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Tree-Representation of Directed Circuits

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ABSTRACT. We prove that a strongly connected directed graph $G=(V,E)$ has a spanning tree T so that each fundamental circuit belonging to T is a directed circuit if and only if G has precisely $|E|-|V|+1$ directed circuits. Another characterization of such directed graphs will also be provided in terms of forbidden minors.

1. Introduction, Preliminaries

A *join (strong join)* J of an undirected graph is a subset of edges so that $|J \cap C| \leq |C|/2$ ($|J \cap C| < |C|/2$) for every circuit C of the graph.

The investigations of joins was initiated by P. Sole and T. Zaslavsky while the problem of determining a maximum strong join is due to D. Welsh [1990] . In [Frank, 1992] a min-max theorem was provided for the maximum cardinality of a join along with a polynomial time algorithm to compute the largest join. A. Fraenkel and M. Loebl [1991] proved that the maximum strong join problem is NP-complete even if the graph is planar and bipartite. We proved in [Frank, Jordán and Szigeti, 1992] that for every graph the maximum cardinality of a strong join is at most $\lfloor (|V|-1)/2 \rfloor$ and provided an algorithm to decide if a given bipartite graph is extreme, that is, it attains this bound.

Suppose that a bipartite graph $B=(U,V;F)$ has a perfect matching M so that for every element e of M an edge parallel to e also belongs to G. In this case clearly no element of M may belong to any strong join and the maximum strong join problem can be reformulated as follows.

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Define a directed graph $G = (V, E)$ so that $uv \in E$ if $uv' \in F$ where v' denotes the node in U for which $vv' \in M$. It is not difficult to prove that B is extreme if and only if G has a spanning tree T so that every fundamental circuit belonging to T is a directed circuit. (*A fundamental circuit* is one having precisely one non-tree edge). We shall call such a tree a *circuit-representing tree* or, in short, a *CR-tree*. It is also true that the set of edges in B corresponding to the edges of a CR-tree of G is a maximum strong join of B. The digraph D_2 on two nodes with two parallel edges in both directions clearly has no CR-tree.

The purpose of the present paper is to provide characterizations for digraphs having a CR-tree as well as a polynomial time algorithm to find a CR-tree if there is any.

Let $G = (V, E)$ be a directed graph. For $X \subseteq V$ let $\delta(X)$ denote the number of edges leaving X. G is called *strongly connected* if there is a directed path from u to v for every $u, v \in V$. This is equivalent to saying that $\delta(X) \geq 1$ for every $\emptyset \neq X \subset V$. We call a set X *tight* if $\delta(X) = 1$. Let T be a spanning tree of G and $e = xy$ an edge of T. Then $T - e$ has two components. Define $T(e)$ to be the node-set of the component of $T - e$ containing x. It is easy to see that T is a CR-tree if and only if $T(e)$ is tight for every edge e of T.

By an *ear-decomposition* of G we mean a sequence $\mathcal{P} := \{P_1, P_2, \ldots, P_t\}$ where P_1 is a circuit of G, each other P_i is a path in G so that each edge of G belongs to precisely one P_i $(i = 1, \ldots, t)$ and precisely the end-nodes of P_i $(i = 2, \ldots, t)$ belong to $P_1 \cup \ldots \cup P_{i-1}$. Each path P_i is supposed to be simple except that the two end-nodes may coincide. The number t of paths is called the *length* of the decomposition.

It is well-known that a digraph G has an ear-decomposition if and only if G is strongly connected. Moreover, for any strongly connected subgraph $H = (U, A)$ of G any ear-decomposition of H is the starting segment of an ear-decomposition of G. The length of an ear-decomposition depends only on the graph and equals $|E| - |V| + 1$. It also follows easily that every strongly connected digraph $G = (V, E)$ has at least $|E| - |V| + 1$ directed circuits.

2. Characterizations of CR-trees

Let $G = (V, E)$ be a strongly connected digraph. We call a simple directed path $P := \{v_0, e_1, v_1, e_2, \ldots, e_k, v_k\}$ *unique* if P is the only simple path from v_1 to v_k. We consider the empty set and a path $\{v_0\}$ as *trivial* unique paths.

PROPOSITION 2.1 *A non-trivial path P is unique if and only if there is a family $\{X_1, \ldots, X_k\}$ of tight sets for which $X_1 \subset X_2 \subset \ldots \subset X_k$ and e_i leaves X_i for every i, $1 \leq i \leq k$.*

Proof. Suppose first the existence of such a family. Let, indirectly, P' be another simple path from v_1 to v_k. Then there is a first edge e_i of P not belonging to P'. Since there is an edge e of P' leaving X_i, we conclude that $\varrho(X_i) \geq 2$, contradicting the tightness of X_i.

Assume now that P is unique. For each $i, 1 \leq i \leq k$ let X_i denote the set of nodes reachable from $\{v_1, \ldots, v_{i-1}\}$ without using the edge e_i. From the definition $X_i \subseteq X_{i+1}$. We claim that $v_j \notin X_i$ for $i < j$, or equivalently, there is no path in $G - e_i$ from $\{v_1, \ldots, v_{i-1}\}$ to $\{v_i, \ldots, v_{k+1}\}$. Indeed, if such a path P' existed, choose it minimal and let s and t denote the first and last node of P', respectively. By the minimality no internal node of P' belongs to P. Hence by replacing the segment of P from s to t by P' we would obtain another simple path from v_1 to v_{k+1}, contradicting the uniqueness of P.

Since the only edge leaving X_i is e_i, each X_i is tight and the family $\{X_1, \ldots, X_k\}$ satisfies the requirements.

□□□

Note that the proof above can easily be turned into a polynomial-time algorithm that either finds two distinct paths from v_1 to v_k or constructs the family $\{X_1, \ldots, X_k\}$ in question.

Let us call an edge $e = xy \in E$ *uni-cyclic* if e is contained in exactly one directed circuit and *multi-cyclic* otherwise. We call an edge $e = xy$ *essential* if $G - e$ is not strongly connected. Otherwise e is *non-essential*. In other words, $e = xy \in E$ is uni-cyclic if there is a unique path from y to x and e is essential if $\{x, e, y\}$ is a unique path. Therefore these properties can be tested in polynomial-time.

PROPOSITION 2.2 *Every directed subpath of a CR-tree T is unique.*

Proof. Let $P := \{v_0, e_1, v_1, e_2, \ldots, e_k, v_k\}$ be a subpath of T. Recall that $T(e)$ denotes the node-set of the component of $T - e$ containing the tail of e. Since T is a CR-tree, the only edge leaving $T(e)$ is e, that is, $T(e)$ is tight for each $e \in T$. Hence the family $\{T(e_i) : i = 1, \ldots, k\}$ satisfies the properties in Proposition 2.1 and therefore P is unique.

□

THEOREM 2.3 *Let T be a spanning tree of a strongly connected digraph $G = (V, E)$. The following are equivalent.*

(a) T is a CR-tree,
(b) Every directed circuit is a fundamental circuit,
(c) Every non-tree edge is uni-cyclic.

Proof. (a→b) Let T be a CR-tree. Suppose (b) fails to hold, that is, there is a directed circuit C which is not fundamental. Then, for an edge $e = xy \in C - T$, the subpath of T from y to x is directed but not unique as $C - e$ is another path from y to x. This contradicts Proposition 2.2.

(b→c) Let C be an arbitrary circuit containing a non-tree edge e. By (b) C is the fundamental circuit belonging to e, that is, e is uni-cyclic.

(c→a) If (a) is not true, then there is a non-tree edge $e = xy$ so that its fundamental circuit is not directed. Then there exists a circuit C containing e and this C contains another non-tree edge $f = uv$. Since both e and f are uni-cyclic, both paths $C - e$ and $C - f$ are unique. By Proposition 2.1 there is a tight set X (resp., Y) so that e enters X (f enters Y) and f (e) is the only edge leaving X (Y). Therefore no edge leaves $X \cup Y$ and $X \cap Y$. Since G is strongly connected, $X \cup Y = V$ and $X \cap Y = \emptyset$, that is, $X = V - Y$. We can conclude that e is the only edge entering X and f is the only edge leaving X contradicting the fact that T is a spanning tree.

□□□

3. Graphs with CR-trees

In this section we provide three characterizations for digraphs $G = (V, E)$ having CR-trees. We can assume that there is no cut-edge in G. Indeed, any cut-edge e belongs to every spanning tree and to no directed circuit. Hence G has a CR-tree precisely if G/e has a CR-tree where G/e denotes a digraph arising from G by contracting e.

A second observation is that G cannot have a CR-tree if G is not strongly connected. Indeed, let T be a CR-tree of G. Every edge of $G - T$ belongs to a directed circuit, namely to its fundamental circuit. Since there is no cut-edge, every element of T belongs to a certain fundamental circuit. Hence G is strongly connected.

Henceforth we assume that G is strongly connected.

THEOREM 3.1 *A strongly connected digraph $G = (V, E)$ has a CR-tree if and only if the set K of multi-cyclic edges forms a forest. Moreover if K is a forest, any spanning tree including K is a CR-tree.*

Proof. Suppose first that T is a CR-tree of G. By Theorem 2.3 every non-tree edge is uni-cyclic, that is, K is a subset of T, and hence K is a forest.

Conversely, suppose that K is a forest. Let T be any spanning tree including K. Now property (c) in Theorem 2.3 holds and hence T is a CR-tree.

□□□

Since we can check in polynomial time if an edge is uni-cyclic or not, Theorem 3.1 suggests an algorithm to decide if a digraph has a CR-tree. A disadvantage of the theorem is that the necessity of the condition is not very straightforward. We provide two other characterizations to overcome this drawback. We will need the following:

PROPOSITION 3.2 *If P is a unique path in G and G has a CR-tree, then G has a CR-tree including P.*

Proof. Let $P := \{v_0, e_1, v_1, e_2, \dots, e_k, v_k\}$. By Proposition 2.1 every subpath of P is unique. By induction we may assume that there is a CR-tree T of G containing each e_i ($i = 1, \dots, k-1$). By Theorem 2.3 each non-tree edge is uni-cyclic.

If $e_k \in T$, we are done. So suppose that $e_k \notin T$ and let C denote the fundamental circuit belonging to e_k. Since T is a CR-tree, C is directed. By Proposition 2.1 there is a tight set X containing $v_1, \dots, v_k$ and not containing v_{k+1}. There is a (unique) edge $f \in C - P$ entering X. Because e_k is the only edge leaving X and G has only fundamental circuits, f is uni-cyclic. Hence $T' := T - f + e_k$ is a tree containing all uni-cyclic edges. By Theorem 2.3 T' is a CR-tree and includes P.

□

For a strongly connected digraph $G = (V, E)$ denote $\kappa(G) := |E| - |V| + 1$. Let $\mathcal{P} := \{P_1, P_2, \dots, P_t\}$ be an ear-decomposition of G and let G_i ($i = 1, \dots, t$) denote the union of the first i members of $\mathcal{P}$. By induction it follows that $\kappa(G_i) = i$ for $1 \le i \le t$. Since G_i is strongly connected, every P_i is a subset of a directed circuit C_i of G_i. Let $R_i := C_i - P_i$ ($i = 2, 3, \dots, t$). Clearly, each G_i has at least $\kappa(G_i)$ directed circuits.

THEOREM 3.3 *For a strongly connected digraph $G = (V, E)$ the following are equivalent:*
(a) G has a CR-tree,

(b) *G has precisely $\kappa(G)$ directed circuits,*
(c) *R_i is a unique path in G_{i-1} $(i = 2, 3, \ldots, t)$.*

Proof. The equivalence of (b) and (c) is straightforward.

$(a \to b)$ If G has a CR-tree T, then every directed circuit of G is a fundamental circuit by Theorem 2.3. Since there are $\kappa(G)$ non-tree edges in G, the total number of directed circuits is $\kappa(G)$.

$(b \to a)$ Apply induction on $t = \kappa(G)$. If $\kappa(G) = 1$, then G is a circuit and $G - e$ is a CR-tree of G for any edge e of G. Let $\kappa(G) > 1$ and assume, by induction, that G_{t-1} has a CR-tree T_{t-1} and R_t is unique in G_{t-1}. By Proposition 3.2 there is a CR-tree T_{t-1} of G_{t-1} including R_t. Then $T := T_{t-1} \cup P_t - e$ is a CR-tree of G for any edge e of P_t. □□□

Finally, we exhibit a minor-type characterization. Let us introduce three operations of a strongly connected graph $G = (V, E)$.

$(\alpha\ )$ Contracting a multi-cyclic edge e,
(β) Deleting a non-essential edge f,
(γ) Restriction to a strongly connected induced subgraph $G_\gamma = (V', E')$.

PROPOSITION 3.4 *If G has a CR-tree, then each of the operations $(\alpha\ )$, (β), (γ) results in a strongly connected digraph having a CR-tree.*

Proof. Let G_α, G_β, G_γ denote the resulting digraphs. Clearly, each of them is strongly connected. Let T be a CR-tree of G. By Theorem 3.1 e belongs to T. Hence T/e is a CR-tree of G_α.

By Proposition 2.2 every edge of T is essential. Hence $f \notin T$ and T is a CR-tree of G_β, as well.

Finally, we show that the restriction T' of T to V' is a CR-tree. This is clearly true if T' is a tree. Suppose T' is not connected and let $X \subset V'$ be a set, $\emptyset \neq X \neq V$, so that there is no edge of T' connecting X and $V' - X$. Since G_γ is strongly connected, there is an edge e from X to $V' - X$. Let C be a directed circuit in G_γ containing e. Now C is not a fundamental circuit of G, therefore G cannot have a CR-tree by Theorem 2.3, a contradiction. □□□

Recall that D_2 denotes the digraph on two nodes with two parallel edges in both directions.

THEOREM 3.5 *A strongly connected digraph $G = (V, E)$ has a CR-tree if and only if D_2 cannot be obtained from G by successively applying operations $(\alpha), (\beta), (\gamma)$.*

Proof. Since D_2 has no CR-tree, the preceding proposition prove the "only if" part.

Suppose now that G is a counter-example to the "if" part with a minimum number of edges. Then G has no CR-tree and cannot be reduced to D_2. Therefore

$$\textit{none of } G_\alpha, G_\beta, G_\gamma \textit{can be reduced to } D_2. \qquad (*)$$

Let $\mathcal{P} := \{P_1, P_2, \ldots, P_t\}$ be an ear-decomposition of G, as before. We use the notation of Theorem 3.3. Now $t > 1$. Let x and y denote the first and last node of P_t, respectively.

CLAIM 1 *$x \neq y$ and there are two paths Q_1, Q_2 in G_{t-1} from y to x.*

Proof. G_{t-1} arises from G by operation (γ). The minimality of G and (*) imply that G_{t-1} has a CR-tree. It follows that R_t cannot be unique in G_{t-1} for otherwise there is a CR-tree T' of G_{t-1} including R_t (by Proposition 3.2) and then $T' \cup P_t - e$ would be a CR-tree of G for any edge $e \in P_t$. □

CLAIM 2 *Both Q_1 and Q_2 consist of one edge.*

Proof. Suppose, indirectly, that Q_1, say, has more than one edge. Let e and f be the first and last edge of Q_1, respectively. Then it is easy to check that at least one of these edges, say e, has the property that in G/e there are at least two paths from y to x. G/e arises from from G_{t-1}/e by adding P_t. By Theorem 3.3 G/e does not have a CR-tree.

On the other hand e is multi-cyclic in G since e belongs to a circuit of G_{t-1} and belongs to a circuit including P_t. By (*) and the minimality of G , $G_\alpha := G/e$ has a CR-tree, a contradiction. □

Let e_i denote the only edge of Q_i $(i = 1, 2)$ and let Q be a path in G_{t-1} from x to y. The union of P_t and Q is a circuit C. Clearly every edge of P_t and Q is multi-cyclic. First erase all nodes not in C (by operation (γ)). Apply then operation (α) to all but one edges of P_t and of Q. This way we get a digraph on two nodes with at least two parallel edges in both directions. In such a graph all edges are non-essential. Thus D_2 can be obtained by operation (β), contradicting the assumption on G.

□□□

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