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Tree-compositions and orientations

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ABSTRACT

A tree-composition is a tree-like family that serves to describe the obstacles to k-edge-connected orientability of mixed graphs. Here we derive a structural result on tree-compositions that gives rise to a simple algorithm for computing an obstacle when the orientation does not exist.

As another application, we show a min–max theorem on the minimal in-degree of a given node set in a k-edge-connected orientation of an undirected graph. This min–max formula can be simplified in the special case of $k = 1$.

1. Introduction

Several results and algorithms in submodular optimization are based on the following fact [6,13]. For every crossing submodular function b for which the base polyhedron $B(b)$ is non-empty, there exists a unique fully submodular function $b^{\downarrow}$ for which $B(b^{\downarrow}) = B(b)$. The function $b^{\downarrow}$ is called the full (lower) truncation of b. (For other definitions, see Preliminaries and the beginning of each section.) The following result of [7] provides a relatively simple formula for $b^{\downarrow}$. Here and in some other theorems, we assume $b(V) = 0$. The general case can be derived from the case $b(V) = 0$, though the formulas get a bit more involved.

Theorem 1.1. *Let b be a crossing submodular set-function on the subsets of a ground set V for which $b(V) = 0$ and $B(b) \neq \emptyset$. Then, for $\emptyset \neq Z \subseteq V$,*

$$b^{\downarrow}(Z) = \min\left\{\sum_{X\in\mathcal{T}} b(X) : \mathcal{T} \text{ a tree-composition of } Z\right\}. \qquad (1)$$

This result was originally proved by using the uncrossing procedure. In Section 4 we exhibit a different approach in which the minimizing tree-composition is given directly without using uncrossing. This simplifies the way how a minimizing tree-composition can be found algorithmically (see Section 5). The proof is based on a structural result on tree-compositions that can be used to find a sub-tree-composition of a special family (see Section 3).

In Section 6, after recalling the original applications of [7], we give a new proof of a graph orientation theorem of [5]. In Section 7 a new application is described in which we show that the maximum size of a tree-composition of T complying with a 2-edge-connected bipartite graph $G = (S, T; E)$ is equal to the minimum number of edges entering T in a strongly connected orientation of G.

2. Preliminaries

All the graphs considered in this paper are loopless but may contain parallel edges. Let $G = (V, E)$ be an undirected graph. For $v \in V$, the number of edges incident to v is denoted by $d(v)$ or $d_G(v)$. (We use this notation also for hypergraphs and also for families that could be interpreted as hypergraphs.) For $X \subseteq V$ we denote by $G[X]$ the subgraph induced by X and by $i(X)$ the number of edges in $G[X]$. For $X, Y \subseteq V$, we denote by $d(X, Y)$ or $d_G(X, Y)$ the number of edges connecting $X - Y$ and $Y - X$. For a partition $\mathcal{P}$ of the node set V, $e(\mathcal{P})$ denotes the number of edges connecting distinct members of $\mathcal{P}$.

For a directed graph $D = (V, A)$ and $v \in V$, $\varrho(v)$ or $\varrho_D(v)$ denotes the in-degree of v and $\delta(v)$ or $\delta_D(v)$ the out-degree of v. We denote by $d(X)$, $\varrho(X)$, $\delta(X)$ or $d_G(X)$, $\varrho_D(X)$, $\delta_D(X)$ the degree, in-degree and out-degree of a subset $X \subseteq V$, respectively. For a given function $x : A \to \mathbb{R}$ on the edge-set, $\varrho_x(X) := \sum\{x(e) : e \text{ enters } X\}$ and $\delta_x(X) := \sum\{x(e) : e \text{ leaves } X\}$ for $X \subseteq V$.

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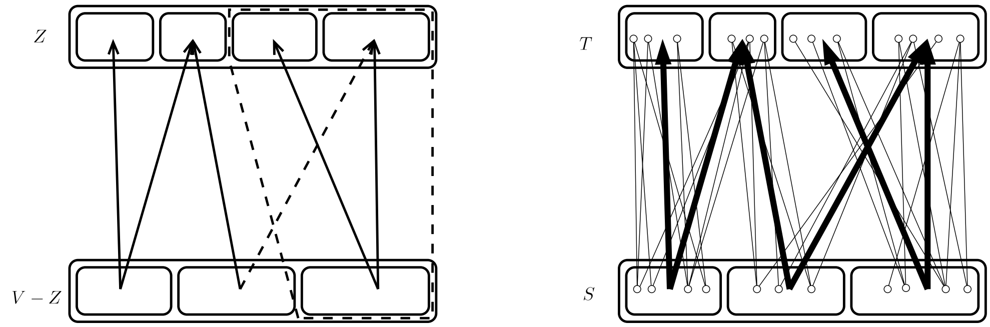

(a) The tree-representation of a tree-composition $\mathcal{T}$ of Z where the node set of the tree is formed by a partition of Z and a partition of $V-Z$ and the dashed edge of the tree represents the dashed member of $\mathcal{T}$. (For simpleness the other members of $\mathcal{T}$ are not shown in this figure.)

(b) The thick directed edges give the tree-representation of a tree-composition of T that complies with the bipartite graph formed by the thin edges.

Fig. 1. Tree-compositions.

Two subsets A and $B \subseteq V$ are *crossing* if $A - B \neq \emptyset$, $B - A \neq \emptyset$, $A \cap B \neq \emptyset$, $V - (A \cup B) \neq \emptyset$. A family $\mathcal{F}$ of subsets of V is a *crossing family* if $A \cap B \in \mathcal{F}$ and $A \cup B \in \mathcal{F}$ for any two crossing members $A, B \in \mathcal{F}$. A family $\mathcal{F}$ is called *cross-free* if there are no crossing pairs in $\mathcal{F}$.

Let $T = (U, A)$ be a directed tree. For an edge $e \in A$, let U_e denote the component of $T - e$ that contains the head of e. In [3] Edmonds and Giles proved the following representation of cross-free families:

Lemma 2.1. *For every cross-free family $\mathcal{F}$ on a ground set V, there exists a directed tree $T = (U, A)$ and a map $\varphi : V \to U$ so that the sets in $\mathcal{F}$ and the edges of T are in a one-to-one correspondence, as follows. For every edge $e \in A$, the corresponding set $\mathcal{F}$ is $\varphi^{-1}(U_e)$.* □

We denote the edge of the tree representing a set F by e_F.

Remark 2.2. 1. It is easy to see that for $F \in \mathcal{F}$ a representing tree of $\mathcal{F} - \{F\}$ is T/e_F with the map φ' that arises from φ by combining it with the natural identifying map $\pi : U \to U/e_F$.

2. It can be proved by induction that a directed tree $F = (U, A)$ admits a *level function* $\pi : U \to \mathbb{Z}_+$ so that $\pi(v) - \pi(u) = 1$ for every $uv \in A$. If (T, φ) is a tree-representation of a cross-free family $\mathcal{F}$ on the ground set V and π is a level function of T, then one can get $\pi(\varphi(v)) - \pi(\varphi(v')) = d_{\mathcal{F}}(v) - d_{\mathcal{F}}(v')$ for every $v, v' \in V$ after proving the statement when $\varphi(v)\varphi(v') \in A$.

A family $\mathcal{K}$ on the ground set V is a *composition of* $\emptyset \neq X \subseteq V$ if there is an integer $\Delta \in \mathbb{Z}_+$ for which every element of X is contained in exactly $\Delta + 1$ members of $\mathcal{K}$ and every element of $V - X$ is contained in exactly Δ members of $\mathcal{K}$. Note that a composition of V is a regular hypergraph while a composition of a proper subset $\emptyset \neq Z \subset V$ becomes a regular hypergraph by adding $V - Z$ to it. The integer Δ is called the *ground-degree of* $\mathcal{K}$ and is denoted by $\Delta(\mathcal{K})$. Thus the ground-degree of an r-regular hypergraph is $r - 1$. Observe that the difference between the ground-degree and the maximum degree of a composition is 1.

Special compositions are the following. A *partition* of $Z \subseteq V$ is a family formed by disjoint sets $Z_1, Z_2, \ldots, Z_t$ with $\bigcup_{i=1}^t Z_i = Z$. We call a partition of any subset of V a *subpartition* of V. If $\{V_1, V_2, \ldots, V_t\}$ is a partition of V, then $\{V - V_1, V - V_2, \ldots, V - V_t\}$ is called a *co-partition* of V. Lemma 2.1 and Remark 2.2 imply the following observation [8]:

Lemma 2.3. *Let $\mathcal{K}$ be a cross-free composition of its ground set V. Then $\mathcal{K}$ can be partitioned into partitions and co-partitions of V.*

Proof. If $\mathcal{K} = \emptyset$, then we are done. Assume that $\mathcal{K} \neq \emptyset$. By induction, it suffices to show that $\mathcal{K}$ includes a partition or a co-partition.

Let $T = (U, A)$ be a tree representing $\mathcal{K}$ along with the map $\varphi : V \to U$. By Remark 2.2(2), all the nodes in $\varphi(V) \subseteq U$ have the same level in the tree. Since the tree has no edges between two nodes on the same level and it has at least one edge, there are at least two levels. Therefore, one of the minimum and maximum level consists of nodes v for which $\varphi^{-1}(v) = \emptyset$. If the level of v is the minimum (maximum, respectively) and $\varphi^{-1}(v) = \emptyset$, then the edges exiting (entering, respectively) v represent a partition (co-partition, respectively) of V. □

Let $\{Z_1, Z_2, \ldots, Z_t\}$ be a partition of $Z \subseteq V$, and let $\{Z_i^1, Z_i^2, \ldots, Z_i^{t_i}\}$ be a partition of $V - Z_i$ ($i = 1, \ldots, t$). Then the set-system $\mathcal{D} := \{V - Z_i^j : 1 \le i \le t, 1 \le j \le t_i\}$ is called a *double-partition of* Z. This $\mathcal{D}$ is a composition of Z with ground-degree $\sum_{i=1}^t (t_i - 1)$. If each $t_i = 1$, then $\mathcal{D}$ is a partition of Z.

For $\emptyset \neq Z \subset V$, a cross-free double-partition $\mathcal{T}$ of Z on the ground set V is called a *tree-composition of* Z if $u \in \varphi(V - Z)$, $v \in \varphi(Z)$ for each edge uv of a representing directed tree of $\mathcal{T}$. By Remark 2.2(2) and the definition of double-partitions, $\pi(u) = \pi(u') = \pi(v) + 1$ holds for every $u, u' \in \varphi(Z)$ and $v \in \varphi(V - Z)$ where π denotes the level function of a representing tree (T, φ). Thus $\varphi(Z)$ and $\varphi(V - Z)$ are disjoint. Moreover, $\varphi^{-1}(u) \neq \emptyset$ for any node u of the tree since a tree is connected and each edge of the representing tree connects $\varphi(Z)$ and $\varphi(V - Z)$. Therefore, a tree-representation can be constructed by taking the members a partition of Z and a partition of $V - Z$ as its node set, mapping each node with φ to the set containing it and taking the edges with tails in $\varphi(V - Z)$ and heads in $\varphi(Z)$ (see Fig. 1(a)). From now on we will use this tree-representation of a tree-composition. Note that a cross-free family $\mathcal{T}$ with such a tree-representation is always a tree-composition. To prove this, one needs to prove that $\mathcal{T}$ is a double-partition. This follows from the fact that the edges of the tree entering a node $u \in \varphi(z)$ represents the complement of a co-partition of $V - \varphi^{-1}(u)$. We will say that the partitions and co-partitions of the ground set V are the *tree-compositions of* V. While a double-partition may consist of $\Omega(|V|^2)$ elements, a tree-composition always has at most $|V| - 1$ elements.

Assume that we are also given a bipartite graph $G = (S, T; E)$ and $V = S \cup T$. We say that a *tree-composition $\mathcal{T}$ of T complies with* G if $\varphi(s)\varphi(t) \in A$ for every edge $st \in E$ with $s \in S$, $t \in T$, where $F = (U, A)$ is a directed tree representing $\mathcal{T}$ with the surjective map $\varphi : (S \cup T) \to U$ (see Fig. 1(b)).

Unless otherwise stated, we assume that a set-function is zero on the empty set. For a vector $x \in \mathbb{R}^V$ and $X \subseteq V$, let $\widetilde{x}(X) :=$

$\sum_{v\in X} x(v)$. We also use the notation $\widetilde{h}(\mathcal{F}) := \sum_{i=1}^{q} h(F_i)$ for a set-function $h : 2^V \to \mathbb{R}$ and for a family $\mathcal{F} := \{F_1, F_2, \dots, F_q\}$ of subsets $F_i \subseteq V$. A set-function $b : 2^V \to \mathbb{R} \cup \{\infty\}$ is *(crossing) submodular* if $b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y)$ for every (crossing) $X, Y \subseteq V$. A set-function $p : 2^V \to \mathbb{R} \cup \{-\infty\}$ is *(crossing) supermodular* if $p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$ for every (crossing) $X, Y \subseteq V$. In some cases when it is important to highlight that the function is submodular (respectively supermodular) on every pair of sets, we call it *fully submodular* (respectively *fully supermodular*).

Let b be a set-function on V for which $b(V) < \infty$. The *base-polyhedron* $B(b)$ defined by b is as follows.

$$B(b) := \{x \in \mathbb{R}^V : \widetilde{x}(X) \leq b(X)\ (X \subset V),\ \widetilde{x}(V) = b(V)\}.$$

This polyhedron is used mainly when b is a fully (intersecting, crossing) *sub*modular function. For a supermodular function p, a related polyhedron $B'(p)$ is considered:

$$B'(p) := \{x \in \mathbb{R}^V : \widetilde{x}(X) \geq p(X)\ (X \subset V),\ \widetilde{x}(V) = p(V)\}.$$

For an arbitrary set-function $h : 2^V \to \mathbb{R} \cup \{+\infty, -\infty\}$ for which $h(V)$ is finite, we define the *complement* $\overline{h}$ of h by the following formula.

$$\overline{h}(X) := h(V) - h(V - X).$$

Obviously, $\overline{h}(\emptyset) = 0$, $\overline{h}(V) = h(V)$ and $\overline{\overline{h}} = h$. The complement p of a (crossing) submodular function b is (crossing) supermodular and $B'(p) = B(b)$. Therefore, the results on (crossing) submodular functions can automatically be transformed into ones on (crossing) supermodular functions by using the complement.

It is known that $B(b)$ is a non-empty integer polyhedron whenever b is fully submodular and integer-valued. A basic theorem of Fujishige [13] characterizes non-emptiness of $B(b)$ for crossing submodular b.

Theorem 2.4 (*Fujishige*)**.** *Let b be a crossing submodular function on the subsets of V for which $b(V)$ is finite. The polyhedron $B(b)$ is non-empty if and only if both*

$$\sum_i b(V_i) \geq b(V) \quad \text{and} \quad \sum_i \overline{b}(V_i) \leq \overline{b}(V) \tag{2}$$

hold for every partition $\mathcal{F} = \{V_1, \dots, V_t\}$ of V. □

There is another important result on crossing submodular functions. If b is a crossing submodular function on the subsets of V and $B(b) \neq \emptyset$, then there exists a fully submodular function $b^{\downarrow}$ for which $B(b) = B(b^{\downarrow})$. This fact appeared implicitly in an algorithm for finding submodular flows confined by crossing submodular functions [6] and was formulated explicitly by Fujishige in [13]. The function $b^{\downarrow}$ is unique by the following theorem (see for example in [2]).

Theorem 2.5. *Let b^* be a submodular function on the subsets of V. Then $b^*(Z) = \max\{\widetilde{m}(Z) : m \in B(b^*)\}$ for every $Z \subseteq V$. If b^* is integer-valued, then the maximum is achieved by an integer vector. An analogous statement holds for supermodular functions.* □

This unique submodular function is the *full (lower) truncation of* b. It is also known [13,12,7] that if $b(V) = 0$ and $B(b) \neq \emptyset$, then $b^{\downarrow}$ can be expressed by the following formula for every $\emptyset \neq Z \subseteq V$:

$$b^{\downarrow}(Z) = \min\{\widetilde{b}(\mathcal{D}) : \mathcal{D} \text{ a double-partition of } Z\}. \tag{3}$$

Similarly, if p is a crossing supermodular function on the subsets of V and $B'(p) \neq \emptyset$, then there exists a unique fully supermodular $p^{\uparrow}$, called the *full (upper) truncation of* p, for which $B'(p) = B'(p^{\uparrow})$. If $B'(p) \neq \emptyset$, then $p^{\uparrow}$ can be expressed by the following formula of [13] for every $\emptyset \neq Z \subseteq V$:

$$p^{\uparrow}(Z) = \max\{\widetilde{p}(\mathcal{D}) - \Delta(\mathcal{D})p(V) : \mathcal{D} \text{ a double-partition of } Z\}.$$

3. Tree-compositions

By a *$z\bar{s}$-set* we mean a set containing z and not containing s. Let Z be a non-empty proper subset of a ground set V. A family $\mathcal{Z}$ of subsets of V is f *Z-separating* if it contains a $z\bar{s}$-set for every pair $\{z, s\}$ of elements with $z \in Z$ and $s \in V - Z$. Here, $\mathcal{Z}$ is said to be *minimal* for this property if no proper subfamily of $\mathcal{Z}$ is Z-separating.

Our first goal is to show that a crossing Z-separating family $\mathcal{F}$ always includes a cross-free Z-separating subfamily $\mathcal{Z}$. We prove this by describing a direct construction of $\mathcal{Z}$ that does not rely on the uncrossing technique. Since $\mathcal{F}$ is crossing, the intersection $M_{z\bar{s}}$ of all $z\bar{s}$-sets of $\mathcal{F}$ belongs to $\mathcal{F}$ for every choice of $z \in Z, s \in V - Z$. Also, if some members of $\mathcal{F}$ form a connected hypergraph on a subset $U \subset V$, then U belongs to $\mathcal{F}$. It follows for every $s \in V - Z$ that the connected components of the hypergraph $H_s := (V, \{M_{z\bar{s}} : z \in Z\})$ intersecting Z form a subpartition $\mathcal{P}_s$ of $V - s$. By construction, $\mathcal{P}_s \subseteq \mathcal{F}$ and $\mathcal{P}_s$ covers Z.

Theorem 3.1. *Let Z be a non-empty proper subset of a ground set V and let $\mathcal{F}$ be a crossing Z-separating family of subsets of V. Then $\mathcal{F}$ includes a cross-free Z-separating subfamily. Namely, $\mathcal{Z} := \bigcup\{\mathcal{P}_s : s \in V - Z\}$ is such a subfamily.*

Proof. Since $\mathcal{P}_s$ is a subpartition of $V - s$ covering Z for each $s \in V - Z$, the family $\mathcal{Z}$ is Z-separating. We have to prove that $\mathcal{Z}$ is cross-free.

Claim 3.2. *Let $s_1, s_2 \in V - Z$ and $z \in Z$ be elements for which $M_{z\bar{s}_1} \neq M_{z\bar{s}_2}$. Then $s_1 \in M_{z\bar{s}_2}$ and $s_2 \in M_{z\bar{s}_1}$.*

Proof. Suppose by contradiction that, say, $s_1 \notin M_{z\bar{s}_2}$. Since $M_{z\bar{s}_1}$ is a minimal $z\bar{s}_1$-set in $\mathcal{F}$, $M_{z\bar{s}_1} \subseteq M_{z\bar{s}_2}$. But $M_{z\bar{s}_2}$ is also a minimal $z\bar{s}_2$-set in $\mathcal{F}$ from which $M_{z\bar{s}_1} = M_{z\bar{s}_2}$, contradicting the hypothesis of the claim. □

For a contradiction, suppose that $\mathcal{Z}$ contains two crossing sets F_1 and F_2. Since the members of $\mathcal{P}_s$ for a given $s \in V - Z$ are disjoint, there are distinct elements $s_1, s_2 \in V - Z$ so that $F_1 \in \mathcal{P}_{s_1}$ and $F_2 \in \mathcal{P}_{s_2}$.

Case 1. $s_2 \in F_1$ and $s_1 \in F_2$. Then there is an element $z \in Z \cap F_1$ for which $s_2 \in M_{z\bar{s}_1}$. Since $s_2 \notin M_{z\bar{s}_2}$, we have $M_{z\bar{s}_1} \neq M_{z\bar{s}_2}$ and hence Claim 3.2 implies that $s_1 \in M_{z\bar{s}_2}$. Since F_2 is a connected component of H_{s_2} containing s_1, we conclude that $M_{z\bar{s}_2} \subseteq F_2$, and in particular, $z \in F_2$. As F_1 and F_2 are crossing, $F_1 \cap F_2 \in \mathcal{F}$, contradicting the minimality of $M_{z\bar{s}_1}$.

Case 2. $s_2 \notin F_1$ or $s_1 \notin F_2$. By symmetry, we may assume that $s_2 \notin F_1$. By Claim 3.2, $M_{z\bar{s}_1} = M_{z\bar{s}_2}$ for every $z \in F_1 \cap Z$. Hence, as F_1 is a connected component of H_{s_1}, $F_1 = \cup(M_{z\bar{s}_1} : z \in F_1 \cap Z) = \cup(M_{z\bar{s}_2} : z \in F_1 \cap Z)$ and therefore there is a component of H_{s_2} including F_1, contradicting the assumption that the connected component F_2 of H_{s_2} crosses F_1. □

Remark. In the construction of the subpartition $\mathcal{P}_s$ for a given $s \in V - Z$, we considered the connected components of the hypergraph H_s intersecting Z. One may feel that it would be more natural, and certainly simpler, to define a subpartition $\mathcal{P}'_s$ of $V - s$ by considering the maximal members of $\mathcal{F}$ not containing s and intersecting Z. Fig. 2, however, shows that the family $\mathcal{Z}' := \bigcup\{\mathcal{P}'_s : s \in V - Z\}$ is not cross-free.

We claim that a tree-composition $\mathcal{T}$ of Z is a cross-free Z-separating family. Indeed for $s \in V - Z$ and $z \in Z$, there is an undirected path between $\varphi(s)$ and $\varphi(z)$ on the representing tree that contains at least one edge with the proper direction (namely the edge of the path exiting $\varphi(s)$) that represents a $z\bar{s}$-set. Moreover, $\mathcal{T}$ is minimal since an edge uv of the tree represents the only $z\bar{s}$-set for $z \in \varphi^{-1}(v)$ and $s \in \varphi^{-1}(u)$. The following result shows that the converse is also true.

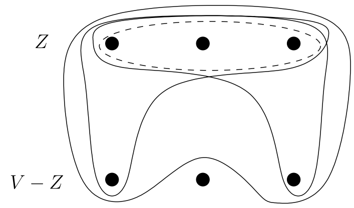

Fig. 2. A family of 4 sets where the maximal $z\bar{s}$-sets do not form a cross-free family.

Theorem 3.3. *For a non-empty proper subset Z of V, a minimal cross-free Z-separating family $\mathcal{Z}$ is a tree-composition of Z.*

Proof. By Lemma 2.1, $\mathcal{Z}$ has a tree-representation (T,φ) where $T=(U,A)$ is a directed tree. Since $\mathcal{Z}$ is Z-separating, we must have $\varphi(s)\neq\varphi(z)$ whenever $s\in V-Z$ and $z\in Z$.

Claim 3.4. *No edge of T enters $\varphi(s)$ for every $s\in V-Z$. No edge of T leaves $\varphi(z)$ for every $z\in Z$.*

Proof. We prove only the first half as it immediately implies the second one by reversing the orientation of T and complementing each member of $\mathcal{Z}$.

Assume for a contradiction that there is an edge $f=u\varphi(s)\in A$ of T entering $\varphi(s)$. Let Z_f be the member of $\mathcal{Z}$ which is represented by f. By Remark 2.2(1), $T':=T/f$ represents the family $\mathcal{Z}':=\mathcal{Z}-\{Z_f\}$. The minimality of $\mathcal{Z}$ shows that $\mathcal{Z}'$ is not Z-separating, that is, there are elements $z\in Z$ and $s'\in V-Z$ such that Z_f is the only member of $\mathcal{Z}$ for which $z\in Z_f$ and $s'\in V-Z_f$. It follows that, going along the unique (undirected) path P of T from $\varphi(s')$ to $\varphi(z)$, the only forward edge is f. Therefore, each edge of the subpath of P connecting $\varphi(s)$ and $\varphi(z)$ is oriented toward $\varphi(s)$, implying that $\mathcal{Z}$ does not contain a $z\bar{s}$-set, a contradiction. □

Let $A'\subseteq A$ denote the subset of edges of T leaving $\varphi(V-Z)$. Let $s\in V-Z$ and $z\in Z$ and consider the unique path P of T connecting $\varphi(s)$ and $\varphi(z)$. By Claim 3.4, its first edge f at $\varphi(s)$ leaves $\varphi(s)$ and hence Z_f is a $z\bar{s}$-set where Z_f denotes the member of $\mathcal{Z}$ represented by f. Therefore, the minimality of $\mathcal{Z}$ implies that $A'=A$, that is, every edge of T leaves $\varphi(V-Z)$. Analogously, every edge of T enters $\varphi(Z)$. Therefore, T is a tree such that each of its edges is of form $\varphi(s)\varphi(z)$ for some $s\in V-Z$ and $z\in Z$, that is, $\mathcal{Z}$ is a tree-composition of Z. □

By combining Theorems 3.1 and 3.3, we obtain the following corollary.

Theorem 3.5. *For a given non-empty proper subset Z of V, a crossing and Z-separating family $\mathcal{Z}$ of subsets of V includes a tree-composition of Z.* □

4. Computing the full truncation of *b*

As an application of Theorem 3.5, we provide a simple proof of Theorem 1.1.

Proof. Since a tree-composition is a special double-partition by definition, (3) implies that $b^{\downarrow}(Z)\leq\min\{\sum_{F\in\mathcal{T}}b(F):\mathcal{T}$ a tree-composition of $Z\}$, therefore, we need to show a tree-composition for which equality holds.

Obviously, $b^{\downarrow}(V)=b(V)$, and $b(V)=\min\{\widetilde{b}(\mathcal{T}):\mathcal{T}$ a tree-composition of $V\}$ by Theorem 2.4. Hence from now on we can assume that Z is a proper subset of V.

$B(b)=B(b^{\downarrow})$ by definition. Theorem 2.5 implies that there is an element m of $B(b)$ for which $\widetilde{m}(Z)(:=\sum_{z\in Z}m(z))=b^{\downarrow}(Z)$. Call a subset $X\subset V$ *m-tight* if $\widetilde{m}(X)=b(X)$ and let $\mathcal{F}$ be the family of m-tight sets. Then $\mathcal{F}$ is a crossing set system by submodularity: $\widetilde{m}(X)+\widetilde{m}(Y)=b(X)+b(Y)\geq b(X\cap Y)+b(X\cup Y)\geq\widetilde{m}(X\cap Y)+\widetilde{m}(X\cup Y)=\widetilde{m}(X)+\widetilde{m}(Y)$ whenever $X,Y\subseteq V$ are crossing.

Claim 4.1. *There exists an m-tight $t\bar{s}$-set for every $s\in V-Z$, $t\in Z$, so $\mathcal{F}$ is Z-separating.*

Proof. If there is an $s\in V-Z$, $t\in Z$ for which no m-tight $t\bar{s}$-set exists, then for $\varepsilon:=\min\{b(X)-\widetilde{m}(X):X$ is a $t\bar{s}$-set$\}$ the vector m' for which $m'(s):=m(s)-\varepsilon$, $m'(t):=m(t)+\varepsilon$, $m'(v):=m(v)$ $(v\in V-\{s,t\})$ would belong to $B(b)$ but would not belong to $B(b^{\downarrow})$ that is a contradiction. □

By Theorem 3.5 there is a tree-composition $\mathcal{T}$ of Z consisting of m-tight sets. Thus for this tree-composition, $\sum_{X\in\mathcal{T}}b(X)=\sum_{X\in\mathcal{T}}\widetilde{m}(X)=\Delta(\mathcal{T})\widetilde{m}(V-Z)+(\Delta(\mathcal{T})+1)\widetilde{m}(Z)=\Delta(\mathcal{T})\widetilde{m}(V)+\widetilde{m}(Z)=\Delta(\mathcal{T})b(V)+\widetilde{m}(Z)=0+b^{\downarrow}(Z)$. □

With a similar proof one can get the following.

Theorem 4.2. *Let p be a crossing supermodular function for which $B'(p)\neq\emptyset$. Then*

$$p^{\uparrow}(Z)=\max\Big\{\widetilde{p}(\mathcal{T})-\Delta(\mathcal{T})p(V)\Big\},$$

where the maximum is taken over all tree-compositions $\mathcal{T}$ of Z. □

5. Algorithmic aspect

With the bi-truncation algorithm [12,16] one can compute the value of the full truncation of a crossing sub- or supermodular function on a set $Z\subseteq V$ provided a subroutine is available that computes $\min\{b(Y)-\widetilde{a}(Y):Y\subseteq X\}$ and the minimizing set for a vector $a\in\mathbb{R}^V$ and a subset $X\subseteq V$. In several applications this minimizing oracle can be obtained via a flow-algorithm hence its running time is usually $\vartheta=O(|V|^3)$. If $B(b)\neq\emptyset$, then the algorithm outputs a vector $m\in B(b)$ with maximum value of $\widetilde{m}(Z)$, hence $\widetilde{m}(Z)=b^{\downarrow}(Z)$ by Theorem 2.5. However, the algorithm does not compute the minimizing tree-composition.

Now we will give an algorithm for computing the minimizing tree-composition in formula (1) if we are given a vector $m\in B(b)$ with maximum value of $\widetilde{m}(Z)$. The proof of Theorem 1.1 implies that if $\widetilde{m}(Z)$ is maximum for $m\in B(b)$, then the m-tight sets form a crossing Z-separating family $\mathcal{F}$. By using the method described in the beginning of Section 3, one can get a cross-free Z-separating family $\mathcal{Z}$ if the minimal m-tight sets are computable. Therefore, assume that the minimizing oracle used in the bi-truncation algorithm can compute also the minimizing set of $\min\{b(Y)-\widetilde{a}(Y):v\in Y\subseteq X\}$ for a vector $a\in\mathbb{R}^V$ and a subset $X\subseteq V$ in running time ϑ. Then the minimal $z\bar{s}$-set can be calculated using this oracle with $a:=m$, $X:=V-s$ and $v:=z$. To get $\mathcal{Z}$ one needs to run the minimizing oracle $O(|V|^2)$ times and after that some search algorithm is needed that runs in $O(|V|^3)$ time. Thus we get $\mathcal{Z}$ in $O(|V|^2\vartheta+|V|^3)$ time. After that we need to omit some elements of $\mathcal{Z}$ to get a minimal Z-separating family that needs $O(|V|^4)$ running time. The family we get will be a tree-composition of Z by Theorem 3.3. Therefore, the total running time of calculating the minimizing tree-composition is $O(|V|^2\vartheta+|V|^4)$ that is $O(|V|^5)$ in practice if the minimizing oracle is given by a flow-algorithm.

Remark. With the bi-truncation algorithm one can get a double-partition that minimizes (3). One can also read out an algorithm from the proof of Theorem 1.1 given in [7] to obtain a minimizing tree-composition, but this algorithm needs to uncross the double-partition provided by the bi-truncation algorithm. The *uncrossing method* means that we keep replacing two crossing members of $\mathcal{F}$ with their union and intersection until the current family gets cross-free. An uncrossing step increases the value of $\sum_{F\in\mathcal{F}}|F|^2$, and hence the running time of the uncrossing method on a double-partition of size $\Omega(|V|^2)$ is more than the running time of the bi-truncation algorithm. More precisely, the bi-truncation algorithm (for finite-valued b) runs in time $O(|V|^2\vartheta + |V|^3)$ where ϑ is the running time of the minimizing oracle used in the algorithm, while the uncrossing procedure for such a family needs $O(|V|^7)$ running time. (If b can have infinite values, then the running time of the bi-truncation algorithm is $O(|V|^3\vartheta)$.) With an adaptation of Fleiner's method [4], this bound for the running time of the uncrossing procedure can be lowered to $O(|V|^5)$.

6. Orientations

A graph $G = (V, E)$ is *k-edge-connected* if $d_G(X) \geq k$ for every $\emptyset \neq X \subset V$. A digraph $D = (V, A)$ is *k-edge-connected* if $\varrho_D(X) \geq k$ for every $\emptyset \neq X \subset V$. By Menger's theorem, this is equivalent to the following: for any two nodes u and v there is k edge-disjoint paths from u to v. A 1-edge-connected digraph is called *strongly connected*. Robbins' theorem [18] states that a graph G has a strongly connected orientation if and only if G is 2-edge-connected. A natural extension to mixed graphs was given by Boesch and Tindell in [1].

A significantly deeper extension of Robbins' theorem is due to Nash-Williams [17] who proved that a $2k$-edge-connected graph has a k-edge-connected orientation. Perhaps surprisingly, the problem of finding a k-edge-connected orientation of a mixed graph is much more complex since in this case the necessary and sufficient condition relies on tree-compositions.

Suppose that $M = (V, E \cup F)$ is a mixed graph that consists of an undirected graph $G = (V, E)$ and a digraph $H = (V, F)$. We want to find an orientation $\overrightarrow{G} = (V, \overrightarrow{E})$ of G so that the digraph $(V, \overrightarrow{E} \cup F)$ is k-edge-connected. This is equivalent to requiring that the orientation of G covers h_k where

$$h_k(X) := \begin{cases} k - \varrho_H(X) & \text{if } \emptyset \neq X \subset V, \\ 0 & \text{if } X \in \{\emptyset, V\}, \end{cases}$$

and an orientation $\overrightarrow{G}$ is said to *cover* h if $\varrho_{\overrightarrow{G}}(X) \geq h(X)$ for all $X \subseteq V$.

For a graph $G = (V, E)$, a set-function $h : 2^V \to \mathbb{R}$ is *(crossing) G-supermodular* if $h(X) + h(Y) \leq h(X \cap Y) + h(X \cup Y) + d_G(X, Y)$ for every (crossing) $X, Y \subseteq V$. It is easy to see that h_k is crossing G-supermodular, hence the following theorem of [7] gives a necessary and sufficient condition to the k-edge-connected orientability problem for mixed graphs.

For an edge $e = uv$ and for a family $\mathcal{F}$ let $w_e(\mathcal{F})$ denote the maximum of the number of $u\bar{v}$-sets and the number of the $v\bar{u}$-sets.

Theorem 6.1 ([7]). *Let $G = (V, E)$ be a graph and let $h : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a crossing G-supermodular function for which $h(V) = 0$. There is an orientation of G covering h if and only if*

$$\sum_{e \in E} w_e(\mathcal{T}) \geq \widetilde{h}(\mathcal{T}), \tag{4}$$

holds for every tree-composition $\mathcal{T}$ of each subset of V. □

Theorem 6.1 was derived in [7] from the submodular flow feasibility theorem. Submodular flows were introduced and investigated by Edmonds and Giles in [3]. Let $D = (V, A)$ be a directed graph, $f : A \to \mathbb{Z} \cup \{-\infty\}$, $g : A \to \mathbb{Z} \cup \{\infty\}$ two integer-valued bounding functions for which $f \leq g$. Moreover, we are given a crossing submodular set-function $b : 2^V \to \mathbb{Z} \cup \{\infty\}$ for which $b(\emptyset) = 0$ and $b(V)$ is finite. A function (or vector) $x : A \to \mathbb{R}$ is called a *submodular flow* or *subflow* confined by b if $\Psi_x(Z) := \varrho_x(Z) - \delta_x(Z) \leq b(Z)$ for every $Z \subseteq V$, where $\varrho_x(Z) := \sum\{x(uv) : u \in V - Z, v \in Z, uv \in A\}$, $\delta_x(Z) := \sum\{x(uv) : u \in Z, v \in V - Z, uv \in A\}$. Since $\Psi_x(V) = 0$ we can assume that $b(V) = 0$. A subflow x is *feasible* if $f \leq x \leq g$.

The basic case of the submodular flow feasibility theorem is when b^* is fully submodular and integer-valued with $b^*(\emptyset) = b^*(V) = 0$ was proved in [6]: *given a digraph $D = (V, A)$ and bounding functions $f : A \to \mathbb{Z}, g : A \to \mathbb{Z}$ with $f \leq g$, there is an integer feasible submodular flow confined by b^* if and only if*

$$\varrho_f - \delta_g \leq b^*. \tag{5}$$

By combining this result with Theorem 1.1, one obtains the general characterization:

Theorem 6.2 ([7]). *Let b be a crossing submodular function for which $b(V) = 0$. There is an integer feasible submodular flow confined by b if and only if*

$$\varrho_f(Z) - \delta_g(Z) \leq \widetilde{b}(\mathcal{T}) \tag{6}$$

for every nonempty $Z \subseteq V$ and every tree-composition $\mathcal{T}$ of Z. □

Note that if b is fully supermodular, then there are several algorithms for finding a feasible (integer) submodular flow (for a survey see [14]). These algorithms can also be applied to compute a feasible subflow when b is crossing submodular. But, in the case when no feasible submodular flow exists, extra work is needed to compute a violating tree-composition (see [7,10]). With the algorithm described in Section 5, it is simpler to find this and hence the present method simplifies the finding of an obstacle if a mixed graph has no k-edge-connected orientation.

In [5] it was proved that in the special case when $h \geq 0$, (4) is required only for tree-compositions of V, that is, for partitions and co-partitions of V. Here we show how Theorem 6.1 implies this special case.

Theorem 6.3. *Let $G = (V, E)$ be a graph and let $h : 2^V \to \mathbb{Z}_+$ be a crossing G-supermodular function for which $h(V) = 0$. There is an orientation of G covering h if and only if*

$$e(\mathcal{P}) \geq \sum_{i=1}^{q} h(V_i) \quad \text{and} \quad e(\mathcal{P}) \geq \sum_{i=1}^{q} h(V - V_i) \tag{7}$$

hold for every partition $\mathcal{P} = \{V_1, V_2, \ldots, V_q\}$ of V.

Proof. As the necessity of (7) is straightforward, we only prove its sufficiency. Assume that G has no orientation covering h. By Theorem 6.1 there is a tree-composition $\mathcal{T}$ of $X \subseteq V$ with $\sum_{e \in E} w_e(\mathcal{T}) < \widetilde{h}(\mathcal{T})$. Then $\mathcal{T}' := \mathcal{T} \cup \{V - X\}$ is a composition of V. Assume for a contradiction that $w_e(\mathcal{T}') > w_e(\mathcal{T})$ for an edge $e = uv$. Then e may have exactly one endpoint, say u, in $V - X$. However, in this case, $d_{\mathcal{T}}(u) = d_{\mathcal{T}}(v) - 1$, thus after subtracting the number of sets containing both u and v, we get that the number of $v\bar{u}$-sets is more than the number of $u\bar{v}$-sets in $\mathcal{T}$. Hence $w_e(\mathcal{T}') = w_e(\mathcal{T})$, a contradiction. Thus $\sum_{e \in E} w_e(\mathcal{T}') = \sum_{e \in E} w_e(\mathcal{T})$. Therefore, since h is non-negative, $\sum_{e \in E} w_e(\mathcal{T}') = \sum_{e \in E} w_e(\mathcal{T}) < \widetilde{h}(\mathcal{T}) \leq \widetilde{h}(\mathcal{T}')$.

By uncrossing $\mathcal{T}'$, we get a cross-free composition $\mathcal{K}$ of V for which $\sum_{e \in E} w_e(\mathcal{K}) < \widetilde{h}(\mathcal{K})$ holds since h is crossing-G-supermodular. By Lemma 2.3, $\mathcal{K}$ can be partitioned into partitions and co-partitions of V and hence at least one of them violates (7). □

Note that $\mathcal{T}'$ has only $O(|V|)$ members hence the running time of the uncrossing procedure is less than the running time of the bi-truncation algorithm. Hence with the present method one can find an orientation covering h or a partition violating (7) in the running time of the bi-truncation algorithm. This is the best known running time for this problem that can also be achieved by the algorithm that can be read out from the following proof of Theorem 6.3 given in [9]:

Proof. Let $p(X) = h(X) + i(X)$. It can be shown that p is crossing supermodular and that an integer vector in the base-polyhedron $B'(p)$ is the in-degree vector of an orientation covering h. A little calculation shows that (7) implies the conditions of Fujishige's theorem (see Theorem 2.4). Hence there exists an integer vector $z \in B'(p)$. By the Orientation lemma of Hakimi [15] there exists an orientation of G with in-degree vector z, completing the proof. □

7. Tree-compositions of bipartite graphs

A theorem of [11] gives a min–max formula for the minimum in-degree of T in a strongly connected orientation of a bipartite graph $G = (S, T; E)$. We sharpen this theorem by using the notion of tree-compositions, as follows.

Theorem 7.1. *Let $G = (S, T; E)$ be a 2-edge connected bipartite graph. Then*

$$\min\left\{\varrho_{\vec{G}}(T) : \vec{G} \text{ strongly connected}\right\}$$
$$= \max\left\{|\mathcal{T}| : \mathcal{T} \text{ a tree-composition of } T \text{ complying with } G\right\}.$$

We are going to derive this theorem from the following more general result.

Theorem 7.2. *Let $G = (V, E)$ be a graph and let $h : 2^V \to \mathbb{Z}_+$ be a crossing G-supermodular function with $h(V) = 0$ for which (7) holds. For a given subset $\emptyset \neq T \subset V$,*

$$\min\left\{\varrho_{\vec{G}}(T) : \vec{G} \text{ covers } h\right\}$$
$$= \max\left\{\widetilde{h}(\mathcal{T}) + \widetilde{i}(\mathcal{T}) - \Delta(\mathcal{T})|E| - i(T) : \mathcal{T} \text{ a tree-composition of } T\right\}.$$

In Section 6 the problem of finding an orientation covering a G-supermodular function was formulated as a submodular flow problem. Therefore, the fundamental result of Edmonds and Giles on total dual integrality of the linear system describing a submodular flow polyhedron implies a min–max result for the minimum of $\varrho_{\vec{G}}(T)$. The point in Theorem 7.2 is that a min–max formula could be given in a relatively simple and compact form.

Proof. In the second proof of Theorem 6.3, and let $p := i + h$ it was shown that $B'(p) \neq \emptyset$ follows by (7) for $p = i + h$. Observe that an integer vector $x \in B'(p)$ for which $\widetilde{x}(T) = \min\{\widetilde{m}(T) : m \in B'(p)\}$, forms the in-degree vector of an orientation of G covering h for which $\varrho(T)$ is minimal. Thus the theorem follows basically from Theorems 2.5 and 4.2. □

Theorem 7.2 can be used to prove the most important corollaries of Theorem 6.3. For example (with $h(X) := k$ for $\emptyset \neq X \subset V$) we can find a k-edge-connected orientation of a $2k$-edge-connected graph in which the in-degree of a given subset of nodes $T \subseteq V$ is minimal. We get the following theorem:

Theorem 7.3. *Let $G = (V, E)$ be a $2k$-edge-connected graph, and let $\emptyset \neq T \subset V$. Then*

$$\min\left\{\varrho_{\vec{G}}(T) : \vec{G} \text{ } k\text{-edge-connected}\right\}$$
$$= \max\left\{\widetilde{i}(\mathcal{T}) + k|\mathcal{T}| - \Delta(\mathcal{T})|E| - i(T) : \mathcal{T} \text{ a tree-composition of } T\right\}. \quad \square \tag{8}$$

Proof of Theorem 7.1. Consider the case of Theorem 7.3 when $G = (S, T; E)$ is a bipartite graph and $k = 1$. In this case we show that there is a tree-composition of T complying with G maximizing (8). Note that if one edge e of G is induced by $u_{\mathcal{T}}(e)$ members of a tree-composition $\mathcal{T}$ of T, then $u_{\mathcal{T}}(e) \leq \Delta(\mathcal{T})$ since its endpoint in S is covered by $\Delta(\mathcal{T})$ members of $\mathcal{T}$. If $\mathcal{T}$ is a tree-composition that complies with the graph, then $u_{\mathcal{T}}(e) = \Delta(\mathcal{T})$ for every $e \in E$. Let the deficit of an edge be $\Delta(\mathcal{T}) - u_{\mathcal{T}}(e)$ and $\gamma(\mathcal{T}) := \sum_{F \in \mathcal{T}} \Delta(\mathcal{T})|E| - i(F)$. Therefore, $\gamma(\mathcal{T})$ is the sum of the deficits. Note also that $i(T) = 0$.

Let $\mathcal{T}$ be a tree-composition of T maximizing (8) for which $\gamma(\mathcal{T})$ is minimum. Let $F = (U_S \cup U_T, A)$ be the directed tree representing $\mathcal{T}$ along with the map $\varphi : V \to U_S \cup U_T$.

Claim 7.4. *$\mathcal{T}$ is complying with G.*

Proof. For a contradiction, assume that $\varphi(s^*)\varphi(t^*) \notin A$ for $s^* \in S, t^* \in T, s^*t^* \in E$. Let P be the undirected path of length $2r + 1$ between $\varphi(s^*)$ and $\varphi(t^*)$. Shrink the set $V(P) \cap U_T$ in F and let F' be the resulting tree and $\mathcal{T}'$ the tree-composition of T that is represented by F'. It is easy to see that $|\mathcal{T}'| = |\mathcal{T}| - r$ and $\gamma(\mathcal{T}') \leq \gamma(\mathcal{T}) - r$, since the deficit of the edge s^*t^* becomes 0 from r and the deficit of the other edges does not increase. Therefore, $\widetilde{i}(\mathcal{T}) + |\mathcal{T}| - \Delta(\mathcal{T})|E| = |\mathcal{T}| - \gamma(\mathcal{T}) \leq |\mathcal{T}'| + r - (\gamma(\mathcal{T}') + r) = \widetilde{i}(\mathcal{T}') + |\mathcal{T}'| - \Delta(\mathcal{T}')|E|$ thus $\mathcal{T}'$ is also a maximizing tree-composition of T with smaller deficit, a contradiction. □

This completes the proof of Theorem 7.1. □

Finally, we note that using Theorem 7.1 one can simplify formula (8) for $k = 1$ with the following idea. Let $G = (V, E)$ be a 2-edge-connected graph, $\emptyset \neq T \subset V$ and $\mathcal{C}$ be the set system formed by the components of $G[T]$ and $G[V - T]$. Now the graph $G/\mathcal{C}$ (that is the graph that arises from G by contracting each member of $\mathcal{C}$) is bipartite and a strongly connected orientation of G determines a strongly connected orientation of $G/\mathcal{C}$. Conversely, any strongly connected orientation of $G/\mathcal{C}$ can be extended to a strongly connected orientation of G. This follows immediately from a theorem of Boesch and Tindell [1] stating that a mixed graph has a strongly connected orientation if and only if there is no cut-edge and there is no one-way cut. Therefore, it suffices to find a strongly connected orientation of the bipartite graph $G/\mathcal{C}$ where the in-degree of $T/\mathcal{C}$ is minimum.

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