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# On Chain and Antichain Families of a Partially Ordered Set

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A common generalization of the theorems of Greene and Greene and Kleitman is presented. This yields some insight into the relation of optimal chain and antichain families of a partially ordered set. The fundamental device is the minimal cost flow algorithm of Ford and Fulkerson.

## INTRODUCTION

Greene and Kleitman [6], while investigating generalizations of Dilworth's theorem, found a nice formula for the maximum cardinality of the union of $\alpha$ antichains in a partially ordered set. Previously, Greene [4] had proved a similar min–max theorem concerning chains instead of antichains. Moreover, he discovered a number of deep and interesting features of chains and antichains. An excellent survey can be found in [5].

This paper has two purposes. A theorem will be proved which is a common generalization of the theorems of Greene and Kleitman [6] and Greene [4] on the one hand, and an algorithm will be described for finding an optimal set of $\gamma$ chains and $\alpha$ antichains on the other.

In our procedure the basic idea is that the elegant proof of Dilworth's theorem given by Fulkerson [3] can be generalized. It will turn out that the problem of finding $\gamma$ chains of largest union is equivalent to a minimal cost flow problem. To solve this we apply the method of Ford and Fulkerson [2] which solves not only the $\gamma$ chains problem but also the $\alpha$ antichains problem at the same time. An analysis of the Ford–Fulkerson algorithm leads us to the theorem in question. Some other results of Greene can also easily be established in this framework.



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## 1. Preliminaries and the Main Result

Let $P = \{p_1, p_2 \cdots p_n\}$ be a finite partially ordered set. A *chain* $C$ is a totally ordered subset of $P$; an *antichain* $A$ is a set of mutually unrelated elements of $P$. Let $\mathbf{c}$ and $\mathbf{a}$ be the cardinalities of the largest chain and antichain, respectively. Dilworth's theorem [1] states that $P$ can be partitioned into $\mathbf{a}$ chains. A dual version of Dilworth's theorem states that $P$ can be partitioned into $\mathbf{c}$ antichains.

If $L$ is a collection of sets we set $\bigcup L = \{x: x \in A \text{ for some } A \in L\}$. $|X|$ denotes the cardinality of the set $X$.

Let $\mathscr{A}$ and $\mathscr{C}$ be the sets of all unordered sets of pairwise disjoint nonempty antichains and chains, respectively. We shall refer to a member $\mathscr{A}_\alpha = \{A_1, A_2, ..., A_\alpha\}$ of $\mathscr{A}$ and to a member of $\mathscr{C}_\gamma = \{C_1, C_2, ..., C_\gamma\}$ of $\mathscr{C}$ as an antichain and a chain family, respectively.

Denote $\mathbf{a}_\alpha = \max |\bigcup \mathscr{A}_\alpha|$ and $\mathbf{c}_\gamma = \max |\bigcup \mathscr{C}_\gamma|$, where the maximum is taken over all antichain families consisting of $\alpha$ antichains and chain families consisting of $\gamma$ chains, respectively. (Note that $\mathbf{a}_1 = \mathbf{a}$ and $\mathbf{c}_1 = \mathbf{c}$.)

By Dilworth's theorem $\mathbf{c}_\mathbf{a} = n$ and by its dual $\mathbf{a}_\mathbf{c} = n$, therefore $1 \leqslant \gamma \leqslant \mathbf{a}$ and $1 \leqslant \alpha \leqslant \mathbf{c}$ are assumed.

Theorem 1a. (Greene and Kleitman [6]).

$$\mathbf{a}_\alpha = \min \sum_{i=1}^{q} \min(|C_i|, \alpha),$$

*where the first minimum runs over all chain partitions* $\{C_1, C_2, ..., C_q\}$ *of* $P$.

Theorem 2a (Greene [4]).

$$\mathbf{c}_\gamma = \min \sum_{i=1}^{q} \min(|A_i|, \gamma),$$

*where the first minimum runs over all antichain partitions* $\{A_1, A_2, ..., A_q\}$ *of* $P$.

Before stating these theorems in another way we need the following.

Definition. We call a chain family $\mathscr{C}_\gamma = \{C_1, C_2, ..., C_\gamma\}$ and an antichain family $\mathscr{A}_\alpha = \{A_1, A_2, ..., A_\alpha\}$ *orthogonal* if

(a) $P = (\bigcup \mathscr{A}_\alpha) \cup (\bigcup \mathscr{C}_\gamma)$ and

(b) $A_i \cap C_j \neq 0$ for $1 \leqslant i \leqslant \alpha$, $1 \leqslant j \leqslant \gamma$.

It can easily be checked that if there exist collections $\mathscr{A}_\alpha$ and $\mathscr{C}_\gamma$ which are orthogonal then $\mathscr{A}_\alpha$ is optimal, i.e., $|\bigcup \mathscr{A}_\alpha| = \mathbf{a}_\alpha$ and Theorem 1a is true for this $\alpha$, and, similarly, $\mathscr{C}_\gamma$ is optimal, i.e., $|\bigcup \mathscr{C}_\gamma| = \mathbf{c}_\gamma$ and Theorem 2a is true for this $\gamma$. Hence the next two theorems imply the above-mentioned results:



Theorem 1b. *For each* $\alpha$, $1 \leqslant \alpha \leqslant \mathbf{c}$, *there exist an antichain family* $\mathscr{A}_\alpha$ *and a chain family* $\mathscr{C}_\gamma$, *for some* $\gamma$, *which are orthogonal.*

Theorem 2b. *For each* $\gamma$, $1 \leqslant \gamma \leqslant \mathbf{a}$, *there exist a chain family* $\mathscr{C}_\gamma$ *and an antichain family* $\mathscr{A}_\alpha$, *for some* $\alpha$, *which are orthogonal.*

Now we are in the position to state our main result.

Theorem 3. *There exists a sequence*

$$\mathscr{C}_\mathbf{a} \mid \mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_{i_1} \mid \mathscr{C}_{\mathbf{a}-1}, \mathscr{C}_{\mathbf{a}-2}, ..., \mathscr{C}_{\mathbf{a}-j_1} \mid \mathscr{A}_{i_1+1} \cdots \mathscr{A}_{i_2} \mid \mathscr{C}_{\mathbf{a}-j_1-1} \cdots \mathscr{C}_{\mathbf{a}-j_2} \mid \cdots$$

*which arises as a combination of two sequences* $\mathscr{C}_\mathbf{a}, \mathscr{C}_{\mathbf{a}-1}, ..., \mathscr{C}_1$ *and* $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_\mathbf{c}$, *where* $\mathscr{C}_j \in \mathscr{C}$ *and* $\mathscr{A}_i \in \mathscr{A}$, *with the property that any member of the sequence (whether* $\mathscr{C}_j$ *or* $\mathscr{A}_i$*) is orthogonal to the last member of other type preceeding it. (That is,* $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_{i_1}$ *are orthogonal to* $\mathscr{C}_\mathbf{a}$, *and* $\mathscr{C}_{\mathbf{a}-1}, \mathscr{C}_{\mathbf{a}-2}, ..., \mathscr{C}_{\mathbf{a}-j_1}$ *are orthogonal to* $\mathscr{A}_{i_1}$, *and so on.)*

Observe that the $\mathscr{C}_j$'s are arranged by decreasing indices while $\mathscr{A}_i$'s by increasing ones. Thus the last member of the sequence is either $\mathscr{C}_1$ or $\mathscr{A}_\mathbf{a}$. Theorems 2a and 2b follow immediately from Theorem 3.

## 2. The Algorithm of Ford and Fulkerson

In the proof we shall need the minimal cost flow algorithm of Ford and Fulkerson [2, p. 113]; thus, before proving Theorem 3 we briefly summarize this algorithm.

Assume given a network $G = (V, E)$ having two specified vertices: a source $s$ and a sink $t$. Non-negative integral cost $a(xy)$ and positive integral capacity $c(xy)$ are assigned to each arc $(xy)$. The task is to look for a minimal cost flow $f(xy)$ from $s$ to $t$, having a flow value $v$ given in advance.

The algorithm solves this problem for all the possible flow values $v$. It invokes dual variables $\pi(x)$ assigned to the vertices of $G$. This so-called potential function (or briefly potential) is non-negative integer valued and $\pi(s) = 0$ throughout the process. The current $\pi(t) = p$ is called the *potential value*.

Suppose we have a flow $f(xy)$ (satisfying the capacity restriction) of value $v$, and a potential $\pi(x)$ of value $p$. Then using the notation $\bar{a}(xy) = a(xy) + \pi(x) - \pi(y)$, the following estimation holds for the flow cost:

$$\sum_{(xy)\in E} a(xy)f(xy)$$
$$= \sum_{(xy)\in E} (\pi(y)-\pi(x))f(xy) + \sum_{(xy)\in E} \bar{a}(xy)f(xy)$$
$$= pv + \sum_{(xy)\in E^-} \bar{a}(xy)f(xy) + \sum_{(xy)\in E^+} \bar{a}(xy)f(xy)$$
$$\geqslant pv + \sum_{(xy)\in E^-} \bar{a}(xy)\,c(xy) + \sum_{(xy)\in E^+} \bar{a}(xy)\cdot 0$$
$$= pv + \sum_{(xy)\in E^-} \bar{a}(xy)\,c(xy),$$

where $E^+$ is the set of edges $(xy)$ having $\bar{a}(xy) > 0$ and $E^-$ is defined similarly.

From this we can see that the flow in question surely has minimal cost among the flows of value $v$ if the above inequality is fulfilled with equality. This is equivalent to the next criteria:

$$\pi(y) - \pi(x) < a(xy) \quad \text{implies} \quad f(xy) = 0 \tag{1}$$
$$\pi(y) - \pi(x) > a(xy) \quad \text{implies} \quad f(xy) = c(xy) \tag{2}$$

The algorithm begins with zero potential and zero flow. In a general step a path, leading from $s$ to $t$, is sought by a labeling process on the network $G'$ consisting of those edges $(xy)$ for which (i) $\bar{a}(xy) = 0$ and $f(xy) < c(xy)$ or (ii) $\bar{a}(yx) = 0$ and $f(yx) > 0$. This path either exists or not. Accordingly, there are two types of steps:

a. If a path exists, a new flow can be obtained by means of this path. The new flow value is greater by one than that of the preceeding one, while the potential is unchanged.

b. In the other case a new potential can be obtained in such a way that $\pi(x)$ is increased by one on the set of those vertices which cannot be reached by a path starting from $s$ in $G'$. The new potential value is greater by one than that of the preceeding one, while the flow is unchanged.

The algorithm consists of the repeated applications of the general step.

A fundamental feature of the algorithm is that optimality criteria (1) and (2) are maintained throughout the computation. It is important to know that the flow cost increases by the current potential value $p$ when the flow value is increased by one. Furthermore $0 \leqslant \pi(x) \leqslant p$ during the whole process for each $x \in V$.

We shall refer to a stage of the algorithm by the pair $(v, p)$ consisting of the current flow and potential values.



## 3. Proof of Theorem 3.

Associate a network $G = (V, E)$ with $P$ as follows. Let $V = \{s, t, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$, $E = \{(sx_i): i = 1, 2, \ldots, n\} \cup \{(y_i t): i = 1, 2, \ldots, n\} \cup \{(x_i y_j): \text{if } p_i \geqslant p_j\}$.

All of the arc capacities $c(e)$ are equal to one, while the costs $a(e)$ are:

$$a(e) = 1 \quad \text{if} \quad e = (x_i y_i)$$
$$= 0 \quad \text{otherwise.}$$

Now apply the procedure of Ford and Fulkerson for this network and consider a stage of the computation.

Firstly we associate a chain family $\mathscr{C}_\gamma$, where $\gamma = n - v$, with the current flow as follows. If the flow cost is $d$ $(d \geqslant 0)$ then the flow is one on $d$ edges of type $(x_i y_i)$, say $f(x_1 y_1) = f(x_2 y_2) = \cdots = f(x_d y_d) = 1$. The edges $(x_i y_j)$ $(i \neq j)$, for which $f(x_i y_j) = 1$, form an independent set of edges. This defines a chain partition of the subset $P' = \{p_{d+1}, p_{d+2} \cdots p_n\}$ as in the proof of Dilworth's theorem given by Fulkerson (see [2, p. 62, Lemma 8.1]). The number of chains is $|P'| - (v - d) = n - v = \gamma$. Let these chains be denoted by $C_1, C_2, \ldots, C_\gamma$ and $\mathscr{C}_\gamma = \{C_1, C_2, \ldots, C_\gamma\}$. Note that $\mathscr{C}_\gamma$ does not depend on the potential.

Secondly we associate an antichain family $\mathscr{A}_\alpha$, where $\alpha = p$, with the current potential. Let $P_i = \{p_j: \pi(x_j) < \pi(y_j) = i\}$ and let $A_i$ consist of the maximal elements of $P_i$ for $i = 1, 2, \ldots, p$. Let $\mathscr{A}_\alpha = \{A_1, A_2, \ldots, A_\alpha\}$. Note that $\mathscr{A}_\alpha$ does not depend on the flow.

The following lemma is the key to our proof.

Lemma. *The above families $\mathscr{C}_\gamma$ and $\mathscr{A}_\alpha$ are orthogonal.*

*Proof.* (a) Let $p_j \notin \bigcup \mathscr{C}_\gamma$; equivalently $f(x_j y_j) = 1$. Then $\pi(y_j) \geqslant 1 + \pi(x_j)$ by (1), i.e., $p_j \in P_i$, where $i = \pi(y_j)$. If, indirectly, $p_j \notin A_i$ then there is a $p_m$ in $P_i$, greater than $p_j$. Now $\pi(y_j) = \pi(y_m) = i$ and $\pi(x_m) < i$, thus $\pi(y_j) - \pi(x_m) > 0$. Applying (2) to the edge $(x_m y_j)$ we get $f(x_m y_j) = 1$. This is impossible since the capacity of the unique edge $(y_j t)$ starting from $y_j$ is one, but $f(x_m y_j) = f(x_j y_j) = 1$.

(b) Let $C_j \in \mathscr{C}_\gamma$ and let $C_j$ consist of vertices $p_{d+1} > p_{d+2} > \cdots > p_{d+b}$ $(b \geqslant 1)$. Now we have $f(x_{d+h-1} y_{d+h}) = 1$ for $h = 2, 3, \ldots, b$ and $f(x_{d+h} y_{d+h}) = 0$ for $h = 1, 2, \ldots, b$ and $f(y_{d+1} t) = f(sx_{d+b}) = 0$.

From $f(y_{d+1} t) = 0$ and $\pi(t) = p$ we get $\pi(y_{d+1}) \geqslant p$ by (2) and thus $\pi(y_{d+1}) = p$. Similarly $f(sx_{d+b}) = 0$ implies $\pi(x_{d+b}) = 0$.

Furthermore $f(x_{d+h} y_{d+h}) = 0$ implies $\pi(y_{d+h}) \leqslant 1 + \pi(x_{d+h})$ by (2) and finally $f(x_{d+h-1} y_{d+h}) = 1$ implies $\pi(y_{d+h}) \geqslant \pi(x_{d+h-1})$ by (1).

These statements show that there exists an element $p_q$ of $C_j$ for which

$\pi(y_q) = i$ and $\pi(x_q) < i$ for each $i = 1, 2, \ldots, p$. Let $p_q$ denote the greatest element of $C_j$ with this property for a fixed $i$. We show that $p_q \in A_i$. By the definition of $p_q$, it is in $P_i$. Assume, indirectly, that there exists a $p_m$ in $P_i$ greater than $p_q$. Then $\pi(y_q) = \pi(y_m) = i$, $\pi(x_m) < i$, and thus $\pi(y_q) - \pi(x_m) > 0$. From this we get $f(x_m y_q) = 1$ by (2). However, this means $p_m \in C_j$ contradicting the choice of $p_q$. ∎

Now suppose that the Ford–Fulkerson algorithm has run as follows. Starting with $\pi(x) \equiv 0$ and $f(xy) \equiv 0$ the flow value increases to $k_0$, then the potential value increases to $i_1$, then the flow value increases to $k_1 \cdots$ finally the potential value increases to $i_s$ and the flow value increases to $k_s$ $(0 \leqslant k_0 < k_1 < \cdots < k_s,\ 0 < i_1 < i_2 < \cdots < i_s)$.

The algorithm terminates when the maximal flow value is attained. In our case this value is equal to $n$, i.e., $k_s = n$. Let $\mathbf{a} = n - k_0$ and $j_i = k_i - k_0$ for $i = 1, 2, \ldots, s$.

By the lemma, a chain family $\mathscr{C}_\mathbf{a}$ and an antichain family $\mathscr{A}_1$, which are orthogonal, belong to the stage $(v, p) = (k_0, 1)$ of the algorithm. Then the potential value increases one by one to $i_1$, as mentioned. Antichain families $\mathscr{A}_2, \mathscr{A}_3, \ldots, \mathscr{A}_{i_1}$ belonging to the intermediate stages are orthogonal to the unchanged $\mathscr{C}_\mathbf{a}$. Then the flow value increases one by one to $k_1$. Chain families $\mathscr{C}_{\mathbf{a}-1}, \mathscr{C}_{\mathbf{a}-2}, \ldots, \mathscr{C}_{\mathbf{a}-j_1}$ belonging to the intermediate stages are orthogonal to the unchanged $\mathscr{A}_{i_1}$, etc. ∎

## 4. Some Consequences

It easy to check that the pairs $(v, p)$ occuring in the course of the algorithm, and thus the sequences $k_0, k_1, \ldots, k_s$ and $i_1, i_2, \ldots, i_s$, depend only on $P$ itself and not on the run of the algorithm. Picture them as points of coordinates $v$ and $p$ in a coordinate system.

For example, consider the poset illustrated in Fig. 1.

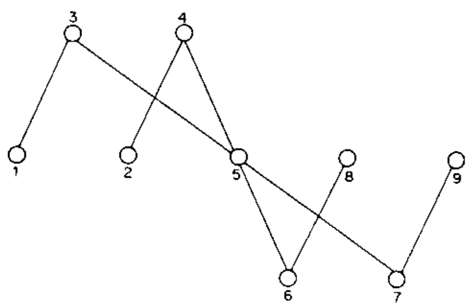


Figure 1



Applying the algorithm, we obtain the sequences $\{k_j\}$ and $\{i_j\}$ mentioned above: $k_0 = 4$, $k_1 = 6$, $k_2 = 8$, $k_3 = 9\ (= n)$ and $i_1 = 1$, $i_2 = 2$, $i_3 = 3$. Picturing the points of coordinates $(v, p)$, we get Fig. 2.

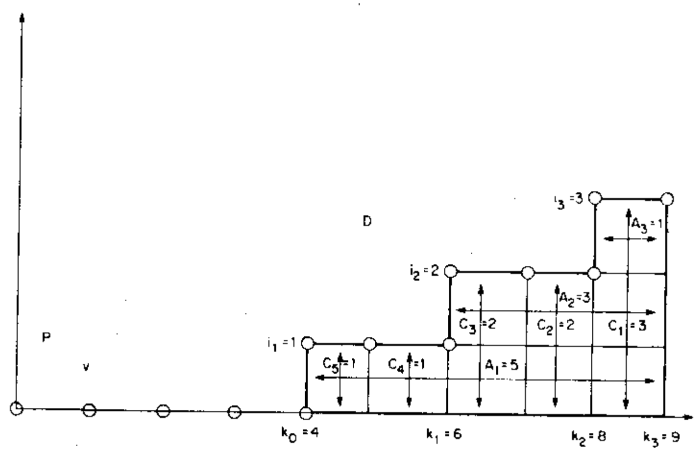


Figure 2

The sequence guaranteed by the theorem is:

$$\mathscr{C}_5 \mid \mathscr{A}_1 \mid \mathscr{C}_4, \mathscr{C}_3 \mid \mathscr{A}_2 \mid \mathscr{C}_2, \mathscr{C}_1 \mid \mathscr{A}_3,$$

where

$$\begin{aligned}
\mathscr{C}_5 &= \{13, 24, 5, 68, 79\} \\
\mathscr{A}_1 &= \{12589\} \\
\mathscr{C}_4 &= \{13, 24, 68, 79\} \\
\mathscr{C}_3 &= \{13, 456, 79\} \\
\mathscr{A}_2 &= \{1267, 3489\} \\
\mathscr{C}_2 &= \{13, 456\} \\
\mathscr{C}_1 &= \{456\} \\
\mathscr{A}_3 &= \{1267, 3489, 5\}.
\end{aligned}$$

Hence we can see: $\mathbf{a}_1 = 5$, $\mathbf{a}_2 = 8$, $\mathbf{a}_3 = 9$ and $\mathbf{c}_1 = 3$, $\mathbf{c}_2 = 5$, $\mathbf{c}_3 = 7$, $\mathbf{c}_4 = 8$, $\mathbf{c}_5 = 9$.

Domain $D$ bounded by the heavy line gives more information about $P$. We call $D$ the *kilter domain* of $P$.

Form the difference sequences of sequences $\mathbf{c}_j$ and $\mathbf{a}_i$. These are $\mathbf{C}_1 = \mathbf{c}_1$, $\mathbf{C}_j = \mathbf{c}_j - \mathbf{c}_{j-1}$ for $2 \leqslant j \leqslant \mathbf{a}$ and $\mathbf{A}_1 = \mathbf{a}_1$, $\mathbf{A}_i = \mathbf{a}_i - \mathbf{a}_{i-1}$ for $2 \leqslant i \leqslant \mathbf{c}$. Obviously $\sum \mathbf{C}_j = \sum \mathbf{A}_i = n$.

The difference sequences have a quite transparent meaning in the kilter domain. If one of the steps of the algorithm is a flow increasing from stage $(v, p)$ to $(v+1, p)$ $(v \geqslant k_0)$ then the flow cost increases by $p$. Hence $\mathbf{C}_{n-v} = p$ by the lemma. Since $p$ never decreases during the algorithm, $\{\mathbf{C}_j\}$ forms a monotone decreasing sequence, furthermore the $\mathbf{C}_j$'s are the heights of the columns of the kilter domain of $P$.

If one of the steps of the algorithm is a potential increasing from stage $(v, p)$ to $(v, p+1)$ then the common chain family $\mathscr{C}_{n-v}$ belonging to these stages is orthogonal to both $\mathscr{A}_p$ and $\mathscr{A}_{p+1}$, therefore $\mathbf{A}_{p+1} = n - v$. Since $v$ does not decrease in the course of the algorithm, the sequence $\{\mathbf{A}_i\}$ is monotone decreasing, furthermore the $\mathbf{A}_i$'s are just the length of the rows of the kilter domain of $P$.

Thus we have obtained a theorem of Greene:

THEOREM 4 [4]. *The sequences* $\{\mathbf{C}_j\}$ and $\{\mathbf{A}_i\}$ *are monotone decreasing and form conjugate partitions of the number* $n$.

Another interesting consequence of Theorem 3 is the so-called $t$-phenomenon (transition phenomenon) for chains and antichains.

THEOREM 5 [5]. *For* $\alpha \geqslant 0$, *there exists a chain partition* $C_1, C_2 \cdots C_q$ *of* $P$ *such that*

$$\mathbf{a}_\alpha = \sum_i \min(|C_i|, \alpha)$$

and

$$\mathbf{a}_{\alpha+1} = \sum_i \min(|C_i|, \alpha+1).$$

*Proof.* Consider the antichain family $\mathscr{A}_{\alpha+1}$ in the sequence guaranteed by Theorem 3. Let $\mathscr{C}_j$ be the last chain family preceeding $\mathscr{A}_{\alpha+1}$. Then $\mathscr{C}_j$ is orthogonal both $\mathscr{A}_{\alpha+1}$ and $\mathscr{A}_\alpha$ (either $\mathscr{A}_{\alpha+1}$ preceeds $\mathscr{C}_j$ or not). Therefore the chain partition $\mathscr{C}_j'$ satisfies the requirements where $\mathscr{C}_j'$ consists of the members of $\mathscr{C}_j$ and some one element chains so that it should form a partition of $P$. ∎

The counterpart of Theorem 5 follows in a similar way:

THEOREM 6 [5]. *For* $\gamma \geqslant 0$, *there exists an antichain partition* $A_1, A_2 \cdots A_q$ *of* $P$ *such that*

$$\mathbf{c}_\gamma = \sum_i \min(|A_i|, \gamma)$$



*and*

$$\mathbf{c}_{\gamma+1} = \sum_i \min(|A_i|, \gamma+1).$$

Hoffman and his co-workers gave some further interesting examples for the $t$-phenomenon (and this name itself is also due to Hoffman) [7, 8].

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