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Discrete Mathematics 308 (2008) 2647–2648
www.elsevier.com/locate/disc

Note

A note on degree-constrained subgraphs

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Received 11 December 2006; received in revised form 24 March 2007; accepted 28 May 2007
Available online 8 June 2007

Abstract

Elementary proofs are presented for two graph theoretic results, originally proved by H. Shirazi and J. Verstraëte using the combinatorial Nullstellensatz.

Keywords: Degree-constrained subgraphs; Combinatorial Nullstellensatz

In an undirected graph $G = (V, E)$ we denote by $d_G(v)$ the degree of $v \in V$. If $F(v) \subseteq \mathbb{N}$ is a set of forbidden degrees for every $v \in V$, then a subgraph $G' = (V, E')$ of G is called F-avoiding if $d_{G'}(v) \notin F(v)$ for all $v \in V$.

Theorem 1 (*Shirazi and Verstraëte [5]*). *If $G = (V, E)$ is an undirected graph and*

$$|F(v)| \leqslant d_G(v)/2 \quad \textit{for every node } v, \tag{1}$$

then G has an F-avoiding subgraph.

Theorem 1 appeared first under the name Louigi's conjecture in [1]. A version with $d_G(v)/2$ replaced by $d_G(v)/12$ was given in [1], while $d_G(v)/8$ was proved in [2]. Louigi's conjecture was first settled in the affirmative by Shirazi and Verstraëte [5]. Their proof is based on the combinatorial Nullstellensatz of Alon [3]. We give an elementary proof, which uses Theorem 2 below. In a directed graph $D = (V, \vec{E})$ we denote by $\varrho_D(v)$ the in-degree of $v \in V$.

Theorem 2. *If $G = (V, E)$ is an undirected graph and it has an orientation D for which $\varrho_D(v) \geqslant |F(v)|$ for every node v, then G has an F-avoiding subgraph.*

Proof. For an undirected edge e, let $\vec{e}$ denote the corresponding directed edge of D. We use induction on the number of edges. If 0 is not a forbidden degree at any node, then the empty subgraph $(V, \emptyset)$ is F-avoiding. Suppose that $0 \in F(t)$

[1] Supported by the Hungarian National Foundation for Scientific Research, OTKA K60802, TS 049788, and by European MCRTN Adonet, Contract Grant no. 504438.

[2] Research was done while the author visited the EGRES Group. Supported by European MCRTN Adonet, Contract Grant no. 504438.
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0012-365X/$ - see front matter
doi:10.1016/j.disc.2007.05.021

for a node t. Then $\varrho_D(t) \geqslant |F(t)| \geqslant 1$ and hence there is an edge $e = st$ of G for which $\vec{e}$ is directed toward t. Let $G^- = G - e$ and $D^- = D - \vec{e}$. Define F^- as follows. Let $F^-(t) = \{i - 1 : i \in F(t) \backslash \{0\}\}$, $F^-(s) = \{i - 1 : i \in F(s) \backslash \{0\}\}$, and for $z \in V - \{s, t\}$ let $F^-(z) = F(z)$. Since $|F^-(t)| = |F(t)| - 1$, $\varrho_{D^-}(v) \geqslant |F^-(v)|$ holds for every node v. By induction, there is an F^--avoiding subgraph G'' of G^-. By the construction of F^-, the subgraph $G' := G'' + e$ of G is F-avoiding. □

Proof of Theorem 1. It is well-known that every undirected graph G has an orientation D in which

$$\varrho_D(v) \geqslant \lfloor d_G(v)/2 \rfloor \quad \text{for every node } v. \tag{2}$$

Indeed, by adding a new node z to G and joining z to every node of G with odd degree, we obtain a graph G^+ in which every degree is even. Hence G^+ decomposes into edge-disjoint circuits and therefore it has an orientation in which the in-degree of every node equals its out-degree. The restriction of this orientation to G satisfies (2). (An orientation with property (2) is also used in [5].) Therefore Theorem 2 implies Theorem 1. □

Hakimi [4] proved that, given a function $f : V \to \mathbb{Z}_+$, an undirected graph G has an orientation for which $\varrho(v) \geqslant f(v)$ for every node v if and only if $e_G(X) \geqslant \sum[f(v) : v \in X]$ holds for every subset $X \subseteq V$, where $e_G(X)$ denotes the number of edges with at least one end-node in X. By combining this with Theorem 2, one obtains the following.

Corollary 3. *If $G = (V, E)$ is an undirected graph and $e_G(X) \geqslant \sum[|F(v)| : v \in X]$ holds for every subset $X \subseteq V$, then G has an F-avoiding subgraph.*

Along with Theorem 1, the following result was also proved in [5] via the Combinatorial Nullstellensatz. A graph is called *empty* if it has no edges.

Theorem 4 (*Shirazi and Verstraëte [5]*). *If $G = (V, E)$ is an undirected graph, $0 \notin F(v)$ for all $v \in V$ and $\sum_{v \in V} |F(v)| < |E|$, then G has a nonempty F-avoiding subgraph G'.*

Proof. Again, we use induction on the number of edges. If $d_G(v) \notin F(v)$ for all $v \in V$, then the nonempty $G' = G$ will do. Otherwise there exists a node $t \in V$ where $d_G(t) \in F(t)$. As $0 \notin F(v)$, there is an edge e of G incident to t. Let $G^- = G - e$, let $F^-(t) = F(t) \backslash \{d_G(t)\}$ and for $z \in V - \{t\}$ let $F^-(z) = F(z)$. By induction, there is a nonempty F^--avoiding subgraph G' of G^-. As $d_{G'}(t) < d_G(t)$, this G' is also F-avoiding. □

We remark that Theorems 2 and 4 clearly hold for hypergraphs, as well, with the same proofs. Combining this with the hypergraph variant of Hakimi's theorem, one concludes that also Corollary 3 applies to hypergraphs. However, in Theorem 1 one should replace the denominator 2 by the rank of the hypergraph (that is, the maximum size of a hyperedge). This is already observed by Shirazi and Verstraëte [5]. Note also that both proofs give rise to polynomial algorithms: such algorithms were not known before.

Acknowledgments

The second author would like to thank the Egerváry Research Group for their hospitality.

Note added in proof

After submitting the paper, the authors learned that Adrian Bondy also formulated and proved Theorem 2. His proof goes along the same line as ours.

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