorem concerning k^- edge-connectivity. Recall the definition of p_k in (1) by which a digraph is k^--ec if and only of $\varrho(X)\geq p_k(X)$ for every subset X of nodes. We say that a digraph is **minimally k^--edge-connected** (with respect to a given root s) if it is k^--ec but leaving out any edge destroys k^--edge-connectivity.

Theorem 3.3. *Let $D=(V,A)$ be a minimally k^--edge-connected digraph with respect to a root node s. If $|V|\geq 2$, then there is a node $r\in V-s$ and an edge $f=ur\in A$ entering r so that $\varrho(r)=k$, $\delta(r)=k-1$ and $D-f$ is k^--edge-connected in $V-r$.*

Proof. Observe first that for $k=1$ a minimally k^--ec digraph is nothing but a spanning arborescence of root s and the theorem asserts that such a digraph contains a node of in-degree one and out-degree zero which is clearly true. Therefore we assume that $k\geq 2$, which implies, in particular, that D is strongly connected.

We call a subset $X\subseteq V$ **tight** if $\varrho(X)=p_k(X)$. A node r of D and the set $\{r\}$ as well will be called **special** if $\varrho(r)=k$, $\delta(r)=k-1$. A special set is clearly tight. (Since $\delta(s)\geq k$, s is not special.) By the minimality of D, every edge enters a tight set. We will show that there is an edges $f=ur\in A$ so that the head r of f is special and the singleton $\{r\}$ is the only tight set entered by f. For such an edge f, $D-f$ is k^--ec in $V-r$, that is, f satisfies the requirements of the theorem. So assume indirectly that every edge enters a non-special tight set.

We need some preparatory claims.

Claim A. *For crossing sets X and Y, one has $p_k(X)+p_k(Y)=p_k(X\cap Y)+p_k(X\cup Y)$ and $p_k(X)+p_k(Y)\leq p_k(X-Y)+p_k(Y-X)$.* ∎

Claim B. *The intersection of two crossing tight sets is not special.*

Proof. Let X and Y be the two sets. By (5) we have $\varrho(X)+\varrho(Y)=p_k(X)+p_k(Y)\leq p_k(X-Y)+p_k(Y-X)\leq \varrho(X-Y)+\varrho(Y-X)=\varrho(X)+\varrho(Y)-\bar{d}(X,Y)-[\varrho(X\cap Y)-\delta(X\cap Y)]$ from which $\varrho(X\cap Y)\leq \delta(X\cap Y)$ follows and hence $X\cap Y$ cannot be special. ∎

Claim C. *For crossing tight sets X and Y, both $X\cap Y$ and $X\cup Y$ are tight. Moreover, $d(X,Y)=0$ holds.*

Proof. By (4) we have $\varrho(X)+\varrho(Y)=p_k(X)+p_k(Y)=p_k(X\cap Y)+p_k(X\cup Y)\leq \varrho(X\cap Y)+\varrho(X\cup Y)=\varrho(X)+\varrho(Y)-d(X,Y)$ from which equality holds everywhere, and the claim follows. ∎

Let us turn to the proof of the theorem and recall that there is a set $\mathcal{T}$ of non-special tight sets so that every edge of D enters a member of $\mathcal{T}$. Assume that $|\mathcal{T}|$ is minimal and, subject to this,

$$\sum(|Z|^2 : Z\in\mathcal{T}) \text{ is maximum.} \tag{14}$$

Claim D. *$\mathcal{T}$ contains no two crossing members.*

Proof. Suppose indirectly that X and Y are two crossing members of $\mathcal{T}$. By Claim B, $X\cap Y$ is not special. By Claim C, $X\cap Y$ and $X\cup Y$ are tight. Hence $\mathcal{T}':=\mathcal{T}-\{X,Y\}\cup\{X\cap Y,X\cup Y\}$ is a family of non-special tight sets. Since $d(X,Y)=0$, every edge of D enters a member of $\mathcal{T}'$, as well. By the minimality of $\mathcal{T}$, we cannot have $|\mathcal{T}'|<|\mathcal{T}|$, that is, $X\cap Y, X\cup Y\notin\mathcal{T}$ and so $|\mathcal{T}'|=|\mathcal{T}|$. Since X and Y are crossing, $|X|^2+|Y|^2<|X\cap Y|^2+|X\cup Y|^2$, contradicting (14). ∎

Let $\mathcal{K}:=\{X\in\mathcal{T}:s\notin X\}$ and $\mathcal{L}:=\{V-X:X\in\mathcal{T},s\in X\}$. Then $\mathcal{K}$ contains no special set, $\varrho(X)=k$ for every $X\in\mathcal{K}$, and $\delta(X)=k-1$ for every $X\in\mathcal{L}$. Let $\mathcal{C}$ denote the union of $\mathcal{K}$ and $\mathcal{L}$ in the sense that if X is a set belonging to both $\mathcal{K}$ and $\mathcal{L}$, then $\mathcal{C}$ includes two copies of X. Now $\mathcal{C}$ is a laminar family of subsets of $V-s$, and every edge e of D is **covered** by $\mathcal{C}$ in the sense that e enters a member of $\mathcal{K}$ or leaves a member of $\mathcal{L}$. Let us choose families $\mathcal{K}$ and $\mathcal{L}$ so as to satisfy all these properties and so that $\sum(|X|:X\in\mathcal{C})$ is minimum.

Claim E. *There is no node $v\in V$ for which $\{v\}\in\mathcal{K}$ and $\{v\}\in\mathcal{L}$.*

Proof. $v\in\mathcal{L}$ implies $\delta(v)=k-1$. $v\in\mathcal{K}$ implies $\varrho(v)=k$, that is, v would be special, contradicting the assumption on $\mathcal{K}$. ∎

Claim F. *There is a non-singleton member Z of $\mathcal{C}$.*

Proof. Suppose indirectly that every member of $\mathcal{C}$ is a singleton. Let $K=\{v:\{v\}\in\mathcal{K}\}$. Since D is strongly connected and $|V|\geq 2$, there is an edge $e=st$ leaving s. Edge e cannot leave any member of $\mathcal{L}$ since these members are subsets of $V-s$. Therefore e must enter a member of $\mathcal{K}$, that is, K is non-empty. By the strong connectivity of D, there is an edge e' leaving K. By Claim E, no element of K, as a singleton, is a member of $\mathcal{L}$, and hence e' neither enters a member of $\mathcal{K}$ nor leaves a member of $\mathcal{L}$, contradicting the assumption. ∎

Let Z be a non-singleton member of $\mathcal{C}$ which is minimal in the sense that every proper subset of Z belonging to $\mathcal{C}$ is a singleton.

Claim G. *Z induces a strongly connected digraph.*

Proof. Suppose indirectly that there is a nonempty subset $X\subset Z$ so that there is no edge in D from X to $Z-X$. If $Z\in\mathcal{K}$, then replace Z in $\mathcal{K}$ by $Z-X$. Since $k\leq\varrho(Z-X)\leq\varrho(Z)=k$ we have $\varrho(Z-X)=k$. Furthermore, $Z-X$ cannot be special since every edge entering Z enters $Z-X$ as well and hence every edge entering X leaves $Z-X$ from which $k\leq\varrho(X)\leq\delta(Z-X)$.

If $Z\in\mathcal{L}$, then replace Z in $\mathcal{L}$ by X. Since $k-1\leq\delta(X)\leq\delta(Z)=k-1$, we have $\delta(X)=k-1$. In both cases we obtain a laminar family satisfying the requirements for $\mathcal{C}$ and this contradicts the minimal choice of $\mathcal{C}$. ∎

To conclude the proof of the theorem, we distinguish two cases.

Case 1. $Z\in\mathcal{K}$.

There must be an element v of Z for which $\{v\}\notin\mathcal{K}$, for otherwise Z could be left out from $\mathcal{K}$. We claim that $\{u\}\notin\mathcal{K}$ for every $u\in Z$. For otherwise $X:=\{x\in Z:\{x\}\in\mathcal{K}\}$ is a non-empty, proper subset of Z, so by Claim G there is an edge $e=xy$ with $x\in X, y\in Z-X$, and then e cannot be covered by $\mathcal{C}$ (using that $\{x\}$ is not in $\mathcal{L}$ by Claim E).

It follows that every edge uv with $u,v\in Z$ leaves a member of $\mathcal{L}$ which is a singleton by the minimal choice of Z, and hence, by Claim G, $\{v\}$ is in $\mathcal{L}$ for every $v\in Z$. Then we have $k=\varrho(Z)=\sum(\varrho(v):v\in Z)-i(Z)\geq k|Z|-i(Z)$ and $k-1\leq\delta(Z)=\sum(\delta(v):v\in Z)-i(Z)=(k-1)|Z|-i(Z)$ from which $(k-1)(|Z|-1)\geq i(Z)\geq k(|Z|-1)$, a contradiction. Therefore Case 1 cannot occur.

Case 2. $Z\in\mathcal{L}$.

There must be an element v of Z for which $\{v\}\notin\mathcal{L}$, for otherwise Z can be left out from $\mathcal{L}$. We claim that $\{u\}\notin\mathcal{L}$ for every $u\in Z$. For otherwise $X:=\{x\in Z:\{x\}\in\mathcal{L}\}$ is a non-empty, proper subset of Z, so by Claim G there is an edge $e=yx$ with $x\in X, y\in Z-X$, and then e cannot be covered by $\mathcal{C}$ (using that $\{x\}$ is not in $\mathcal{K}$ by Claim E).

It follows that every edge uv with $u,v\in Z$ must enter a member of $\mathcal{K}$, which is a singleton, by the minimal choice of Z, and hence, by Claim G, $\{v\}$ is in $\mathcal{K}$ for every $v\in Z$. Therefore, as $\mathcal{K}$ contains no special sets, no element of Z is special, from which $\delta(v)\geq k$ holds for every $v\in Z$.

We have $k-1=\delta(Z)=\sum(\delta(v):v\in Z)-i(Z)\geq k|Z|-i(Z)$ and $k\leq\varrho(Z)=\sum(\varrho(v):v\in Z)-i(Z)=k|Z|-i(Z)$ from which $k-1\geq k$, a contradiction. Therefore Case 2 cannot occur either, and this contradiction completes the proof of the theorem. ∎∎

We are now in a position to formulate the main result of this section.

Theorem 3.4. *Let $D=(V,A)$ be a digraph with a root-node s and let $k\geq 2$ be an integer. Then D is k^--edge-connected (with respect to s) if and only if D can be constructed from s by a sequence of the following operations: (j) add a new edge, (jj) pinch $k-1$ existing edges with a new node r and add a new edge entering r and leaving an existing node.*

Proof. It is easy to see that both operations (j) and (jj) preserve k^--edge-connectivity.

To see the converse, suppose that D is k^--ec. If D has no edges, then D has the only node s. Suppose now that A is non-empty and assume by induction that every k^--ec digraph, having a fewer number of edges than D has, is **constructible** in the sense that it can be constructed as described in the theorem.

If D has an edge f so that $D' := D - f$ is k^--ec, then D' is constructible and then we obtain D form D' by adding back f, that is, by operation (j). Therefore we may assume that the deletion of any edge destroys k^--edge-connectivity.

By Theorem 3.3 there is a special node $r \in V - s$ and an edge $f = ur$ of D so that $D' := D - f$ is k^--edge-connected in $U := V - r$. In D' both the in-degree and the out-degree of r is k so we may apply Theorem 2.6 to D'. The digraph $D_1 + H$ arising from the complete splitting ensured by Theorem 2.6 is l^--ec. (Here $D_1 := D - r$ and $H = (V, F)$ is the digraph of split-off edges.) Now $D_1 + H$ has $k+1$ fewer edges than D has so $D_1 + H$ is constructible by induction.

Clearly, D arises from $D_1 + H$ by pinching first F with r and adding then $f = ur$, that is, D arises from $D_1 + H$ by applying operation (jj), proving that D is also constructible. ∎

Remarks. Note that using network flow algorithms, it is possible to check in polynomial time for every edge $f = ur$ of D whether f can be deleted from D without violating k^--edge-connectivity in $V - r$. If so, we can leave out f and hence we may assume that every edge of D enters a tight set. A flow algorithm can also compute the (unique) largest tight set X entered by f. Edge f satisfies the requirements of Theorem 3.3 if and only if $X = \{r\}$ and $\delta(X) = k - 1$. Since Theorem 3.3 ensures the existence of such an edge, we can find one.

The proof of Theorem 2.5 in [6] is algorithmic and gives rise to a combinatorial strongly polynomial algorithm provided an oracle is available to decide whether a digraph $D' = (V, A')$ with in-degree function ϱ' satisfies $\varrho'(X) \geq p(X)$ for every subset $X \subseteq V$. We applied Theorem 2.5 to a function p defined by $p(X) := (p_k(X) - \varrho_1(X))^+$ and in this case the oracle can indeed be constructed via a network flow algorithm. Hence we can find in polynomial time a digraph $H = (U, F)$ for which $D_1 + H$ is k^--ec in U. By applying this method at most $|A|$ times, one can find the sequence of operations (j) and (jj) guaranteed by Theorem 3.4.

4. The main result

Our main result is as follows.

Theorem 4.1. *Let $G = (V, E)$ be an undirected graph with $n \geq 1$ nodes and let l be a positive integer. The following properties are equivalent.*

(P1) *G has an l-edge-connected T-odd orientation for every G-even subset $T \subseteq V$,*

(P2) *G is $(2l+2)^-$-edge-connected, that is,*

$$e_G(\mathcal{F}) \geq (l+1)t - 1 \tag{15}$$

for every partition $\mathcal{F} := \{V_1, V_2, \dots, V_t\}$ of V,

(P3) *G can be constructed from a node by a sequence of the following operations: (i) add a new edge, (ii) pinch a subset F of l (distinct) existing edges with a new node r and connect r with an arbitrary existing node u (that may or may not be an end-node of an element of F).*

Note that, by applying Theorem 2.3 to $k = l+1$, we obtain that property (P2) is equivalent to (P2′), and by applying Theorem 1.1 of Tutte to $k = l+1$, we obtain that property (P2) is equivalent to (P2″), where

(P2′) *G has an orientation which is $(l+1)^-$-edge-connected,*

(P2″) *$G - J$ contains $l+1$ edge-disjoint spanning trees for every choice of an l-element subset J of edges.*

Proof. (P1)→(P2). Let $\mathcal{F} := \{V_1, \dots, V_t\}$ be a partition of V. For $j = 2, \dots, t$, choose an element t_j of V_j if $l + i(V_j)$ is even. Furthermore, if the number of chosen elements plus $|E|$ is odd, then choose an element t_1 of V_1. Let T be the set of chosen elements. Then T is G-even and, by (P1), G has an l-ec T-odd orientation. For every $j = 2, \dots, t$, $\varrho(V_j) \geq l$, and we claim that equality cannot occur. Indeed, if $l + i(V_j) = \varrho(V_j) + i(V_j) = \sum(\varrho(v) : v \in V_j) \equiv |V_j \cap T|$ (mod 2), then $l + i(V_j) + |V_j \cap T|$ would be even contradicting the definition of T. Therefore, for every $j = 2, \dots, t$ we have $\varrho(V_j) \geq l+1$ and also $\varrho(V_1) \geq l$. Hence $e_G(\mathcal{F}) = \sum_{j=1}^{t} \varrho(V_j) \geq (l+1)(t-1) + l = (l+1)t - 1$, that is, (15) holds and (P2) follows.

(P2)→(P3) Let s be any node of G. By Theorem 2.3, G has an $(l+1)^-$-ec orientation, denoted by $D = (V, A)$. By Theorem 3.4, D can be constructed from s by a sequence of operations (j) and (jj). The corresponding sequence of operations (i) and (ii) results in G, that is, (P3) holds.

(P3)$\to$(P1). We use induction on the number of edges. There is nothing to prove if G has no edges so suppose that E is non-empty. Let T be a G-even subset of V. Let G' denote the graph from which G is obtained by one of the operations (i) and (ii). By induction, we may assume that G' has an l-ec T'-odd orientation for every G'-even set T'.

Suppose first that G arises from G' by adding a new edge $f=xy$. Let $T':=T\oplus\{y\}$ (where $\oplus$ stands for the symmetric difference). Clearly, T' is G'-even. By induction, there exists an l-ec T'-odd orientation of G'. By orienting edge e from x to y, we obtain an l-ec T-odd orientation of G.

Second, suppose that G arises from G' by operation (ii), that is, by pinching F with r and adding ur. In case $r\in T$, define $T':=T-r$ if l is even and $T':=(T-r)\oplus\{u\}$ if l is odd. In case $r\notin T$, define $T':=T\oplus\{u\}$ if l is even and $T'=T$ if l is odd. Then T' is G'-even and, by induction, G' has an l-ec T'-odd orientation $\vec{G}'$. Orient the undirected edge ur from u to r if either l is even and $r\in T$ or else l is odd and $r\notin T$. Otherwise (that is, if either l is odd and $r\in T$ or else l is even and $r\notin T$) orient ur from r to u.

Furthermore, if an element xy of F is oriented in $\vec{G}'$ from x to y, then the two corresponding edges rx, ry of G be oriented from x to r and from r to y, respectively. Obviously, the resulting orientation of G is l-ec and T-odd. ∎

We remark that property (P2″) can be derived directly from (P3) without invoking Tutte's theorem. Since both (P1) and (P2′) are about the existence of certain orientations of G, it is tempting to try to find a direct, short proof of their equivalence, or at least one of the two opposite implications. We did not succeed even in the special case $l=1$.

Since any $(2k)$-ec graph is $(2k)^-$-ec, one has the following corollary.

Corollary 4.2. *A $(2l+2)$-edge-connected undirected graph $G=(V,E)$ with $|E|+|V|$ even has an l-edge-connected orientation so that the in-degree of every node is odd.* ∎

We do not know any simple proof of this result even in the special case of $l=1$.

5. Remarks and problems

In Theorem 4.1, property (P2) may be viewed as a co-NP characterization of property (P1). This means that there is a polynomially checkable certificate for the negation of (P1), namely, a partition $\mathcal{F}$ of V violating (15). In fact, if (P2) fails to hold for a partition, then a concrete G-even subset T can be constructed (as proved in $(P1)\to(P2)$) for which no l-ec T-odd orientation exists.

Also, property (P3) may be viewed as an NP characterization of (P1). This time a polynomially checkable certificate for the truth of (P1) is a sequence of operations (i) and (ii) which actually builds up G from an initial node. Moreover, if a building procedure for G described in (P3) is available, then one can easily find (as described in the proof of $(P3)\to(P1)$ above) an l-ec T-odd orientation of G for any specified G-even set T.

In the introduction it was mentioned that, generalizing Nebesky's theorem, [5] provided a necessary and sufficient condition for the existence of a rooted l-out-connected and T-odd orientation of a graph where T is a specified G-even subset. The counterpart of Theorem 4.1 concerning rooted l-out-connectivity was also given in [5] but this is much easier than Theorem 4.1. Namely, *an undirected graph $G=(V,E)$ with a root-node s has an s-rooted l-out-connected and T-odd orientation for every G-even subset T of nodes if and only if $e_G(\mathcal{F})\ge(l+1)(t-1)$ holds for every partition $\mathcal{F}:=\{V_1,V_2,\dots,V_t\}$ of V.* The necessity of this condition is rather straightforward and can be proved along the same line as the necessity of (15) was proved in Theorem 4.1. An easy, direct proof of sufficiency was given in [5]. For completeness, we cite this proof. Let T be any fixed G-even subset. By Tutte's Theorem 1.1 the condition is equivalent to the existence of $l+1$ edge-disjoint spanning trees. Orient the edges not in these trees arbitrarily. Among the $l+1$ trees take l and orient each to become an s-rooted arborescence, ensuring this way the s-rooted l-out-connectivity. Finally, the remaining tree, being a connected graph, can be oriented so as to meet the parity prescription on the in-degrees.

As far as algorithmic aspects are considered, for a given graph $G=(V,E)$, the question whether there is a partition $\mathcal{F}$ violating (15) or else G can be built up as described in property (P3) of Theorem 4.1 can be answered algorithmically as follows. The proof of Theorem 2.3 described in [3] is algorithmic, and gives rise to a combinatorial strongly polynomial time algorithm in the special case $p=p_k$ for finding either a k^--ec orientation of G or else a partition violating (3) (which is equivalent to (15) for $k=l+1$). As we have mentioned at the end of Section 3, finding a sequence of operations (j) and (jj) to build D, and hence a sequence of operations (i) and (ii) to build G, can also be done in polynomial time.

Naturally, this is just an outline of the existence of a combinatorial polynomial time algorithm for finding either a partition of V violating (15) or else a sequence of operations (i) and (ii) to build G, and leaves room for improvements to get a more decent bound on the complexity.

In this work an edge-connectivity and parity constrained graph orientation problem has been solved. As a main tool of the solution, we introduced the notions of k^--edge-connectivity of digraphs and $(2k)^-$-edge-connectivity of undirected graphs which may be considered as refinements of the known edge-connectivity notion of graphs and digraphs. This naturally gives rise to the problem of investigating further refinements.

For integers k and l (among which l may be negative), we say an undirected graph $G=(V,E)$ to be (k,l)-**partition-connected**, in short, (k,l)-**pc** if the number $e_G(\mathcal{F})$ of cross edges is at least $k(t-1)+l$ for every partition $\mathcal{F}$ of V into non-empty parts where $t=|\mathcal{F}|\geq 2$. It is not difficult to show that for $l\geq k$, a graph G is (k,l)-pc if and only if G is $(k+l)$-edge-connected. Note that $(k,k-1)$-partition-connectivity is the same as k^--edge-connectivity.

By Theorem 1.1 of Tutte, $(k,0)$-partition-connectivity is equivalent to the existence of k edge-disjoint spanning trees. For negative values of l, it can be proved with standard matroidal tools that G is (k,l)-pc if and only if G can be made $(k,0)$-partition-connected by adding $|l|$ new edges. That is, for negative l, every (k,l)-pc graph arises from a $(k,0)$-pc graph by deleting $|l|$ edges.

For non-negative integers $l\leq k$, a digraph $D=(V,A)$ is called (k,l)-**edge-connected** if D has a node s so that $\varrho(X)\geq k$ and $\delta(X)\geq l$ for every non-empty subset $X\subseteq V-s$. For $l=k$, $l=0$, and $l=k-1$, this is, respectively, equivalent to k-edge-connectivity, rooted k-out-connectivity, and k^--edge-connectivity.

Fortunately, Theorem 2.2 may be applied again and, by choosing function p to be $p(X):=k$ if $\emptyset\subset X\subseteq V-s$, $p(X):=l$ if $s\in X\subset V$, $p(\emptyset):=p(V):=0$, we obtain the following generalization of Theorem 2.3.

Theorem 5.1. *For nonnegative integers $l\leq k$, an undirected graph G has a (k,l)-edge-connected orientation if and only if G is (k,l)-partition-connected.*

It would be interesting to find constructive characterizations for (k,l)-ec digraphs and (k,l)-pc undirected graphs. We propose the following conjectures.

Conjecture 5.2. Let $0\leq l<k$ be integers. A directed graph $D=(V,A)$ is (k,l)-edge-connected if and only if it can be built form a node by the following two operations: (j) add a new edge, (jj) pinch i $(l\leq i<k)$ existing edges with a new node r, and add $k-i$ new edges entering r and leaving existing nodes.

Conjecture 5.3. Let $0\leq l<k$ be integers. An undirected graph $G=(V,E)$ is (k,l)-partition-connected if and only if it can be built from a node by the following two operations: (j) add a new edge, (jj) pinch i $(l\leq i<k)$ existing edges with a new node r, and add $k-i$ new edges connecting r with existing nodes.

By Theorem 5.1, the second conjecture follows from the first one. For $l=0$, Conjecture 5.2 is known to follow by an easy elementary construction from Theorem 3.1 of Mader. For $l=k-1$ the statement of Conjecture 5.2 is the same as Theorem 3.4. In a recent paper [7], we proved Conjecture 5.2 for $l=1$. The smallest open case is $l=2$, $k=4$.

Theorem 2.6 is a result on splitting off preserving $(k,k-1)$-edge-connectivity. A generalization of this concerning (k,l)-edge-connectivity $(0<l<k)$ follows from Theorem 2.5 in an analogous way as Theorem 2.6 did:

Theorem 5.4. *Let $l<k$ be positive integers. Let $D'=(U+r,A')$ be a directed graph with a root node $s\in U$ and a special node r for which $\varrho'(r)=\delta'(r)$. Suppose that D' is (k,l)-edge-connected in U. Then there is a complete splitting at r resulting in a (k,l)-edge-connected digraph on node set U.*

Therefore the only obstacle in proving Conjecture 5.2 is the lack of an extension of Theorem 3.3. We formulate it as a conjecture.

Conjecture 5.5. Let $l<k$ be positive integers. Let $D=(V,A)$ be a minimally (k,l)-edge-connected digraph with respect to a root node s. If $|V|\geq 2$, then there is a node $r\in V-s$ with $\varrho(r)=k$, $\delta(r)<k$ and a subset F of $\varrho(r)-\delta(r)$ edges entering r so that $D-F$ is (k,l)-edge-connected in $V-r$.

The technique applied in the proof of Theorem 3.3 may be used to derive the following consequence of this conjecture: *Where $l<k$, a minimally (k,l)-ec digraph with at least two nodes contains a node $r\in V-s$ with $\varrho(r)=k$, $\delta(r)<k$.*

6. New proof of Theorems 2.2 and 2.5

In this section we include new proofs of the two general theorems in Section 2 concerning supermodular functions. The goal, on one hand, is to make the paper self-contained, and to provide proof-techniques different from the existing ones on the other. We note however that the relatively short proof below of Theorem 2.2, unlike its original proof, is not algorithmic.

Proof of Theorem 2.2. As the necessity of (6) and (7) is straightforward, we prove the sufficiency only. Let us start with the first part of the theorem and assume that (6) and (7) hold. For a subset $X \subseteq V$, we use $e_G(X)$ to denote the number of edges of G with at least one end-node in X. (For a partition $\mathcal{F}$ consisting of $V-X$ and of the singletons of X, clearly, $e_G(X)=e_G(\mathcal{F})$.) We will need the following well-known and easy claim.

Claim 6.1. *Given $m: V \to \mathbf{Z}_+$, there is an orientation of G for which $\varrho(v)=m(v)$ for every $v \in V$ if and only if $m(V)=|E|$ and $m(X) \le e_G(X)$ for every $X \subseteq V$.*

Proof. The necessity is trivial. To prove sufficiency it suffices to show the existence of an orientation for which $\varrho(v) \ge m(v)$ for every $v \in V$ since then, by $m(V)=|E|$, equality must hold everywhere. Start with any orientation. If it has a deficient node s, that is, $\varrho(s) < m(s)$, then the set X of nodes reachable from s in the given orientation must contain a node t with $\varrho(t) > m(t)$, for otherwise X would violate the condition (as $m(X) > \sum(\varrho(v): v \in X) = e_G(X) - \delta_G(X) = e_G(X)$). By reorienting each edge of a directed path from s to t we obtain a better orientation. ∎

Let $\mathcal{R}$ be a family of (not necessarily distinct) non-empty proper subsets of V. We say that $\mathcal{R}$ is **regular** if each node of V belongs to the same number of elements of $\mathcal{R}$. Clearly, both a partition and a co-partition of V are regular cross-free families.

Claim 6.2. *A regular cross-free family $\mathcal{R}$ includes a partition or a co-partition of V.*

Proof. Let $\bar{\mathcal{R}} := \{X : V-X \in \mathcal{R}\}$. Then $\bar{\mathcal{R}}$ is also regular and cross-free, and the claim is equivalent to stating that $\mathcal{R}$ or $\bar{\mathcal{R}}$ includes a partition of V.

Let Z be a minimal member of $\mathcal{R} \cup \bar{\mathcal{R}}$ (with respect to containment). By symmetry, we may assume that $Z \in \mathcal{R}$. We are going to show that $\mathcal{R}$ includes a partition of $V-Z$. This partition along with $\{Z\}$ will provide the desired partition of V.

Every member X of $\mathcal{R}$ either includes Z or is disjoint from Z, for otherwise X would cross Z by the minimal choice of Z. Therefore the regularity of $\mathcal{R}$ implies that every element of $V-Z$ belongs to a member of $\mathcal{R}$ disjoint from Z, and the maximal members of $\mathcal{R}$ disjoint from Z do not cross, that is, they form a partition of $V-Z$. ∎

Let $e_G(\mathcal{R}) := \frac{1}{2}\sum(d_G(X): X \in \mathcal{R})$. (When $\mathcal{R}$ is a partition, then this sum is the number of edges connecting different parts.) By Claim 6.2, any cross-free, regular family $\mathcal{R}$ can be decomposed into partitions and co-partitions. Hence (6) and (7) imply

$$\sum(p(X) : X \in \mathcal{R}) \le e_G(\mathcal{R}). \tag{16}$$

Actually, (16) holds for every regular family $\mathcal{R}$. To see this, suppose indirectly that there is a regular family $\mathcal{R}$ violating (16) and assume that, given $|\mathcal{R}|$, $\sum(|X|^2 : X \in \mathcal{R})$ is as large as possible. Since $\mathcal{R}$ cannot be cross-free, it contains two crossing members S and T. Let $\mathcal{R}'$ denote the family arising from $\mathcal{R}$ by replacing S and T by $S \cap T$ and $S \cup T$. (This operation may be called an **uncrossing step.**) Clearly $\mathcal{R}'$ is regular. By the crossing supermodularity of p, $\sum(p(X): X \in \mathcal{R}') \ge \sum(p(X): X \in \mathcal{R})$. By the submodularity of d_G, $e_G(\mathcal{R}') \le e_G(\mathcal{R})$, that is, $\mathcal{R}'$ also violates (16), contradicting the extremal choice of $\mathcal{R}$ as $|\mathcal{R}'|=|\mathcal{R}|$ and $\sum(|X|^2 : X \in \mathcal{R}') > \sum(|X|^2 : X \in \mathcal{R})$.

We say that a regular family is **tight** if (16) is satisfied with equality. It follows from the argument above that an uncrossing step preserves tightness of a regular family. Also, if a tight regular family $\mathcal{R}$ includes a regular subfamily $\mathcal{R}_1$, then $\mathcal{R}_2 := \mathcal{R} - \mathcal{R}_1$ is also regular, so they both satisfy (16) and hence both $\mathcal{R}_i$'s $(i=1,2)$ are tight.

We may assume that every singleton $\{v\}$ belongs to a tight partition $\mathcal{R}_v$. For otherwise, revise p by increasing its value on $\{v\}$ as much as possible without violating (6) or (7). Clearly, such a modification results in a crossing supermodular function and cannot destroy (7) unless (6) is destroyed. Therefore when $p(v)$ cannot be increased anymore without violating (6) and (7), then there is a tight partition including $\{v\}$ as a member. The union $\mathcal{R}_0$ of tight partitions $\mathcal{R}_v$ $(v \in V)$ is tight and so is the regular cross-free family $\mathcal{R}$ arising from $\mathcal{R}_0$ after applying uncrossing steps as long as possible. Since an uncrossing step never removes singletons, $\mathcal{R}$ includes all singletons $\{v\}$ $(v \in V)$ from which we can conclude that partition $\{\{v\} : v \in V\}$ is tight.

Let $m(v) := p(v)$ $(v \in V)$. The tightness of partition $\{\{v\} : v \in V\}$ is equivalent to $m(V)=|E|$. Let $\emptyset \subset X \subset V$ and let $\mathcal{F}$ be the partition consisting of $V-X$ and the singletons of X. By (6) and by $p(V-X) \ge 0$ we have $m(X) = \sum(m(v): v \in X) = \sum(p(v): v \in X) \le \sum(p(v): v \in X) + p(V-X) \le e_G(\mathcal{F}) = e_G(X)$. By applying Claim 6.1 to m, we obtain an orientation of G for which $\varrho(v) = p(v)$ for every $v \in V$. We claim that this orientation satisfies the requirement of the theorem. Indeed, for a subset X of V, let $\mathcal{F}$ be the partition consisting of X and the singletons of $V-X$. By (6) we have $p(X) + \sum(p(v): v \in V-X) \le e_G(\mathcal{F}) = \varrho(X) + \sum(\varrho(v): v \in V-X) = \varrho(X) + \sum(p(v): v \in V-X)$ from which $p(X) \le \varrho(X)$, as required.

To see the second part of the theorem, suppose that p is monotone decreasing. We only have to show that in this case (6) implies (7). Since $V_{i+1} \subseteq V - V_i$, by the monotonicity of p we have $p(V-V_i) \le p(V_{i+1})$ for $i=1,\dots,t$ (where $V_{t+1} := V_1$), from which $\sum((p(V-V_i) : i=1,\dots,t) \le \sum((p(V_i) : i=1,\dots,t) \le e_G(\mathcal{F})$ and (7) follows.

To see the last part of the theorem, suppose now that p is symmetric. Then (6) and (7) are equivalent so it suffices to show that (6) holds true for every partition of V into t parts provided that (6) holds for every partition of V into 2 parts, which is equivalent to requiring that $d(X) \geq 2p(X)$ for every subset $X \subseteq V$. Let $\mathcal{F}=\{V_1,\dots,V_t\}$ be a partition of V into $t \geq 3$ parts. Then, by using $d(V_i) \geq 2p(V_i)$, we have $e_G(\mathcal{F})=\sum_i d(V_i)/2 \geq \sum_i p(V_i)$, that is, (6) indeed holds. ∎

Proof of Theorem 2.5. Necessity. Suppose there is a digraph H with the required properties. Then $m_i(X)=\sum(\varrho_H(v): v \in X) \geq \varrho_H(X) \geq p(X)$ and $m_o(U-X)=\sum(\delta_H(v): v \in U-X) \geq \delta_H(U-X)=\varrho_H(X) \geq p(X)$, that is, (11) and (12) hold.

Sufficiency. Assume indirectly that no digraph exists with the required properties. Since $m_i(U)=m_o(U)$, it can easily be seen that there is a digraph H (possibly with loops and parallel edges) satisfying (10). Let $q_H(X) := p(X)-\varrho_H(X)$ and $\mu_H := \max(q_H(X): X \subseteq U)$. Let $\mathcal{F}_H := \{X \subset U : q_H(X) = \mu_H\}$. By $p(\emptyset)=0=\varrho_H(\emptyset)$, we have $\mu_H \geq 0$ and since $\mu_H=0$ is equivalent to (9), μ_H must be positive. This implies that $p(X)>0$ for every member X of $\mathcal{F}_H$.

Claim 6.3. *Let X and Y be two crossing members of $\mathcal{F}_H$. Then both $X \cap Y$ and $X \cup Y$ belong to $\mathcal{F}_H$.*

Proof. Since ϱ_H is submodular, p is positively crossing supermodular, and $p(X)>0$ and $p(Y)>0$, we have $\mu_H+\mu_H=q_H(X)+q_H(Y) \leq q_H(X \cap Y)+q_H(X \cup Y) \leq \mu_H+\mu_H$ from which $q_H(X \cap Y)=\mu_H$ and $q_H(X \cup Y)=\mu_H$, and the claim follows. ∎

Assume now that H is chosen in such a way that (*) *μ_H is as small as possible* and, subject to this, (**) *$|\mathcal{F}_H|$ is as small as possible.*

Let K be a minimal member of $\mathcal{F}_H$ and $L \supseteq K$ a maximal member of $\mathcal{F}_H$. There is an edge $e=uv$ of H with $u,v \in K$ for otherwise $m_i(K)=\sum(\varrho_H(z): z \in K)=\varrho_H(K)=p(K)-\mu_H<p(K)$, contradicting (11). There is an edge $f=xy$ of H with $x,y \in U-L$ for otherwise $m_o(U-L)=\sum(\delta_H(z): v \in U-L)=\delta_H(U-L)=\varrho_H(L)=p(L)-\mu_H<p(L)$, contradicting (12).

Revise H by replacing edges e and f by edges uy and xv, and let H' denote the resulting digraph. Clearly, H' satisfies (10), as well. The following is immediate.

Claim 6.4. *If $\varrho_{H'}(X)<\varrho_H(X)$ for some subset $X \subseteq U$, then either uv enters X and xy leaves X or else uv leaves X and xy enters X (and, in particular, none of e and f is a loop).* ∎

This implies $\varrho_{H'}(X) \geq \varrho_H(X)-1$ for every subset $X \subseteq U$. There is no set $X \in \mathcal{F}_H$ for which $\varrho_{H'}(X)=\varrho_H(X)-1$, for otherwise X and K are crossing by Claim 6.4 and then $X \cap K \in \mathcal{F}_H$ by Claim 6.3, contradicting the minimal choice of K. It follows that $\mu_{H'} \leq \mu_H$ and actually here we have equality by assumption (*).

Since $\varrho_{H'}(K)>\varrho_H(K)$, the subset K is not in $\mathcal{F}_{H'}$. By assumption (**), there must be a set X which is in $\mathcal{F}_{H'}-\mathcal{F}_H$. Then $\varrho_{H'}(X)<\varrho_H(X)$, that is, $\varrho_{H'}(X)=\varrho_H(X)-1$. Apply Claim 6.4. By symmetry we may assume that uv enters X and xy leaves X.

We have $q_H(X)+1=q_{H'}(X)=\mu_{H'}=\mu_H$. As $q_{H'}(X)=\mu(H')$ and $q_H(K)=\mu_H$ are positive number, so are $p(X)$ and $p(K)$. Since K and X are crossing, by the minimal choice of K we have $\mu_H+(\mu_H-1)=q_H(K)+q_H(X) \leq q_H(K \cap X)+q_H(K \cup X) \leq (\mu_H-1)+\mu_H$, from which $q_H(K \cup X)=\mu_H$. That is, $X' := K \cup X$ belongs to $\mathcal{F}_H$. Since L and X' are either crossing or $L \subset X'$, we obtain by using Claim 6.3 that $X' \cup L \in \mathcal{F}_H$, in a contradiction with the maximal choice of L. ∎

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András Frank

Department of Operations Research,
Eötvös University,
Pázmány Péter sétány 1/c
Budapest, Hungary, H-1117
and
Traffic Lab Ericsson Hungary,
Laborc u. 1, Budapest,
Hungary H-1037
`frank@cs.elte.hu`

Zoltán Király

Department of Computer Science,
Eötvös University,
Pázmány Péter sétány 1/c
Budapest, Hungary, H-1117
and
Traffic Lab Ericsson Hungary,
Laborc u. 1, Budapest,
Hungary H-1037
`kiraly@cs.elte.hu`