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Rooted k-connections in digraphs☆

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ABSTRACT

The problem of computing a minimum cost subgraph $D' = (V, A')$ of a directed graph $D = (V, A)$ so as to contain k edge-disjoint paths from a specified root $r_0 \in V$ to every other node in V was solved by Edmonds [J. Edmonds, Submodular functions, matroids, and certain polyhedra, in: R. Guy, H. Hanani, N. Sauer, J. Schönheim (Eds.), Combinatorial Structures and their Applications, Gordon and Breach, New York, 1970, pp. 69–87] by an elegant reduction to weighted matroid intersection. A corresponding problem when openly disjoint paths are requested rather than edge-disjoint ones was solved in [A. Frank, É. Tardos, An application of submodular flows, Linear Algebra Appl. 114–115 (1989) 329–348] with the help of submodular flows. Here we show that the use of submodular flows is actually avoidable and even a common generalization of the two rooted k-connection problems reduces to matroid intersection. The approach is based on a new matroid construction extending what Whiteley [W. Whiteley, Some matroids from discrete applied geometry, in: J.E. Bonin, J.G. Oxley, B. Servatius (Eds.), Matroid Theory, in: Contemp. Math., vol. 197, Amer. Math. Soc, Providence, RI, 1996, pp. 171–311] calls count matroids. We also provide a polyhedral description using supermodular functions on bi-sets and this approach enables us to handle more general rooted k-connection problems. For example, with the help of a submodular flow algorithm the following restricted version of the generalized Steiner-network problem is solvable in polynomial time: given a digraph $D = (V, A)$ with a root-node r_0, a terminal set T, and a cost function $c : A \to \mathbf{R}_+$ so that each edge of positive cost has its head in T, find a subgraph $D' = (V, A')$ of D of minimum cost so that there are k openly disjoint paths in D' from r_0 to every node in T.

1. Introduction

Let $D = (V, A)$ be a directed graph. For two specified nodes s and t of D, a (directed) path from s to t is called an st-path. Let $\lambda(s, t; D)$ and $\kappa(s, t; D)$ denote the maximum number of edge-disjoint, respectively, openly disjoint, st-paths. Two st-paths are called **openly disjoint** if their nodes in common are exactly s and t. In particular, k parallel edges from s to t form k openly disjoint paths.

It is well known that $\lambda(s, t; D)$ can be computed via a max-flow min-cut algorithm and even more, given a non-negative cost function on A, the cheapest set of k edge-disjoint paths from s to t can also be computed in strongly polynomial time with the help of a min-cost flow algorithm. There is a well-known and easy node-splitting technique (described, for example, in [8]) to reduce the computation of k openly disjoint st-paths to that of k edge-disjoint st-paths.

Let r_0 be a specified node of D called a root. We will throughout assume that no edge of D enters r_0. The digraph is called **rooted k-edge-connected** (resp., **rooted k-node-connected** or briefly **rooted k-connected**) if $\lambda(r_0, v; D) \geq k$

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(resp., $\kappa(r_0, v; D) \geq k$) holds for every $v \in V - r_0$. By the directed edge-version of Menger's theorem rooted k-edge-connectivity is equivalent to requiring that the in-degree $\varrho_D(X)$ of every non-empty subset $X \subseteq V - r_0$ is at least k. Suppose that D is endowed with a non-negative cost function $c : A \rightarrow \mathbf{R}_+$.

We are interested in the **rooted k-edge-connection** and the **rooted k-node-connection** problems which consist of finding a cheapest subgraph $D' = (V, A')$ of D so that

$$D' \text{ is rooted } k\text{-edge-connected from } r_0 \tag{1}$$

and, respectively,

$$D' \text{ is rooted } k\text{-node-connected from } r_0. \tag{2}$$

When $k = 1$ the two problems coincide and it amounts to finding a minimum cost spanning arborescence. This was first solved by Yong-Jin Chu and Tseng-Hong Liu [1] by using a direct graph-theoretical approach. Fulkerson [9] described a min–max theorem along with a polyhedral description of arborescences. For general $k \geq 1$, two approaches have been known for solving (1). The first one, due to Edmonds [2], consists of showing that there are two matroids on the edge-set of D so that their common bases are exactly the minimal rooted k-edge-connected subgraphs of D. The second one uses polyhedral techniques [10]. Though this is more complicated and its algorithmic solution relies on submodular flows algorithms, it has the advantage of implying several extensions of (1).

The minimum cost rooted k-connected subgraph problem was solved by Frank and Tardos [13] who described a rather complicated way to reduce it to submodular flows: in this sense a polynomial algorithm has been available. Since the k openly disjoint st-paths problem can so easily be reduced, via the node-splitting technique, to that of k edge-disjoint st-paths, it has been tempting to avoid the difficult reduction of [13] by invoking node-splitting. A natural direct approach, however, fails since node-splitting gives rise to new nodes of in-degree 1 and therefore the resulting digraph certainly will not contain k edge-disjoint paths from r_0 to *every* other node (when $k \geq 2$).

The goal of this paper is twofold. First, it will be shown that the rooted k-node-connection problem, like its k-edge-connection counterpart, can also be reduced to matroid intersection, and this will, in fact, be proved for a common generalization of the two versions. In other words, the use of submodular flows is avoidable. Second, by introducing an approach simpler and more natural than the one in [13], a totally dual integral (TDI) description of the rooted k-connected subgraphs will be provided. By extending a method of A. Schrijver, we show that the polyhedron in question is also a submodular flow polyhedron. Again, this second, more complicated framework gives rise to more general root-connection problems. For example, this way one is able to find a cheapest subgraph of a digraph in which there are k edge-disjoint (resp., openly disjoint) paths from r_0 to every node in a specified terminal set $T \subseteq V - r_0$ provided that the head of every edge with positive cost is in T, a requirement satisfied automatically when $T = V - r_0$. In fact, these solutions will be formulated in a general framework that includes both the rooted k-edge-connection and the rooted k-node-connection problems.

The idea behind the extensions is that, while edge-connection problems are often successfully attacked via supermodular set-functions, for node-connections the right tool is supermodular functions defined on certain pairs of subsets called bi-sets. They not only help in formulating the requested polyhedral description but can also be used to extend a matroid construction called count matroids by Whiteley [27].

The paper is organized as follows. The next section describes two known approaches to the rooted k-edge-connection problem: Namely, Edmonds' reduction, via his disjoint arborescences theorem, to matroid intersection and a polyhedral description from [10]. In Section 3, the notion of count matroids on directed graphs is introduced and an independence oracle for them is described. Section 4 shows how the rooted k-connection problem is reduced to matroid intersection while the last section describes a polyhedral description of rooted k-connected subgraphs.

We conclude this introductory section by listing some notions and notation used throughout the paper. We often do not distinguish a one-element set $\{v\}$ (sometimes called a singleton) from its only element v. The union of a set X and a singleton $\{v\}$ is abbreviated by $X + v$ while $X \setminus \{v\}$ by $X - v$. For the difference $X \setminus Y$ of two sets, we write $X - Y$. By a subpartition of S, we mean a partition of a subset of S. $X \subset Y$ means that X is a proper subset of Y, in particular, $\emptyset \subset X$ denotes that X is non-empty.

An undirected edge connecting u and v will be denoted by uv or vu. A directed edge e with head v and tail u is denoted by $e = uv$. It is said to **enter** a subset Z if $v \in Z$, $u \notin Z$. When $u, v \in Z$, the edge uv is **induced** by Z. Induced edges in an undirected graph are defined analogously.

For a digraph $D = (V, A)$, its in-degree function $\varrho := \varrho_D := \varrho_A$ is defined for every subset $Z \subseteq V$ by the number of edges entering Z. For a function $x : A \rightarrow \mathbf{R}$, we write $\varrho_x(Z) := \sum[x(e) : e \text{ enters } Z]$. The out-degree function $\delta := \delta_D := \delta_A$ is defined by $\delta(Z) := \varrho(V - Z)$ and also $\delta_x(Z) := \varrho_x(V - Z)$. The number of edges of D induced by a subset Z is denoted by $i(Z) := i_D(Z) := i_A(Z)$.

By a **branching** we mean a directed forest in which the in-degree of each node is at most one. The set of nodes of in-degree 0 is called the **root-set** of the branching. When the root-set is a singleton, we speak of an **arborescence**. Note that a branching with root-set R is the union of $|R|$ disjoint arborescences.

The non-negative part of a number x is $x^+ := \min\{x, 0\}$. For a function $m : S \rightarrow \mathbf{R}$ on S and for any subset $X \subseteq S$, we use the abbreviation $m(X) := \sum[m(v) : v \in X]$.

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2. Preliminaries on rooted k-edge-connection

2.1. Edmonds' reduction to matroid intersection

Edmonds' algorithmic solution in [2] to the rooted k-edge-connection problem has actually been a by-product of his approach to the problem of finding a cheapest subset of edges of D which is the union of k edge-disjoint spanning arborescences. The link between the two problems is Edmonds' disjoint arborescences theorem [4].

Theorem 2.1 (*Edmonds' Disjoint Arborescences Theorem*). *A digraph $D = (V, A)$ with a specified root r_0 has k edge-disjoint spanning arborescences of root r_0 if and only if*

$$\varrho_D(X) \geq k \quad \text{for every non-empty subset } X \subseteq V - r_0, \tag{3}$$

that is, D is rooted k-edge-connected. •

This theorem (along with the stronger disjoint branchings theorem stating that, *given k non-empty root-sets $R_i \subseteq V$, there are k disjoint branchings B_i of root-set R_i if and only if the in-degree of every non-empty subset $X \subseteq V$ is at least the number of root-sets disjoint from X*) was used in [2]. Its full proof appeared in [4]. Edmonds observed that Theorem 2.1 easily implies the following.

Theorem 2.2. *A digraph $D' = (V, B)$ is the union of k edge-disjoint spanning arborescences of root r_0 if and only if*

$$\varrho(r_0) = 0 \quad \text{and} \quad \varrho(v) = k \quad \text{for every } v \in V - r_0 \tag{4}$$

and the underlying undirected graph of D' is the union of k edge-disjoint spanning trees.

Proof. The necessity of the conditions is straightforward. Their sufficiency follows from Theorem 2.1 once one shows that D' is rooted k-edge-connected. Indeed, for a non-empty subset $X \subseteq V - r_0$, we have $i(X) \leq k(|X| - 1)$ since the underlying graph is the union of k forests and hence $\varrho(X) = \sum[\varrho(v) : v \in X] - i(X) = k|X| - i(X) \geq k|X| - k(|X| - 1) = k$, as required. •

Edmonds and Fulkerson [6] proved that the sum (or union) of some matroids forms a matroid, in particular, the subsets of edges of a graph which are the union of k edge-disjoint spanning trees form the set of bases of a matroid denoted by M_1. Let M_2 denote the partition matroid whose set of bases is defined by (4). Theorem 2.2 implies that finding a cheapest subgraph of a digraph which is the union of k edge-disjoint arborescences is equivalent to computing a cheapest common basis of matroids M_1 and M_2. This can be done with the help of any weighted matroid intersection algorithm. (See, for example, Edmonds' algorithm [5] or a conceptually simpler version in [11].) A matroid intersection algorithm can only be applied if the independence oracles (or equivalent) for the two matroids are indeed available. This is obvious for the partition matroid M_2. As far as M_1 is concerned, Edmonds' [3] polynomial-time algorithm for computing the rank of the sum of matroids provides the requested oracle.

Summing up, Edmonds' approach to compute k edge-disjoint spanning arborescences of root r_0 whose set of edges is of minimum cost is as follows. First, compute a cheapest common basis B of matroids M_1 and M_2. Second, apply any existing algorithm to partition B into k edge-disjoint spanning arborescences. For example, Tarjan [26] observed that the strong form of Edmonds' theorem (that is, the disjoint branchings theorem) has a self-algorithmic nature in the sense that the theorem lends itself to an algorithm for constructing the k disjoint branchings, provided an MFMC-subroutine is available for computing minimum cuts. Lovász' simple proof [20] for Edmonds' theorem also gave rise to such an algorithm.

2.2. Polyhedral description

The second approach uses a more general framework. In handling edge-connectivity optimization problems, it is rather typical that a general result on covering supermodular functions by directed graphs is in the background. For the rooted k-edge-connection problem such a framework can be formulated as follows. A set-function $p : 2^V \to \mathbf{Z}$ is said to satisfy the supermodular inequality on subsets $X, Y \subseteq V$ if

$$p(X) + p(Y) \leq p(X \cup Y) + p(X \cap Y). \tag{5}$$

If this holds whenever $X \cap Y \neq \emptyset$, then p is called **intersecting supermodular**. If (5) is required only for subsets with $p(X) > 0, p(Y) > 0$, and $X \cap Y \neq \emptyset$, then p is **positively intersecting supermodular**.

A typical way to create a positively intersecting supermodular function is to take the 'non-negative part' of an intersecting supermodular one with possible negative values which means replacing each negative value by zero. Example shows, however, that not every non-negative, positively intersecting supermodular function arises this way.

A digraph $D = (V, A)$ (or a function $x : A \to \mathbf{Z}_+ \cup \{\infty\}$ on its edge-set) is said to **cover** p if $\varrho_D(X) \geq p(X)$ (resp., $\varrho_x(X) \geq p(X)$) for every subset $X \subseteq V$. The problem consists of finding a cheapest subgraph of D covering a positively intersecting supermodular function. To formulate the result on covering intersecting supermodular functions, let $g : A \to \mathbf{Z}_+ \cup \{\infty\}$ be a non-negative upper bound on the edges of D that covers p.

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Theorem 2.3 *([10,15]). If p is a positively intersecting supermodular set-function, the linear system*

$$\{0 \leq x \leq g \text{ and } \varrho_x(Z) \geq p(Z) \text{ for every } Z \subset V\} \tag{6}$$

described for $x \in \mathbf{R}^A$ is totally dual integral. In particular, the linear programming problem

$$\min\{cx : x \text{ satisfies } (6)\} \tag{7}$$

has an integer-valued optimum solution and so has its linear programming dual provided c is integer-valued.

Theorem 2.4 *(Schrijver [22]). If p is an intersecting supermodular set-function, the polyhedron R defined by (6) is a submodular flow polyhedron.*

To see how the framework in Theorem 2.3 includes Problem (1), consider the special case when $g \equiv 1$ and p is defined by $p(Z) = k$ for every subset $\emptyset \subset Z \subseteq V - r_0$ and $p(Z) = 0$ otherwise. In this case there is a one-to-one correspondence between the 0–1-valued solutions of (6) and the rooted k-edge-connected subgraphs of D. Therefore a submodular flow algorithm may be used to construct a cheapest rooted k-edge-connected subgraph. More generally, given root-sets $R_1, \dots, R_k$, by choosing p to be the function for which $p(\emptyset) = 0$ and $p(X)$ is the number of root-sets disjoint from $X \neq \emptyset$, one obtains a polyhedral description of subsets of D which contain k disjoint branchings B_i of root-set R_i.

The statement in Schrijver's theorem is not known for positively intersecting supermodular functions. Fortunately, each known application of the framework in Theorem 2.3 requires intersecting supermodular functions. The main reason for the usage of positively intersecting supermodular functions is that the proofs become technically simpler.

Schrijver's method to formulate (6) as a submodular flow problem will be extended in Section 5.2. Since this reduction can be carried out in polynomial time and since there are good (combinatorial) algorithms for submodular flows (for a comprehensive overview, see Fujishige's book [16]), the optimization problem (7) is also solvable in polynomial time.

It was proved in [12] that every (integer) submodular flow polyhedron is the projection (along coordinate axes) of the intersection of two base-polyhedra. The submodular flow polyhedron R above is actually in the 0–1 cube of $\mathbf{R}^A$ and in this case R is the projection of the intersection of the base-polyhedra of two matroids in the space of dimension $2|A|$. In this sense, matroid intersection can be applied to optimize 0–1-valued submodular flows. This reduction, however, is not only significantly more complex than the one outlined in Section 2.1 but its algorithmic realization is also far more complicated.

On the other hand, the polyhedral approach has the advantage that more general rooted k-edge-connection problems can also be handled with its help. For example, let us be given a digraph $D = (V, F_0 \cup A)$ with a root-node r_0 and a terminal set $T \subseteq V - r_0$ so that T contains the head of every edge in A and so that there are k edge-disjoint paths from r_0 to every node $t \in T$. There is also a cost function $c : A \to \mathbf{R}$. The problem is to find a minimum cost subset F of A so that there are k edge-disjoint paths in $(V, F_0 \cup F)$ from r_0 to every node $t \in T$. Note that this problem specializes to the rooted k-edge-connection problem when $T = V - r_0$ and $F_0 = \emptyset$, while it becomes NP-complete even for $k = 1$ if $F_0 = \emptyset$ and the assumption on the head of edges in A (to be in T) is dropped. (This is the directed Steiner-tree problem). The problem is indeed a special case of the framework in Theorem 2.4 when the groundset is chosen to be T and function p is defined by $p(X) := \max\{k - \varrho_{F_0}(X \cup Y) : Y \subseteq V - (T + r_0)\}$ when $X \subseteq T$ is non-empty and $p(\emptyset) = 0$ since this p is easily proved to be intersecting supermodular. (This application was explicitly mentioned in [10] only for the special case $T = V - r_0$.) For an extension, see Section 5.3.

3. Matroids on the edge-set of undirected and directed graphs

In addition to the weighted matroid intersection algorithm, the matroidal approach of Edmonds outlined above needed several other tools: his disjoint arborescence theorem and algorithm, the notion of the matroid sum along with the matroid partition algorithm. Since there is no known decomposition theorem for rooted k-connected digraphs analogous to Theorem 2.1, in order to reduce the rooted k-connection problem to matroid intersection, we must choose a different approach.

A set-function b on a groundset S is called **non-decreasing** if $b(X) \leq b(Y)$ whenever $X \subseteq Y \subseteq S$. It is **fully submodular**, or for short, submodular if the submodular inequality

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y) \tag{8}$$

holds for every two subsets X and Y of S, and b is **intersecting submodular** if (8) holds whenever $X \cap Y \neq \emptyset$.

Theorem 3.1 *(Edmonds). For an integer-valued, non-decreasing, intersecting submodular function b, the set $\{F \subseteq S : |F \cap X| \leq b(X)$ for every non-empty subset $X \subseteq F\}$ forms the set of independent sets of a matroid M_b whose rank function r_b is given by the formula*

$$r_b(Z) = \min\left\{\sum_i b(X_i) + |Z - \cup_i X_i| : \{X_i\} \text{ a subpartition of } Z\right\}. \quad \bullet \tag{9}$$

M_b is called the matroid of b.

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3.1. Count matroids on undirected (hyper)graphs

A widely used way to apply this construction is as follows. Let $G = (V, E)$ be an undirected graph with no loops. For a subset $F \subseteq E$, let $V(F)$ denote the nodes covered by the edges in F. Let $m : V \to \mathbf{Z}_+$ be a non-negative integer-valued function on V and l an integer for which

$$m(u) + m(v) \geq l \quad \text{for every edge } uv \in E. \tag{10}$$

It is easy to see that the set-function $b^\#$ defined by $b^\#(\emptyset) = 0$ and

$$b^\#(F) := m(V(F)) - l \quad (\emptyset \subset F \subseteq E) \tag{11}$$

is a non-decreasing, intersecting submodular function on groundset E (which is actually fully submodular for $l \leq 0$). Note that (10) ensures that $b^\#$ is non-negative. By Theorem 3.1 the set

$$\{F \subseteq E : |F'| \leq b^\#(F') \text{ for every non-empty subset } F' \subseteq F\} \tag{12}$$

forms the independent sets of a matroid. It is easily seen and will be shown for a more general case in the next subsection that it suffices to require the inequality $|F'| \leq b^\#(F')$ only for subsets $F' \subseteq F$ induced by subsets of nodes. Hence one gets the following.

Theorem 3.2. *Suppose that m and l satisfy* (10). *The set*

$$\mathcal{F}^\# := \{F \subseteq E : i_F(X) \leq m(X) - l \text{ for every } X \subseteq V \text{ with } i_F(X) \geq 1\} \tag{13}$$

forms the set of independent sets of a matroid $M_{m,l}(G)$.

The matroids $M_{m,l}$ are called **count matroids** (on graphs) in a paper by Whiteley [27] where several special cases and applications are exhibited along with an extension to hypergraphs $H = (V, \mathcal{E})$. (Here a subset $\mathcal{F} \subseteq \mathcal{E}$ of hyperedges is declared independent if $|\mathcal{F}'| \leq m(\cup(Z : Z \in \mathcal{F}')) - l$ for every non-empty subset $\mathcal{F}' \subseteq \mathcal{F}$. The only difference is that (10) should be replaced by $m(Z) \geq l$ for every hyperedge $Z \in \mathcal{E}$.) For example, for a graph $G = (V, E)$ and a positive integer k, let $m \equiv k$ and $l := k$. Then in the corresponding count matroid $M_k(G) := M_{m,l}(G)$ a subset F of edges is independent if every non-empty subset X of nodes induces at most $k(|X| - 1)$ elements of F. For $k = 1$, this is just the circuit matroid of G. Note that by a theorem of Nash-Williams [21], the independent sets in M_k are those subsets of edges which can be partitioned into k forests. Hence, by the matroid partition theorem of Edmonds and Fulkerson [6], M_k is the matroid union (sum) of k copies of the circuit matroid M_1. However, in order to construct an independence oracle for M_k, this relationship and in particular the matroid partition algorithm is avoidable. For any count matroid such an oracle, relying only on MFMC computations was developed by H. Imai and by K. Sugihara (see the book of Sugihara [24] or papers of Sugihara and Imai [25,19]).

Remark. We indicate that the count matroid $M_k(G)$ allows one to provide another matroid intersection approach to rooted k-edge-connection more flexible than the one outlined in Section 2.1. Namely, let M_2 be the same partition matroid as before, while M_1' is the direct sum of the free matroid on A_0 (the set of edges leaving r_0) and the restriction of $M_k(G)$ to $A^* := A - A_0$. Although this matroid is more free than the matroid M_1 used in Edmonds' reduction, it is still true (and will be proved in a more general context) that the minimal rooted k-edge-connected subgraphs of D are exactly the common independent sets of $k(|V| - 1)$ elements of M_1' and M_2.

3.2. Count matroids on directed graphs

We are going to extend the notion of count matroids for digraphs and this will include count matroids on undirected graphs as a special case. An independence oracle for these more general count matroids will also be outlined.

Given a groundset V, by a **bi-set** $X = (X_O, X_I)$ we mean a pair of subsets X_O, X_I of V for which $\emptyset \subseteq X_I \subseteq X_O \subseteq V$. X_O is the **outer** member of X while X_I is the **inner** member. A bi-set X with $X_I = \emptyset$ or with $X_O = V$ is called **trivial**. When $X_I = \emptyset$ the bi-set is **void**. A function defined on bi-sets will be called a **bi-set function**. We will assume throughout that the bi-set functions in question are integer-valued and that their value on void bi-sets is always zero.

We say that a directed edge $e = uv$ is **induced** by a bi-set $X = (X_O, X_I)$ if the head v of e is in X_I while its tail u is in X_O. For a digraph $D = (V, A)$, let $I_D(X) := I_A(X)$ denote the set of edges induced by X and $i_D(X) := i_A(X) := |I_D(X)|$. Let $D^* = (V^*, A^*)$ be a digraph, $m_O : V^* \to \mathbf{Z}_+$ and $m_I : V^* \to \mathbf{Z}_+$ two functions and l an integer such that

$$m_I(v) + m_O(u) + m_O(v) \geq l \quad \text{for every edge } uv \in A^*. \tag{14}$$

Let

$$\mathcal{F}^\# := \{F \subseteq A^* : i_F(X) \leq m_I(X_I) + m_O(X_O) - l \text{ for each bi-set } X = (X_O, X_I) \text{ with } i_F(X) \geq 1\}. \tag{15}$$

Theorem 3.3. *$\mathcal{F}^\#$ forms the set of independent sets of a matroid on A^*.*

Proof. We are going to define a non-decreasing, intersecting submodular function $b^{\#}$ on A^* and show that the set of independent sets of its matroid is $\mathcal{F}^{\#}$.

For a subset $J \subseteq A^*$ of edges of digraph D^*, let $H(J) := \{v : v \text{ is the head of some edges in } J\}$ and $V(J) := \{u : u \text{ is the head or the tail of some edges in } J\}$. Note that $V(J)$ is actually a set-function on the underlying undirected edge-set and independent of the orientation of the edges. Let

$$b_I(J) := m_I(H(J)) \quad \text{and} \quad b_O(J) := m_O(V(J)). \tag{16}$$

The proof of the following proposition is an easy exercise and is left to the reader.

Proposition 3.4. *Both b_I and b_O are non-decreasing submodular functions on groundset A^*.* •

Define the set-function $b^{\#}$ to be 0 on the empty set and, for $\emptyset \subset J \subseteq A^*$,

$$b^{\#}(J) := b_I(J) + b_O(J) - l. \tag{17}$$

By the proposition, $b^{\#}$ is intersecting submodular. Furthermore, if $uv \in A^*$ is an edge, then (14) implies for the singleton $J := \{uv\}$ that $b^{\#}(J) = m_I(H(J)) + m_O(V(J)) - l = m_I(v) + m_O(u) + m_O(v) - l \geq 0$. This and the non-negativity of m_I and m_O imply that $b^{\#}$ is non-decreasing (and hence non-negative). Let $M^{\#} := M_{b^{\#}}$ denote the matroid of $b^{\#}$ defined in Edmonds' theorem (Theorem 3.1).

Proposition 3.5. *F is independent in $M^{\#}$ if and only if $F \in \mathcal{F}^{\#}$.*

Proof. Let F be independent in $M^{\#}$. Let X be a bi-set for which $i_F(X) \geq 1$ and let $J := I_F(X)$ be the subset of F induced by X. Since $H(J) \subseteq X_I$ and $V(J) \subseteq X_O$, we have $|J| \leq b^{\#}(J) = m_I(H(J)) + m_O(V(J)) - l \leq m_I(X_I) + m_O(X_O) - l$, that is, (14) holds true.

Conversely, if F is dependent in $M^{\#}$, then it has a subset J for which $|J| > b^{\#}(J)$. Let $X_I := H(J)$ and $X_O := V(J)$. Then every element of J is induced by bi-set $X = (X_O, X_I)$ and hence $i_F(X) \geq |J| > b^{\#}(J) = m_I(H(J)) + m_O(V(J)) - l$, that is, (14) is violated. ••

Note that in the special case $m_I \equiv 0$, (15) reads as $\mathcal{F}^{\#} := \{F \subseteq A^* : i_F(X) \leq m_O(X_O) - l \text{ for every bi-set } X = (X_O, X_I) \text{ with } i_F(X) \geq 1\}$. Since $i_F(X_O, X_O) \geq i_F(X_O, X_I)$, it suffices to require the inequality only for bi-sets of type (X_O, X_O), that is, $\mathcal{F}^{\#} = \{F \subseteq A^* : i_F(X_O) \leq m_O(X_O) - l \text{ for every subset } X_O \subseteq V^*\}$. In this case, the orientation of the edges does not play any role and we are back at the count matroids on undirected graphs. (Also, (14) is equivalent to (10).)

3.2.1. Independence oracle for $M^{\#}$

In order to apply count matroids in algorithms, one must have an independence oracle to decide for any input subset X whether X is independent or not in $M^{\#}$. For count matroids on undirected graphs, this can be obtained via MFMC computations (see the book of Sugihara [24] or papers by Sugihara and Imai [25,19]). Here we describe the corresponding oracle for count matroids on digraphs. Instead of using MFMC computations, we rely on an orientation result which is a slight extension of a theorem Hakimi [18] obtained in the special case $\gamma = 0$.

Lemma 3.6. *Let $H = (V, F)$ be an undirected graph, $g' : V \to \mathbf{Z}_+$ an upper-bound function and $\gamma \geq 0$ an integer. It is possible to remove at most γ edges from H so that the remaining graph H' has an orientation with in-degree function ϱ' satisfying $\varrho'(v) \leq g'(v)$ for every node v if and only if*

$$g'(X) + \gamma \geq i_H(X) \tag{18}$$

holds for every subset X of nodes.

Proof. Necessity. If a subgraph $H' = (V, F')$ with $|F - F'| \leq \gamma$ has the required orientation, then $0 \leq \varrho'(X) = \sum[\varrho'(v) : v \in X] - i_{H'}(X) \leq \sum[g'(v) : v \in X] - (i_H(X) - \gamma)$ from which (18) follows.

Sufficiency. Starting with an arbitrary orientation of H, we gradually reduce the 'error-sum' $\sum[(\varrho(v) - g'(v))^+ : v \in V]$ by successively reorienting certain paths. Let Z_0 denote the set of nodes z with $\varrho(z) > g'(z)$ (called bad nodes). If Z_0 is empty, then the current orientation of H itself is good. For a non-empty Z_0, compute the set Z of nodes from which Z_0 is reachable along a directed path in the current orientation. Then $Z_0 \subseteq Z$ and no edge enters Z. If there is a node $u \in Z$ with $\varrho(u) < g'(u)$, then by reorienting any path from u to Z_0 the error-sum becomes smaller. If no such node u exists, then remove $\varrho(z) - g'(z)$ entering edges at every bad node z. In the remaining digraph the in-degree of every node v is at most $g'(v)$. The number of removed edges is $\sum[\varrho(v) - g'(v) : v \in Z_0] = \sum[\varrho(v) - g'(v) : v \in Z] = \varrho(Z) + i_H(Z) - g'(Z) = i_H(Z) - g'(Z) \leq \gamma$. •

Note that the proof of the lemma gives rise to an algorithm of complexity $O(|V||E|^2)$ and since it can be considered as a variation of the alternating path algorithm for flows, the bound can actually be reduced to $O(|V|^3)$.

We assume $l \geq 0$ since only this case is needed for rooted node-connection. (For $l < 0$, see the remark below.) What we actually construct is a subroutine which decides for an input independent set $F' \subseteq A^*$ of $M^{\#}$ and for an input element $f = sz \in A^* - F'$ whether $F := F' + f$ is independent. By repeated applications of this, one can easily decide if an arbitrary subset is independent or not in $M^{\#}$.

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For the digraph $D_F = (V^*, F)$, construct a bipartite undirected graph $G = (V', V''; E)$ as follows. To every node $v \in V^*$, assign a node $v' \in V'$ and a node $v'' \in V''$ which are connected by $m_O(v)$ parallel edges. The set of these edges is denoted by E_1. Furthermore, with every directed edge $e = uv \in F$, we associate an edge $e_G = u'v''$ of G. The set of these edges is denoted by E_2. Let $E := E_1 \cup E_2$ and $V_G := V' \cup V''$. We use the convention that the subsets of V' and V'' corresponding to a subset $X \subseteq V^*$ will be denoted by X' and X'', respectively.

By (14), there are integers $0 \le l(s) \le m_O(s)$ and $0 \le l(z) \le m_O(z) + m_I(z)$ for which $l = l(s) + l(z)$. For example, $l(z) := \min\{l, m_O(z) + m_I(z)\}$ and $l(s) := l - l(z)$ will do.

Define a function $g' : V_G \to \mathbf{Z}_+$ as follows. Let $g'(s') := m_O(s) - l(s)$ and $g'(z'') := m_O(z) + m_I(z) - l(z)$. For $v' \in V' - s'$, let $g'(v') := m_O(v)$, and for $v'' \in V'' - z''$, let $g'(v'') := m_O(v) + m_I(v)$.

Lemma 3.7. *For the independent set $F' \subseteq A^*$ of $M^\#$ and for the edge $f = sz \in A^* - F'$, the set $F := F' + sz$ is independent in $M^\#$ if and only if G has an orientation in which the in-degree of each node x is at most $g'(x)$.*

Proof. Assume first that the required orientation does not exist. By Lemma 3.6 (when applied for $\gamma = 0$) there is a subset $X' \cup Y'' \subseteq V_G$ of nodes for which $i_G(X' \cup Y'') > g'(X' \cup Y'')$. Let $J \subseteq F$ denote the set of those edges $e = uv$ for which $u' \in X'$ and $v'' \in Y''$. Since $X' \cup Y''$ induces $m_O(X \cap Y)$ edges from E_1, we have $|J| + m_O(X \cap Y) = i_G(X' \cup Y'') > g'(X' \cup Y'') = g'(X') + g'(Y'') \ge [m_O(X) - l(s)] + [m_O(Y) + m_I(Y) - l(z)] = m_O(X) + m_O(Y) + m_I(Y) - l$, from which

$$|J| > m_O(X \cup Y) + m_I(Y) - l. \tag{19}$$

If, indirectly, F were independent in $M^\#$, then we would have $|J| \le b^\#(J) = m_I(H(J)) + m_O(V(J)) - l \le m_I(Y) + m_O(X \cup Y) - l$, contradicting (19).

To see the converse, assume that F is dependent in $M^\#$, that is, there is a bi-set $X = (X_O, X_I)$ for which

$$|J| = i_F(X) > m_O(X_O) + m_I(X_I) - l \tag{20}$$

where J denotes the subset of F induced by X.

As F' is independent in $M^\#$, X must induce $f = sz$, that is, $s \in X_O$ and $z \in X_I$. Hence $g'(X_I'') = m_O(X_I) + m_I(X_I) - l(z)$ and $g'(X_O') = m_O(X_O) - l(s)$ from which $g'(X_O' \cup X_I'') = m_I(X_I) + m_O(X_I) + m_O(X_O) - l$. The set $X_O' \cup X_I'' \subseteq V_G$ induces (in G) $m_O(X_I)$ edges from E_1. If, indirectly, the requested orientation does exist, then $|J| + m_O(X_I) = i_G(X_O' \cup X_I'') \le g'(X_O' \cup X_I'') = m_I(X_I) + m_O(X_I) + m_O(X_O) - l$, that is, $|J| \le m_I(X_I) + m_O(X_O) - l$, contradicting (20). •

We can conclude that with the help of the orientation lemma the necessary independence oracle for $M^\#$ is available (and this does not rely on the matroid partition algorithm).

Remark. In the oracle above, we needed the orientation result only in the special case $\gamma = 0$ which is Hakimi's original theorem. The general form has only been included in order to outline an independence oracle for count matroids in case of $l < 0$ which is even simpler than the one above for non-negative l. Although this is not required for our present purposes, for completeness we include it.

Let $\gamma := -l$. For a subset $F \subseteq A^*$, we consider the same bipartite graph as before. (Now there is no special element of F and no a priori assumption is made on the independence of any subset of F.) Define a function $g' : V_G \to \mathbf{Z}_+$ as follows. For $v' \in V'$, let $g'(v') := m_O(v)$, and for $v'' \in V''$, let $g'(v'') := m_O(v) + m_I(v)$. Similarly to the proof of Lemma 3.7, it can be shown that F is independent in $M^\#$ if and only if $\gamma + i_G(X' \cup Y'') \le g'(X' \cup Y'')$ for every subsets $X' \subseteq V'$, $Y'' \subseteq V''$, and this condition can be checked with the help of the orientation lemma (Lemma 3.6).

4. Rooted k-connections via matroid intersection

Turning to our main goal, we show how a common generalization of the rooted k-edge- and k-node-connection problem can be formulated as a matroid intersection. Let $D = (V, A)$ be a digraph with a root-node r_0. Throughout the section, A^* denotes the set of edges induced by $V^* := V - r_0$, that is, $D^* = (V^*, A^*)$ is the digraph $D - r_0$. Let $A_0 := A - A^*$, that is, A^0 is the set of edges with tail r_0.

4.1. (k, g)-foliages

Let $D = (V, F)$ be a digraph and $g : V^* \to \mathbf{Z}_+$ a function. A set of edge-disjoint r_0t-paths is said to be g-**bounded** if each node $v \in V - \{r_0, t\}$ is used by at most $g(v)$ of these paths. We stress that g-boundedness automatically means that the paths are edge-disjoint. Let $\lambda_g(r_0, t; D)$ denote the maximum number of g-bounded r_0t-paths. Note that for large g (say, $g \equiv |F|$) $\lambda_g(r_0, t; D)$ is the maximum number of edge-disjoint r_0t-paths, while for $g \equiv 1$, $\lambda_g(r_0, t; D)$ is the maximum number of openly disjoint r_0t-paths.

A directed edge $a = uv$ **enters** or **covers** a bi-set $X = (X_O, X_I)$ if a enters both X_O and X_I. For a directed graph $D = (V, A)$, $\varrho(X) := \varrho_D(X) := \varrho_A(X)$ denotes the number of edges entering (covering) X. We will need a bi-set function μ_g defined by

$$\mu_g(X) := \sum[g(v) : v \in X_O - X_I]. \tag{21}$$

Proposition 4.1 (*Variation of Menger's Theorem*). *In a digraph $D = (V, F)$ there are k g-bounded r_0t-paths if and only if*

$$\varrho_F(X) \geq k - \mu_g(X) \tag{22}$$

holds for every bi-set $X = (X_O, X_I)$ with $t \in X_I \subseteq X_O \subseteq V^*$.

Proof. Suppose that there are k g-bounded r_0t-paths. Among these paths at most $\mu_g(X)$ use a node from $X_O - X_I$, hence at least $k - \mu_g(X)$ of them must use an edge entering bi-set X and the necessity of (22) follows.

Conversely, suppose that (22) holds. We may assume that there is no edge entering r_0 and no edge leaving t. Define a new digraph $D' := (V' \cup V'', F' \cup E' \cup E'')$, as follows. V' and V'' are disjoint copies of V. For each edge $uv \in F$, let $u'v''$ be a member of F'. For each node $v \in V - \{r_0, t\}$ put $g(v)$ parallel edges from v'' to v' and k parallel edges from v' to v''. The edges from v'' to v' form E', the edges from v' to v'' form E''.

By this construction, if D' includes k edge-disjoint $r_0't''$-paths, then these paths correspond to k g-bounded r_0t-paths in D. If no such paths exist in D', then, by the directed edge-version of Menger's theorem, there is a subset X' of nodes of D' so that $\varrho_{D'}(X) < k$ and $t'' \in X' \subseteq V' \cup V'' - r_0'$. Let $X_I := \{v \in V : v'' \in X'\}$ and let $X_O := \{v \in V : v' \in X'\}$. Due to the edges in E'', $v'' \in X'$ implies $v' \in X'$ and hence $z \in X_I \subseteq X_O \subseteq V^*$. By the construction, we get $k > \varrho_{D'}(X') = \varrho_D(X) + \mu_g(X)$ contradicting (22). •

Note that for $g \equiv k$, (22) is automatically satisfied for bi-sets X with $t \in X_I \subset X_O \subseteq V^*$ and hence (22) is equivalent to requiring that $\varrho(Y) \geq k$ holds for every *subset* Y with $t \in Y \subseteq V^*$.

The proposition immediately implies the following slight extension. Let $D = (V, F)$ be a digraph with a specified root-node r_0 and terminal set $T \subseteq V^*$. Let $g : V^* \to \{1, 2, \dots, k\}$ be a function. We say that D is (k, g)-**connected from** r_0 **to** T if

$$\lambda_g(r_0, t; D) \geq k \text{ holds for every } t \in T. \tag{23}$$

In the special case when $T = V^*$, we call D **rooted** (k, g)-**connected**.

Proposition 4.2. *A digraph $D = (V, F)$ is (k, g)-connected from r_0 to T if and only if*

$$\varrho_F(X) \geq k - \mu_g(X) \tag{24}$$

holds for every bi-set $X = (X_O, X_I)$ with $X_I \cap T \neq \emptyset$ and $X_O \subseteq V^*$. •

A digraph is called a (k, g)-**foliage** (of root r_0) if it is rooted (k, g)-connected but deleting any edge destroys this property.

Proposition 4.3. *Suppose that $D = (V, F)$ is (k, g)-connected from r_0 to T but removing any edge of D destroys this property. Then the in-degree of every node in T is exactly k. In particular, in a (k, g)-foliage the in-degree of every node distinct from r_0 is k.*

Proof. Suppose indirectly that $\varrho(z) > k$ for some $z \in T$. Choose k g-bounded r_0z-paths $P_1, \dots, P_k$. Then there is an edge $e = uz$ not used by these paths. We claim that there are k g-bounded r_0t-paths in $D' := D - e$ for every $t \in T$ and this will contradict the minimality assumption on D. If these paths do not exist for some $t \in T$, then, by Proposition 4.2 there is a bi-set X violating (24) in D'. Since X does not violate (24) in D, it follows that e must enter X and hence $t \in X_I$. But then the existence of paths $P_1, \dots, P_k$ show that X cannot violate (24) in D' either, a contradiction. •

4.2. Foliages as matroid intersection

Let $c : A \to \mathbf{R}_+$ be a non-negative cost function and $g : V^* \to \{1, 2, \dots, k\}$ a bounding function (but now no terminal set T is considered). Since c is non-negative, in order to find a cheapest rooted (k, g)-connected subgraph of D, it suffices to find a cheapest (k, g)-foliage (of root r_0). We are going to show that there are two matroids M_1 and M_2 on A so that their common independent sets of cardinality $k(|V| - 1)$ are exactly the (k, g)-foliages of D.

Recall the notation V^*, A^*, A_0, and consider the count matroid $M^\#$ on A^* determined by $b^\#(X) := m_O(X_O) + m_I(X_I) - l$ where $l := k$ and, for $v \in V^*$, $m_O(v) := g(v)$, $m_I(v) := k - g(v)$. For brevity, we will refer to $M^\#$ as the **master** matroid. An easy calculation shows that $b^\#(X) := k(|X_I| - 1) + \mu_g(X)$, and hence a subset $F \subseteq A^*$ is independent in $M^\#$ if and only if

$$i_F(X) \leq b^\#(X) \tag{25}$$

for every bi-set $X = (X_O, X_I)$ with $\emptyset \subset X_I \subseteq X_O \subseteq V^*$.

Define a matroid M_1 on A to be the direct sum of the free matroid on A_0 (in which, by definition, every subset is independent) and the master matroid $M^\#$. Let M_2 denote the partition matroid on groundset A in which a subset $I \subseteq A$ is independent if $\varrho_I(v) \leq k$ for every node $v \in V^*$ (and $\varrho_I(r_0) = 0$).

Theorem 4.4. *A subgraph $D_B = (V, B)$ of digraph $D = (V, A)$ is a (k, g)-foliage if and only if B is a common independent set of matroids M_1 and M_2 and $|B| = k(n - 1)$ where $n = |V|$.*

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Proof. If D_B is a (k, g)-foliage, then Proposition 4.3 implies that $\varrho_B(v) = k$ and $\varrho_B(r_0) = 0$. Hence D_B has exactly $k(n-1)$ edges and B is a basis in M_2. For any bi-set $X = (X_O, X_I)$ with $\emptyset \subset X_I \subseteq X_O \subseteq V^*$, one has $\varrho_B(X) + \mu_g(X) \geq k$ and hence $i_B(X) = \sum[\varrho_B(v) : v \in X_I] - \varrho_B(X) \leq k|X_I| + \mu_g(X) - k = b^\#(X)$ and thus B is independent in M_1.

Conversely, suppose that a $k(n-1)$-element subset $B \subseteq A$ of edges is independent in both M_1 and M_2. Then $\varrho_B(v) = k$ for every $v \in V^*$ and $\varrho_B(r_0) = 0$. Furthermore, for a bi-set $X = (X_O, X_I)$ with $\emptyset \subset X_I \subseteq X_O \subseteq V^*$, one has $i_B(X) \leq k(|X_I| - 1) + \mu_g(X)$. Therefore $\varrho_B(X) + \mu_g(X) = \sum[\varrho_B(v) : v \in X_I] - i_B(X) + \mu_g(X) = k|X_I| - i_B(X) + \mu_g(X) \geq k|X_I| - k(|X_I| - 1) = k$ and hence Proposition 4.2 implies that $D_B = (V, B)$ is a (k, g)-foliage. •

This matroid approach enables us to handle a variation of the rooted (k, g)-connection problem in which the goal is to find a cheapest rooted (k, g)-connected subgraph obeying a specified upper bound β imposed on the out-degree of root r_0. To this end it suffices again to restrict ourselves to consider (k, g)-foliages and the only change in Theorem 4.4 is that matroid M_1 should be replaced by the direct sum of the master matroid $M^\#$ and the uniform matroid on A_0 in which the subsets of cardinality at most β are the independent sets.

More generally, one can deploy a matroid M_r on the edge-set A_0 and a matroid M_v of rank k on the set of edges entering v for each node $v \in V^*$. Call a (k, g)-foliage **matroid-restricted** if its subset of edges leaving r_0 is independent in M_r and its subset of edges entering v is independent in M_v for each $v \in V^*$.

Let M_1' be the direct sum of M^* and M_r and let M_2' be the direct sum of the $n-1$ matroids M_v $(v \in V^*)$.

Theorem 4.5. *A subgraph $D_B = (V, B)$ of digraph $D = (V, A)$ is a matroid-restricted (k, g)-foliage if and only if B is a common independent set of matroids M_1' and M_2' and $|B| = k(n-1)$ where $n = |V|$.*

Proof. Suppose first that $D_B = (V, B)$ is a matroid-restricted (k, g)-foliage. Since by Theorem 4.4 the edge-set of any (k, g)-foliage is independent in M^* it follows from the definitions that B is independent in both M_1' and M_2'. The reverse implication follows similarly from Theorem 4.4. •

Edmonds' matroid intersection theorem, combined with the rank-formula (9), provides a necessary and sufficient condition for the existence of a matroid-restricted (k, g)-foliage. We formulate this only for the very special case when the only restriction is imposed on the out-degree of r_0.

Theorem 4.6. *In a digraph $D = (V, A)$, there exists a rooted (k, g)-foliage in which the out-degree of root r_0 is at most β if and only if $\sum_i[k - (\varrho_{D^*}(X_i) + \mu_g(X_i))] \leq \beta$ for every set of non-void bi-sets $X_1, \dots, X_q$ whose outer members are subsets of V^* and inner members are pairwise disjoint, where $D^* = D - r_0$. In particular, in a digraph there is a rooted k-edge-connected subgraph in which the out-degree of the root is at most β if and only if $\sum_i[k - \varrho_{D^*}(X_i)] \leq \beta$ holds for every set of pairwise disjoint subsets X_i of V^*.* •

Note that if upper-bound restrictions are given on the out-degree of nodes $v \in V^*$ rather than on the in-degrees, then the problem becomes NP-complete even in the special case $k = 1$, $g \equiv 1$ since the restricted (k, g)-foliages in this case are exactly the Hamiltonian paths of initial node r_0.

5. Covering supermodular bi-set functions by digraphs

In this section we show how Theorems 2.3 and 2.4 concerning supermodular set-functions can be extended to those on supermodular bi-set functions.

Let $\mathcal{P}_2 = \mathcal{P}_2(V)$ denote the set of all bi-sets of V. The intersection $\cap$ and the union $\cup$ of bi-sets is defined in a straightforward manner: for $X, Y \in \mathcal{P}_2$ let $X \cap Y := (X_O \cap Y_O, X_I \cap Y_I)$, $X \cup Y := (X_O \cup Y_O, X_I \cup Y_I)$. We write $X \subseteq Y$ if $X_O \subseteq Y_O, X_I \subseteq Y_I$. This determines a partial order on $\mathcal{P}_2$. Accordingly, when $X \subseteq Y$ or $Y \subseteq X$, we call X and Y **comparable**. A family of pairwise comparable bi-sets is called a **chain**. Two bi-sets are **intersecting** if $X_I \cap Y_I \neq \emptyset$ and **properly intersecting** if, in addition, they are not comparable. A family of bi-sets is called **laminar** if it has no two properly intersecting members. A family $\mathcal{F}$ of bi-sets is **intersecting** if both the union and the intersection of any two intersecting members of $\mathcal{F}$ belong to $\mathcal{F}$. A laminar family is obviously intersecting.

For a bi-set function p, a digraph $D = (V, A)$ is said to cover p if $\varrho_D(X) \geq p(X)$ for every $X \in \mathcal{P}_2(V)$. For a vector $z : A \to \mathbf{R}$, let $\varrho_z(X) := \sum[z(a) : a \in A, a \text{ covers } X]$. A vector $z : A \to \mathbf{R}$ **covers** p if $\varrho_z(X) \geq p(X)$ for every $X \in \mathcal{P}_2(V)$.

A non-negative bi-set function p is said to satisfy the **supermodular inequality** on $X, Y \in \mathcal{P}_2$ if

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y). \tag{26}$$

If the reverse inequality holds, we speak of the **submodular** inequality. p is said to be **fully supermodular** or supermodular if it satisfies the supermodular inequality for every pair of bi-sets X, Y. If (26) holds for intersecting pairs, we speak of **intersecting supermodular** functions. Analogous notions can be introduced for submodular functions. Sometimes (26) is required for those intersecting pairs for which $p(X) > 0$ and $p(Y) > 0$. In this case p is called **positively intersecting supermodular**.

Proposition 5.1. *The in-degree function ϱ_D on $\mathcal{P}_2$ is submodular.* •

5.1. Total dual integrality

Proposition 5.2. *Let $\mathcal{F}$ be a laminar family of bi-sets and $D = (V, A)$ a digraph. Let M be a 0–1 matrix the rows and columns of which correspond to the members of $\mathcal{F}$ and to the edges of D, respectively. An entry of M corresponding to $X \in \mathcal{F}$ and $e \in A$ is 1 if e enters X and zero otherwise. Then M is totally unimodular.*

Proof. Since a subfamily of a laminar family is also laminar, by a characterization of Ghouila-Houri [17], it suffices to prove that there is a uniform 2-colouration of the rows of M, that is, a function $h : \mathcal{F} \to \{-1, +1\}$ so that $|\sum[h(X) : e \text{ enters } X]| \leq 1$ for each edge e of D. (In words, each edge e enters near the same number of 1-coloured and of (-1)-coloured members of $\mathcal{F}$ where near the same means that the two numbers may differ by at most one.)

We may assume that the members of $\mathcal{F}$ are distinct. Indeed, if a bi-set X occurs in at least two copies then, in order to get a uniform 2-colouration of $\mathcal{F}$, remove first two copies of X, get then inductively a uniform 2-colouration of the rest and finally colour the two removed copies of X differently.

Turning to the case when the members of $\mathcal{F}$ are distinct, define first $h(X)$ to be 1 for each maximal member X of $\mathcal{F}$. In a general step take a maximal uncoloured member X of $\mathcal{F}$. By the laminarity, there is a unique smallest coloured member Y for which $X \subset Y$. Define $h(X) := -h(Y)$. By the laminarity, the members of $\mathcal{F}$ entered by an edge e form a chain in which two consecutive members $X \subset Y$ have the property that Y is the unique smallest member of $\mathcal{F}$ that is larger than X. Hence $h(X) = -h(Y)$ and therefore the 2-colouration h is indeed uniform. •

Remark. The matrix M in the theorem can be rather easily shown to be a network matrix. To see this, consider the laminar family of sets consisting of the inner sets of bi-sets in $\mathcal{F}$. It is well known [7] that a laminar family $\mathcal{L}$ of subsets of V can be represented by an arborescence $T = (U, F)$ in the sense that there is a mapping $\varphi : V \to U$ and there is a one-to-one correspondence between $\mathcal{L}$ and F so that each member X of F is $\varphi^{-1}(U(f_X))$ where f_X denotes the edge of T corresponding to X and $U(f_X)$ denotes the subset of nodes of T reachable in T from the head of f_X. Using this representation, one can show that the edges of the arborescence corresponding to the inner sets of those members of $\mathcal{F}$ that are entered by e form a directed path in T and hence M is indeed a network matrix.

The following result is a direct extension of Theorem 2.3 to bi-set functions. Its proof is a rather standard application of the well-known uncrossing technique.

Theorem 5.3. *Let $D = (V, A)$ be a digraph. Let $p : \mathcal{P}_2 \to \mathbf{Z}$ be a positively intersecting supermodular bi-set function and $g_A : A \to \mathbf{Z}_+ \cup \{\infty\}$ a non-negative upper bound on the edges of D that covers p. The linear system*

$$\{0 \leq x \leq g_A \text{ and } \varrho_x(Z) \geq p(Z) \text{ for every bi-set } Z \in \mathcal{P}_2\} \tag{27}$$

described for $x \in \mathbf{R}^A$ is totally dual integral. In particular, the linear programming problem

$$\min\{cx : x \text{ satisfies } (27)\} \tag{28}$$

has an integer-valued optimum solution and so has its linear programming dual provided c is integer-valued.

Proof. Let $c : A \to \mathbf{Z}$ be integer-valued so that the primal optimum is bounded (which, in the present case, is equivalent to requiring that $g_A(e)$ is finite whenever $c(e) < 0$). Let Q denote a 0–1 matrix in which the rows and the columns correspond to the non-trivial members of $\mathcal{P}_2$ and to the edges of D, respectively. An entry of Q corresponding to a bi-set X and edge e is 1 if e covers X and zero otherwise. In what follows, we also denote by p the $|\mathcal{P}_2|$-dimensional vector whose component corresponding to the member $X \in \mathcal{P}_2$ has value $p(X)$.

Then the primal linear programming problem is $\min\{cx : 0 \leq x \leq g_A, Qx \geq p\}$, while its dual is:

$$\max\{yp - zg_A : yQ - z \leq c, y \geq 0, z \geq 0\}, \tag{29}$$

where $z(e)$ denotes the dual variable corresponding to the primal inequality $x(e) \leq g_A(e)$ ($g_A(e)$ is finite).

For a given y, the optimal z is uniquely determined: $z(e) = (yq_e - c(e))^+$, where q_e denotes the column of Q corresponding to edge e. Therefore we can say that a certain y is an optimal solution to (29).

What we have to prove is that the optimum to (29) is attained at an integer vector. Let y_0 be an optimal rational solution. As long as there exist two properly intersecting bi-sets $X = (X_K, X_B)$ and $Y = (Y_K, Y_B)$ with positive $y_0(X)$ and $y_0(Y)$, revise y_0 as follows. Define $\alpha := \min\{y_0(X), y_0(Y)\}$, decrease by α both $y_0(X)$ and $y_0(Y)$, and increase by α both $y_0(X \cap Y)$ and $y_0(X \cup Y)$.

Due to the submodularity of bi-set function ϱ on $\mathcal{P}_2$, the resulting dual vector continues to be feasible. Moreover it is also dual optimal since p is assumed to be positively intersecting supermodular. Let us call such a change in the dual solution an uncrossing step.

Define a linear ordering of the partially ordered set $(\mathcal{P}_2, \subseteq)$ obtained in such a way that if the first $j - 1$ elements of the ordering have already been determined then the subsequent jth element is selected to be minimal among the members of $\mathcal{P}_2$ not yet selected. In this ordering, for arbitrary $X, Y \in \mathcal{P}$, $X \cap Y$ precedes both X and Y while $X \cup Y$ follows both of them. Therefore the following lemma implies that the number of uncrossing steps cannot be infinite.

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Lemma 5.4. *Let $r_1, \ldots, r_n$ be a sequence of non-negative rational numbers. As long as possible, apply the following 4-change step. Select four distinct members for which the two middle ones are positive. Let α denote the minimum of the two middle elements. Decrease by α the value of the two middle elements and increase by α the value of the first and fourth ones. Then after a finite number of 4-change steps the procedure terminates.*

Proof. By multiplying through with the least common denominator, if necessary, we may assume that the sequence consists of integers. Since the first member never decreases, each member remains non-negative and the total sum stays constant, after a finite number of 4-change steps the first member gets fixed and the lemma follows by induction on n. •

We may therefore assume that the set $\mathcal{H}$ of bi-sets for which the y_0-value is positive is laminar. By Proposition 5.2 the submatrix of Q determined by the rows corresponding to the members of $\mathcal{H}$ is totally unimodular. Therefore the optimal dual solution y_0 may be chosen integer-valued, as required. ••

5.2. Relation to submodular flows

In order to have an algorithm for the optimization problem given in Theorem 5.3, we are going to prove that the linear system (27) actually describes a submodular flow polyhedron. Since there are efficient combinatorial algorithms for submodular flows (for rich overviews, see [16,23]) this way we will have one for finding optimal coverings of intersecting supermodular bi-set functions. We remark that, for the special case when p is identically 1 on a given intersecting family of bi-sets and zero otherwise, Theorem 5.3 was algorithmically proved in [14] with the help of a two-phase greedy algorithm.

Let $\hat{D} = (\hat{V}, \hat{A})$ be a digraph, $\mathcal{F}$ an intersecting family of subsets of $\hat{V}$, $b : \mathcal{F} \to \mathbf{Z}$ an intersecting submodular function. Let $\hat{f} : \hat{A} \to \mathbf{Z} \cup \{-\infty\}$ and $\hat{g} : \hat{A} \to \mathbf{Z} \cup \{\infty\}$ be two functions with $\hat{f} \leq \hat{g}$. A function $\hat{x} : \hat{A} \to \mathbf{R}$ is called a **submodular flow** or for short a **subflow** if

$$\varrho_{\hat{x}}(Z) - \delta_{\hat{x}}(Z) \leq b(Z) \quad \text{for every } Z \in \mathcal{F} \tag{30}$$

and

$$\hat{f} \leq \hat{x} \leq \hat{g}.$$

The set of subflows is called a submodular flow (subflow) polyhedron. This notion was originally introduced by Edmonds and Giles [7] for the more general case of crossing submodular functions: here we need only intersecting submodular functions. It is known and easy to show anyway that, for an intersecting supermodular function p on $\mathcal{F}$, the polyhedron defined by the linear system

$$\{\hat{f} \leq \hat{x} \leq \hat{g} \text{ and } \varrho_{\hat{x}}(Z) - \delta_{\hat{x}}(Z) \geq p(Z) \text{ for every } Z \in \mathcal{F}\} \tag{31}$$

is also a subflow polyhedron. In this sense one could speak of supermodular flows as well but we stay at the conventional term of submodular flow even if the polyhedron is defined by a supermodular function. The subflow polyhedron is called **one-way** if the in-degree or the out-degree of every member of $\mathcal{F}$ is zero.

Theorem 5.5. *Let $D = (V, A)$ be a digraph. Let $p : \mathcal{P}_2 \to \mathbf{Z}$ be an intersecting supermodular bi-set function and $g_A : A \to \mathbf{Z}_+ \cup \{\infty\}$ a non-negative upper bound covering p. The polyhedron P defined by*

$$\{x \in \mathbf{R}^A : 0 \leq x \leq g_A \text{ and } \varrho_x(Z) \geq p(Z) \text{ for every bi-set } Z \in \mathcal{P}_2\} \tag{32}$$

is a one-way submodular flow polyhedron.

Proof. It follows from the definition that the intersection of a submodular flow polyhedron with a box is also a submodular flow polyhedron so it suffices to prove the theorem for the special case when $g_A = \infty$.

Construct a digraph $\hat{D} = (U, \hat{A})$ from D as follows. For each edge $e = uv$ of D, subdivide e by a new node u_e and delete uu_e, one of the two newly arising edges. The remaining edge $u_e v$ will be denoted by $\hat{e}$. Here $U = V \cup \hat{V}_A$ where $\hat{V}_A$ denotes the set of subdividing nodes. For any subset $F \subseteq A$, the corresponding subset of edges and subset of nodes of $\hat{D}$ will be denoted by $\hat{F}$ and $\hat{V}_F$, respectively.

Define a family $\mathcal{F}$ of subsets of U and a function $\hat{p}$ on $\mathcal{F}$ as follows. For each non-void bi-set $X \in \mathcal{P}_2(V)$ with finite $p(X)$ and for each subset $F \subseteq I_D(X)$, let $X_I \cup \hat{V}_F$ be a member of $\mathcal{F}$ and let $\hat{p}(X_I \cup \hat{V}_F) := p(X)$.

Claim 5.6. *$\mathcal{F}$ is an intersecting family of sets and $\hat{p}$ is intersecting supermodular.*

Proof. Suppose for bi-sets X, X' and edge-sets $F \subseteq I_D(X)$, $F' \subseteq I_D(X')$ that $Y := X_I \cup \hat{V}_F$ and $Y' := X'_I \cup \hat{V}_{F'}$ are intersecting sets. Then X and X' are also intersecting. It easily follows from the definition that $I_D(X) \cap I_D(X') \subseteq I_D(X \cap X')$ and $I_D(X) \cup I_D(X') \subseteq I_D(X \cup X')$ and hence both Y and Y' are in $\mathcal{F}$. Furthermore, we have $\hat{p}(Y) + \hat{p}(Y') = p(X) + p(X') \leq p(X \cap X') + p(X \cup X') = \hat{p}((X_I \cap X'_I) \cup \hat{V}_{(F \cap F')}) + \hat{p}((X_I \cup X'_I) \cup \hat{V}_{(F \cup F')}) = \hat{p}(Y \cap Y') + \hat{p}(Y \cup Y')$, as required. •

By the construction, no edge of $\hat{D}$ leaves any member of $\mathcal{F}$ and hence $\hat{P} := \{\hat{x} \in \mathbf{R}^{\hat{A}} : \hat{x} \geq 0, \varrho_{\hat{x}}(Z) \geq \hat{p}(Z) \text{ for every } Z \in \mathcal{F}\}$ is a one-way subflow polyhedron. Since the edges of D and $\hat{D}$ correspond to each other, we may speak of the polyhedron P' in $\mathbf{R}^A$ corresponding to $\hat{P}$.

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Claim 5.7. $P = P'$.

Proof. Let $x \in P'$, that is, $\hat{x} \in \hat{P}$. For a non-void bi-set X with finite $p(X)$ and for $F := I_D(X)$ we have $\varrho_x(X) = \varrho_{\hat{x}}(X_I \cup V_F) \geq \hat{p}(X_I \cup V_F) = p(X)$ and hence $x \in P$, from which $P' \subseteq P$.

Conversely, let $x \in P$. For a non-void bi-set X with finite $p(X)$ and for $F \subseteq I_D(X)$ we have $\varrho_{\hat{x}}(X_I \cup V_F) \geq \varrho_x(X) \geq p(X) = \hat{p}(X_I \cup V_F)$ and hence $\hat{x} \in \hat{P}$, from which $P \subseteq P'$. •

By the two claims, the proof of the theorem is complete. ••

Theorem 5.3 has a certain self-refining nature. Given a subset $T \subseteq V$, we say that a bi-set function p is (**positively**) T-**intersecting supermodular** if the supermodular inequality holds for bi-sets X and Y whenever $X_I \cap Y_I \cap T \neq \emptyset$ (and $p(X) > 0$, $p(Y) > 0$).

Proposition 5.8. *For bi-set function p_1, define a bi-set function p on bi-sets $Z = (Z_O, Z_I)$ by*

$$p(Z) := \begin{cases} \max\{p_1(Z_O, Z_I \cup K) : K \subseteq Z_O - T\} & \text{if } Z_I \subseteq T \\ 0 & \text{otherwise.} \end{cases} \tag{33}$$

If p_1 is (positively) T-intersecting supermodular, then so is p.

Proof. Let X and Y be two intersecting bi-sets (for which $p(X) > 0$, $p(Y) > 0$ in case p_1 is positively T-intersecting supermodular). There are subsets $K \subseteq X_O - T$, $L \subseteq Y_O - T$ for which $p(X) = p_1(X')$ and $p(Y) = p_1(Y')$ where $X' = (X_O, X_I \cup K)$ and $Y' = (Y_O, Y_I \cup L)$. Since $(X_I \cup K) \cap (Y_I \cup L) \neq \emptyset$, $K \cap L \subseteq (X_O \cap Y_O) - T$ and $K \cup L \subseteq (X_O \cup Y_O) - T$, therefore $p_1(X' \cap Y') \leq p(X \cap Y)$ and $p_1(X' \cup Y') \leq p(X \cup Y)$. Hence $p(X) + p(Y) = p_1(X') + p_1(Y') \leq p_1(X' \cap Y') + p_1(X' \cup Y') \leq p(X \cap Y) + p(X \cup Y)$, as required. •

Theorem 5.9. *Let $D = (V, A)$ be a digraph and $g_A : A \to \mathbf{Z}_+ \cup \{\infty\}$ an integer-valued function. Let $T \subseteq V$ be a subset of nodes containing the head of every edge of D. Let p_1 be a positively T-intersecting supermodular bi-set function covered by g_A. Then the linear system*

$$\{0 \leq x \leq g_A \text{ and } \varrho_x(X) \geq p_1(X) \text{ for every bi-set } X\} \tag{34}$$

described for $x \in \mathbf{R}^A$ is totally dual integral. The polyhedron defined by (34) *is a submodular flow polyhedron.*

Proof. By Proposition 5.8 the bi-set function p defined in (33) is positively intersecting supermodular. Since every edge has its head in T, a vector $x : A \to \mathbf{R}$ covers p_1 if and only if x covers p. Furthermore, a dual solution y to (32) determines a dual solution y_1 to (34) as follows. For $X = (X_O, X_I)$ with $X_I \subseteq T$ let Y be the bi-set for which $Y_O = X_O$, $X_I \subseteq Y_I$ and $p(X) = p_1(Y)$. Define $y_1(Y) := y(X)$ if Y arises this way and $y_1(Y) := 0$ otherwise. Then y_1 is a dual feasible solution to (34) having the same value as y does. Therefore Theorem 5.3 implies that the system (34) is also TDI. •

5.3. Polyhedral descriptions of rooted (k, g)-connected subgraphs

Let $k \geq 1$ be an integer and $g : V \to \{1, \ldots, k\}$ a function. As an application, we exhibit how the problem of cheapest subgraphs which are (k, g)-connected from r_0 to a terminal set T can be handled polyhedrally and algorithmically provided each edge of positive cost has its head in T.

Theorem 5.10. *Let $H = (V, F_0 \cup A)$ be a digraph with a specified root-node r_0 and terminal set $T \subseteq V - r_0$ so that the head of each edge in A is in T. Suppose that H is (k, g)-connected from r_0 to T. The convex hull of incidence vectors of the edge-sets $F \subseteq A$ for which the subgraph $(V, F_0 \cup F)$ is (k, g)-connected from r_0 to T is equal to the polyhedron*

$$\{x \in \mathbf{R}^A : 0 \leq x \leq 1 \text{ and } \varrho_x(Z) \geq p_1(Z) \text{ for every bi-set } Z\} \tag{35}$$

where p_1 is defined for each bi-set $Z = (Z_O, Z_I)$ by

$$p_1(Z) := \begin{cases} k - \varrho_{F_0}(Z) - \mu_g(Z) & \text{if } Z_I \cap T \neq \emptyset \text{ and } Z_O \subseteq V - r_0 \\ -\infty & \text{otherwise.} \end{cases} \tag{36}$$

Furthermore, the linear system in (35) *is TDI and determines a submodular flow polyhedron.*

Proof. Observe that the function p_1 defined in the theorem is intersecting supermodular and hence Theorem 5.9 can be applied to $D = (V, A)$, p_1, and $g_A \equiv 1$. •

Let us formulate Theorem 5.10 in the special case when $g \equiv k$.

Corollary 5.11. *Let $D = (V, F_0 \cup A)$ be a digraph with a specified root-node r_0 and terminal set $T \subseteq V - r_0$ so that the head of each edge in A is in T. Suppose that D is k-edge-connected from r_0 to T. The convex hull of incidence vectors of the edge-sets $F \subseteq A$ for which the subgraph $(V, F_0 \cup F)$ is k-edge-connected from r_0 to T is equal to the polyhedron*

$$\{x \in \mathbf{R}^A : 0 \leq x \leq 1 \text{ and } \varrho_x(X) \geq k - \varrho_{F_0}(X) \quad \text{for each } X \subseteq V - r_0 \text{ with } X \cap T \neq \emptyset\}. \tag{37}$$

Furthermore, the linear system in (37) *is TDI and determines a submodular flow polyhedron.*

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Let us formulate Theorem 5.10 in the special case when $T = V - r_0$ and $F_0 = \emptyset$.

Corollary 5.12. *Let $D = (V, A)$ be a rooted (k, g)-connected digraph with respect to a root-node r_0. The convex hull of incidence vectors of the edge-sets $F \subseteq A$ for which the subgraph (V, F) is rooted (k, g)-connected is equal to the polyhedron*

$$\{x \in \mathbf{R}^A : 0 \le x \le 1 \text{ and } \varrho_x(Z) \ge k - \mu_g(Z) \quad \text{for each bi-set } Z \text{ with } \emptyset \subset Z_I \subseteq Z_O \subseteq V - r_0\}. \tag{38}$$

Furthermore, the linear system in (38) is TDI and describes a submodular flow polyhedron.

6. Conclusion

In this paper we considered the rooted (k, g)-connection problem which is a common generalization of those of finding a cheapest rooted k-edge-connected and k-node-connected subgraph of a digraph. By extending a known result on rooted k-edge-connectivity, we proved that the general version is also a matroid intersection problem and hence a weighted matroid intersection algorithm may be applied. We also showed that the independence oracle required for the matroids in question can be constructed through an easy graph orientation result. This matroid approach supersedes the only solution to the rooted k-node-connection problem known earlier, which invoked the more complex model of submodular flows.

Moreover, we exhibited TDI descriptions for further generalizations of the rooted (k, g)-connection problem for which the algorithmic solution did invoke submodular flows. For example, the problem of finding a cheapest subgraph of a digraph in which there are kg-bounded paths from a root-node to each element of a terminal set T could be handled this way provided that each edge of positive cost has its head in T. Without this latter restriction, even the special case $k = 1$ involves the NP-complete problem of directed Steiner trees.

The key idea behind our approach was that earlier results on supermodular set-functions could be extended to those on supermodular bi-set functions.

Finally, we remark that the same technique can be used to solve the following extended form of the optimization problem over (k, g)-foliages. Suppose that, in addition to the function g on the node set V, we are also given a function g_A on the edge-set A of D. Call a flow $z : A \to \mathbf{Z}$ from r_0 to a node t node-feasible if $\varrho_z(v) \le g(v)$ for each node $v \in V - \{r_0, t\}$. The generalized problem consists of finding a cheapest vector $x : A \to \mathbf{Z}$ so that $x \le g_A$ and there is a node-feasible flow $x' \le x$ of amount k from r_0 to t for every node t of V. Naturally, in this case one must rely on the intersection of two polymatroids rather than just matroids.

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