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A constrained independent set problem for matroids

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Abstract

In this note, we study a constrained independent set problem for matroids. The problem can be regarded as an ordered version of the matroid parity problem. By a reduction of this problem to matroid intersection, we prove a min–max formula. We show how earlier results of Hefner and Kleinschmidt on the so-called MS-matchings fit in our framework.

Keywords: MS matchings; Matroid intersection; Matroid parity

1. Introduction

In this note, we shall study the following constrained independent set problem for matroids. Let $\mathbf{M} = (V, \mathcal{I})$ be a matroid with $|V|$ even and Π be the partition of V into ordered pairs. An *ideal independent set* is an independent set $I \in \mathcal{I}$ that satisfies the constraint

$$\text{if } (u, v) \in \Pi \text{ and } u \in I, \text{ then } v \in I. \tag{1}$$

Our basic problem, the *ordered matroid parity problem* is to find a maximum cardinality ideal independent set. We shall show that the ordered matroid parity problem can be reduced to matroid intersection, i.e., to the problem of maximizing the size of a common independent set of two matroids.

The ordered matroid parity problem looks similar to the matroid parity problem, i.e., the problem of finding a maximum size independent set I of $\mathbf{M}$ so that

$$\text{if } (u, v) \in \Pi, \text{ then } u \in I \text{ if and only if } v \in I. \tag{2}$$

In contrast to our ordered version, the matroid parity problem includes NP-hard problems. It is even known to be intractable with an ordinary oracle model of matroids, although it is solvable in polynomial time for linearly represented matroids [5].

If $\mathbf{M}$ is the transversal matroid of some bipartite graph $G = (U, V; E)$, then the ordered matroid parity problem is equivalent to the problem of

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finding a matching M of G that has the property that for any $(u, v) \in \Pi$, the vertex v is covered by M whenever u is covered by M. This is a special case of the so-called MS-matching problem introduced by Hefner and Kleinschmidt [4].

The MS-matching problem involves a graph G and a digraph D on the common vertex set W. The task that was originated from a practical man-power scheduling problem was to find a maximum cardinality matching M of G so that no arc of D leaves the set of vertices that are covered by M. It was shown in [4] that an NP-complete version of the satisfiability problem can be reduced to a restricted MS-matching problem in which each weak component of D has size at most three. They also proved that if all weak components of dependence graph D have size at most two, then even the edge-weighted MS-matching problem can be solved in polynomial time. Furthermore, for the above bipartite special case, Hefner [3] has found a min–max formula. Our work is motivated by these latter results and we shall generalize the min–max formula of Hefner to the ordered matroid parity problem.

András Recski has pointed out to us an unexpected application of the ordered matroid parity problem. Recski [6] studied problems on electric networks. Some of these problems are reduced to matroid partition, some others to linear matroid parity and the rest to our ordered matroid parity. This third-type problem can be reduced to linear matroid parity, which is attributed to Bland in [6]. Our result shows that it can also be reduced to the easier matroid intersection problem.

Our work is organized as follows. In Section 2, we prove a min–max formula on the ordered matroid parity problem. We also solve the weighted ordered matroid parity problem in a special case. Then we show that the min–max formula extends that of Hefner in Section 3.

The original version of this note contains generalizations of the main result to supermodular functions and Δ-matroids as well as other related models that are polynomially equivalent to matroid parity [2].

2. Ordered parity problem

Let us fix a matroid $\mathbf{M}$ and a set of ordered pairs Π for the ordered matroid parity problem. Let S be the set of all the second elements and R be the set of all first elements in the pairs in Π. For any $v \in S$, let $\bar{v}$ denote its mate, i.e., $(\bar{v}, v)$ is a pair in Π. For any subset X of S, we denote $\{\bar{v} : v \in X\}$ by $\bar{X}$.

Let $\mathcal{J}$ denote the family of all the ideal independent sets. The weighted ordered matroid parity problem is to find for a given weight function $w : V \to \mathbb{R}_+$ an ideal independent set I that is of maximum weight, i.e., we look for $\max\{w(I)\colon\ I \in \mathcal{J}\}$ where $w(I) := \sum_{v\in I} w(v)$.

A weight function $w : V \to \mathbb{R}_+$ is called *consistent* if $w(u) \leqslant w(v)$ holds for any pair of $u \in R$ and $v \in S$. Clearly, the ordered matroid parity problem is a weighted ordered matroid parity problem for the consistent weight function $w = \mathbf{1}$.

Lemma 2.1. *In the framework of the ordered matroid parity problem, if the weight function w is consistent, then there is an optimal solution I that contains a base of S.*

Proof. Suppose $I \in \mathcal{J}$ is a maximal optimal solution that satisfies $|I \cap S| < r(S)$, and let J be an arbitrary base of S containing $I \cap S$. For any $v \in J \setminus I$, if $I \cup \{v\} \in \mathcal{I}$, then $I \cup \{v\} \in \mathcal{J}$, which contradicts the optimality of I. Therefore, $I \cup \{v\}$ is not independent, and there exists an element $u \in I \setminus S$ such that $I' = I \cup \{v\} \setminus \{u\} \in \mathcal{I}$. Since w is consistent, we have $w(I') \geqslant w(I)$. Then I' is another optimal solution with $|I' \cap S| = |I \cap S| + 1$. Thus, we may assert there is an optimal solution I^0 that satisfies $|I^0 \cap S| = r(S)$. □

Let $\mathbf{M} \cdot S$ denote the restriction of $\mathbf{M}$ to S, and $\mathbf{M}/S$ denote the contraction of $\mathbf{M}$ by S. Lemma 2.1 implies that the ordered parity problem for $\mathbf{M}$ can be reduced to an ordered parity problem for the direct sum $\mathbf{M}^0 = \mathbf{M} \cdot S \oplus \mathbf{M}/S$. We now reduce this problem to the matroid intersection problem.

Let $J \subseteq S$ be an independent set in $\mathbf{M} \cdot S$ such that $\bar{J}$ is also independent in $\mathbf{M}/S$. Let K be an arbitrary base in $\mathbf{M} \cdot S$ containing J. Then $I = \bar{J} \cup K$ satisfies the ordered parity condition. Conversely, an optimal solution I for $\mathbf{M}^0$ must be in this form. Note that $|I| = r(S) + |J|$ holds independently of the choice of K. Therefore, an optimal solution I of the ordered parity problem can be obtained by finding a maximum cardinality J, which is the matroid intersection problem. The following min–max theorem follows from the matroid intersection theorem of Edmonds [1].

Theorem 2.2. *For the ordered parity problem, we have*

$$\max\{|I|: I \in \mathscr{I}\} = \min\{r(X) + r(V \setminus \bar{X}): X \subseteq S\}. \quad (3)$$

Proof. Let J be a maximum cardinality independent set in $\mathbf{M} \cdot S$ such that $\bar{J}$ is also independent in $\mathbf{M}/S$. The matroid intersection theorem implies

$$|J| = \min\{r(X) + r_S(R \setminus \bar{X}): X \subseteq S\} = \min\{r(X) + r(V \setminus \bar{X}): X \subseteq S\} - r(S),$$

where r_S is the rank function of $\mathbf{M}/S$. Since $|I| = |J| + r(S)$, we obtain (3). □

The weighted ordered matroid parity problem for particular weight $w = \chi^R$ is exactly the NP-hard matroid parity problem. But if weight w is consistent, then by Lemma 2.1, the weighted ordered matroid parity problem for $\mathbf{M}$ can be reduced to the weighted ordered matroid parity problem for the above $\mathbf{M}^0$. In what follows, we reduce this latter problem to weighted matroid intersection.

Lemma 2.3. *Let $\mathbf{M}^0$ be the direct sum of matroids $\mathbf{K}$ on R and $\mathbf{L}$ on S. Then the weighted ordered parity problem for $\mathbf{M}^0$ is solvable in polynomial time.*

Proof. Define matroid $\mathbf{K}^0 := \mathbf{K} \oplus \mathbf{F}_S$ where $\mathbf{F}_S$ is the free matroid on S, i.e., each element of S is a coloop in $\mathbf{K}^0$. Let $\mathbf{L}^0$ be the matroid on V so that elements v and $\bar{v}$ are parallel and $\mathbf{L}^0 \cdot S = \mathbf{L}$. Define a weight function w' on V by $w'(s) := w(s)$ and $w'(\bar{s}) := w(s) + w(\bar{s})$ for all $s \in S$.

Let K be a base of $\mathbf{K}$ and $J \subseteq K$. Clearly, if $I = \bar{J} \cup K$ is an independent set of $\mathbf{M}^0 = \mathbf{K} \oplus \mathbf{L}$, then $I' := I \setminus J$ is a common independent set of $\mathbf{K}^0$ and $\mathbf{L}^0$ with $w(I) = w'(I')$. On the other hand, if $I' = \bar{J} \cup L$ is a common independent set of $\mathbf{K}^0$ and $\mathbf{L}^0$ for $J, L \subseteq S$, then J and L are disjoint and $I := I' \cup J$ is an independent set of $\mathbf{M}^0$ satisfying (1) and $w'(I') = w(I)$. That is, if I' is a maximum w'-weight common independent set of $\mathbf{K}^0$ and $\mathbf{L}^0$, then I is a maximum w-weight independent set of $\mathbf{M}^0$ with (1).

Thus, we reduce the weighted ordered parity problem for $\mathbf{M}^0$ to the weighted matroid intersection problem, which can be solved in polynomial time. □

Note that Lemma 2.3 is independent of the consistency of w. Combining Lemmas 2.1 and 2.3, we have the following theorem.

Theorem 2.4. *The weighted ordered matroid parity problem for consistent weight can be solved in polynomial time.*

3. MS-matchings

In this section, we derive the min–max theorem of Hefner on bipartite MS-matchings from Theorem 2.2.

Let $G = (U, V; E)$ be a bipartite graph with the vertex set $W = U \cup V$ and the edge set E. Suppose the vertex set V is of even cardinality and partitioned into ordered pairs Π, and just like in Section 2, R and S denote the set of first and second elements of pairs in Π, respectively. For a subset $M \subseteq E$, we denote by ∂M the set of vertices covered by M. A matching M in G is called an MS-matching if $\bar{v} \in \partial M$ implies $v \in \partial M$ for every ordered pair $(\bar{v}, v) \in \Pi$. The problem of finding a maximum cardinality MS-matching in G is nothing but an ordered parity problem for the transversal matroid on V.

An *MS-cover* is a vector $y \in \mathbf{Z}_+^W$ that satisfies

$$y(u) + y(v) \geqslant 1 \quad \forall (u, v) \in E, \ v \in R, \quad (4)$$

$$y(u) + y(v) - y(\bar{v}) \geqslant 1 \quad \forall (u, v) \in E, \ v \in S. \quad (5)$$

The *value of an MS-cover* y is defined by

$$\mathrm{val}(y) = \sum_{u \in U} y(u) + \sum_{v \in S} y(v).$$

Summing up (4) and (5) for the edges of MS-matching M shows that $|M| \leqslant \mathrm{val}(y)$ for any pair of an MS-matching M and an MS-cover y. Hefner [3] showed that the equality holds for an optimal pair of M and y.

Theorem 3.1 (Hefner [3]). *The maximum cardinality of an MS-matching is equal to the minimum value of an MS-cover.*

We prove this by applying Theorem 2.2 to the transversal matroid. For $Y \subseteq V$, let $\Gamma(Y)$ denote the set of vertices in U adjacent to Y. The rank function τ of the transversal matroid is given by

$$\tau(X) = \min\{|\Gamma(Y)| - |Y|: Y \subseteq X\} + |X|. \quad (6)$$

Let M be a maximum MS-matching. Theorem 2.2 asserts that there exists a subset $X \subseteq S$ such that $|M| = \tau(X) + \tau(V \setminus \bar{X})$. Since $|M| \leqslant \mathrm{val}(y)$ holds for any MS-cover y, the following lemma completes the proof of Theorem 3.1.

Lemma 3.2. *For any* $X \subseteq S$, *there exists an MS-cover* y *such that* $\mathrm{val}(y) = \tau(X) + \tau(V \setminus \bar{X})$.

Proof. Let Y be the unique minimal minimizer that determines $\tau(X)$ in the right-hand side of (6). Similarly, let Z be a minimizer that determines $\tau(V \setminus \bar{X})$. Then we claim that $Y \subseteq Z$.

Note that $|\Gamma(Y)| + |\Gamma(Z)| \geqslant |\Gamma(Y \cap Z)| + |\Gamma(Y \cup Z)|$ and $|Y| + |Z| = |Y \cap Z| + |Y \cup Z|$ hold. Since $|\Gamma(Z)| - |Z| \leqslant |\Gamma(Y \cup Z)| - |Y \cup Z|$, we have $|\Gamma(Y \cap Z)| - |Y \cap Z| \leqslant |\Gamma(Y)| - |Y|$, which implies $Y \subseteq Z$ by the minimality of Y.

We now construct an MS-cover y. For each $u \in U$, we assign

$$y(u) = \begin{cases} 2 & \text{if } u \in \Gamma(Y), \\ 1 & \text{if } u \in \Gamma(Z) \setminus \Gamma(Y), \\ 0 & \text{if } u \in U \setminus \Gamma(Z). \end{cases}$$

For each $v \in R$, we also assign $y(v)$ by

$$y(v) = \begin{cases} 1 & \text{if } v \in R \setminus Z, \\ 0 & \text{if } v \in R \cap Z. \end{cases}$$

Note that y already satisfies (4). For each $v \in S$, we assign $y(v) = z(v) + y(\bar{v})$, where $z(v)$ is defined by

$$z(v) = \begin{cases} 1 & \text{if } v \in S \setminus Z, \\ 0 & \text{if } v \in Z \setminus Y, \\ -1 & \text{if } v \in Y. \end{cases}$$

By definition, $z(u) + z(v) \geqslant 1$ for any $(u, v) \in \Pi$. So the resulting $y \in \mathbf{Z}_+^W$ is an MS-cover and its value is given by

$$\begin{aligned} \mathrm{val}(y) &= |\Gamma(Y)| + |\Gamma(Z)| + |R \setminus Z| + |S \setminus Z| - |Y| \\ &= |\Gamma(Y)| + |\Gamma(Z)| + |V \setminus \bar{X}| + |X| - |Z| - |Y| \\ &= \tau(X) + \tau(V \setminus \bar{X}). \end{aligned}$$

Thus, we obtain an MS-cover y with $\mathrm{val}(y) = \tau(X) + \tau(V \setminus \bar{X})$. □

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