Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2001-05. Published by the Egrerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres . ISSN 1587–4451.

An extension of a theorem of Henneberg and Laman

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February 2001

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Abstract

We give a constructive characterization of graphs which are the union of k spanning trees after *adding* any new edge. This is a generalization of a theorem of Henneberg and Laman who gave the characterization for $k = 2$.

We also give a constructive characterization of graphs which have k edge-disjoint spanning trees after *deleting* any edge of them.

Keywords: graph, constructive characterization, rigidity, packing and covering by trees

1 Introduction

The idea of constructive characterizations in graph theory is not new. The first example is the following theorem of Tutte [15] from 1966. A graph on more than k nodes is said to be k**-node-connected** if after deleting less than k nodes the graph remains connected.

Theorem 1.1 (Tutte). *An undirected graph $G = (V, E)$ is 3-node-connected if and only if G can be obtained from the complete graph on 4 nodes by the following two operations:*

(i) *add a new edge,*

(ii) *take a node z and replace it by two nodes z_1, z_2, put an edge between them, and put edges incident to z_1, z_2 such that the union of the neigbours of them are exactly the original neighbours of z and there are at least two neighbours of z_i for $i = 1, 2$.*

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Research supported by the Hungarian National Foundation for Scientific Research Grant, OTKA T17580. The second author is supported by the Siemens-ZIB Fellowship Program and FKFP grant no. 0143/2001.

In 1976 Lovász [8] proved the following theorem. A graph is said to be **k-edge-connected** if after deleting less than k edges the graph remains connected.

Theorem 1.2 (Lovász). *An undirected graph $G = (V, E)$ is $2k$-edge-connected if and only if G can be obtained from a single node by the following two operations:*

(i) *add a new edge,*

(ii) *add a new node z, subdivide k existing edges by new nodes, then identify the k subdividing nodes with z.*

The (ii) operation in this theorem is called **pinching k edges (with z)**.

Similar constructive characterizations for directed edge-connectivity in directed graphs exist due to Mader [9].

Theorem 1.3 (Mader). *A directed graph $G = (V, E)$ is k-edge-connected if and only if G can be obtained from a single node by the following two operations:*

(i) *add a new edge,*

(ii) *pinch k existing directed edges.*

Another example of this kind of theorems concerns spanning trees. We call an undirected graph **k-tree-connected** if it contains k edge-disjoint spanning trees. It was observed in [3] that a combination of Mader's characterization and Tutte's disjoint tree theorem gives rise to the following.

Theorem 1.4. *An undirected graph $G = (V, E)$ is k-tree-connected if and only if G can be built from a single node by the following two operations:*

(j) *add a new edge,*

(jj) *pinch i $(0 \leq i \leq k-1)$ existing edges with a new node z, and add $k - i$ new edges connecting z with existing nodes.*

For a direct proof see Tay [13].

An undirected graph $G = (V, E)$ is said to be **k-stiff** if it is the union of k edge-disjoint spanning trees after *adding* any new edge, that is, $G + e$ is the union of k edge-disjoint spanning trees for every possible new edge $e = uv$ $(u, v \in V)$ (multiple edges are permitted).

2-stiff or, as often called, minimal generically rigid graphs are important in statics. A framework in the plane is statically rigid if and only if its graph has a minimal generically rigid subgraph. This was proved by Laman [7]. A framework consists of rigid rods and rotatable joints. Its underlying graph is the natural one: the node set is the set of the joints and there is an edge between two nodes if there is a rod between the corresponding two joints. A consequence of the theorem of Laman is that the notion of rigidity is a property of the graph and not only its embedding into the plane with real rods and joints.

According to a theorem of Nash-Williams [12], a graph $G = (V, E)$ is k-stiff if and only if $|E| = k(|V| - 1) - 1$ and $\gamma_G(X) \leq k(|X| - 1) - 1$ for every subset $X \subseteq V$ with $|X| \geq 2$ (where $\gamma_G(X)$ denotes the number of the edges of G whose two end-nodes are in X). By combining theorems of Henneberg [5] and of Laman [7], one obtains the following constructive characterization of 2-stiff graphs.

Theorem 1.5 (Henneberg and Laman). *A graph G is 2-stiff if and only if G can be constructed from one (non-loop) edge by the following two operations:*

(i) *add a new node z and connect z to two distinct existing nodes,*

(ii) *subdivide an existing edge uv by a node z and connect z to an existing node distinct from u and v.*

In this paper we give the generalization of this theorem for arbitrary k. The difficulties which come to the picture for k greater than 2 will be presented in the next section. We note however that 3-stiff graphs have no direct meaning in 3-dimensional rigidity.

In Sect. 3 we will give a corresponding constructive characterization of graphs which have k edge-disjoint spanning trees after *deleting* any edge of them.

For the sake of completeness we give the original theorem of Henneberg [5] and Laman [7].

Theorem 1.6 (Henneberg). *A framework in the plane is minimally rigid if and only if it can be constructed from one rod by the following two operations:*

(i) *add a new joint z and connect z to two distinct existing joints by rods,*

(ii) *subdivide an existing rod uv by a node z and connect z to an existing joint distinct from u and v.*

Theorem 1.7 (Laman). *A framework is minimally rigid if and only if its underlying graph $G = (V, E)$ has the following property: $|E| = k|V| - (k+1)$, $\gamma_G(X) \leq k|X| - (k+1)$ for all $X \subseteq V, |X| \geq 2$.*

2 Construction of k-stiff graphs

Let k be an integer not less than 2. Let K_2^{k-1} denote the graph on two nodes with $k-1$ parallel edges.

Theorem 2.1. *$G = (V, E)$ is a graph. The following are equivalent:*

(1) *G k-stiff.*

(2) *$|E| = k|V| - (k+1)$ and $\gamma_G(X) \leq k|X| - (k+1)$ for all subsets $X \subseteq V$, $|X| \geq 2$.*

(3) *G can be built from K_2^{k-1} by applying the following operation:*

Choose a subset F of i existing edges ($0 \leq i \leq k-1$), pinch the elements of F with a new node z, and add $k-i$ new edges connecting z with other nodes so that there are no k parallel edges in the resulting graph.

The equivalence of (1) and (2) is straightforward by a theorem of Nash-Williams [12]. The fact that (3) implies (2) is an easy exercise.

(2) implies (3). This is the main point of this section.

After some definitions and lemmas, we give a necessary and sufficient condition when the inverse of operation (3) is applicable at node s, which is important for an inductive proof.

A graph which satisfies the conditions in (2) is called a **Laman-graph**.

A graph D on node-set U is called an **admissible graph** if it satisfies the following property:

$$(4) \qquad \gamma_D(X) \leq k|X| - (k+1) \quad \text{for all subsets} \quad X \subseteq U, |X| \geq 2.$$

In the graph $G = (V, E)$ **splitting off a pair of edges** at node s means the operation of replacing su and sv by a new edge connecting u and v.

At node s with degree $k+i$ ($0 \leq i \leq k-1$) **admissible splitting off** j ($1 \leq j \leq k-1$) pairs of edges means j number of splitting off a pair of edges such that the resulting induced subgraph on $V - s$ is an admissible graph (we often leave out the word admissible). If $j = i$, then it is called a **complete splitting off**.

Our goal is to find a node s with degree $k+i$ ($0 \leq i \leq k-1$) in the Laman-graph G, such that i pairs of edges can be split off, that is, the inverse operation of (3) can be applied at the node s in such a way that the resulting graph is also a Laman-graph. This will give our inductive proof.

G'_s will denote the graph that we obtained by splitting off some pairs of edges incident to s. We will use the term **split edge** in G'_s for an edge uv which comes from splitting off edges su and sv.

Definition 2.2. Let b_G denote the following function on the subsets of V with cardinality at least 2:

$$b_G(X) := k|X| - (k+1) - \gamma_G(X).$$

By this definition we have the following: for a graph G, property (2) holds if and only if $b_G(V) = 0$ and $b_G(X) \geq 0$ for all subsets $X \subseteq V, |X| \geq 2$. The graph $G = (U, F)$ is an admissible graph if and only if $b_G(X) \geq 0$ for all subsets $X \subseteq U, |X| \geq 2$.

If $b_G(V) = 0$, then X is said to be a G-**tight** set. From now on we leave G out if it is unambiguous.

All the lemmas below are about admissible graphs.

Lemma 2.3. *Let X and $Y \subseteq V$ and $|X \cap Y| \geq 2$. Then*

$$b(X) + b(Y) = b(X \cap Y) + b(X \cup Y) + d(X, Y).$$

Proof. $b(X) + b(Y) = k|X| - (k+1) - \gamma_G(X) + k|Y| - (k+1) - \gamma_G(Y) = k(|X| + |Y|) - 2(k+1) - (\gamma_G(X \cap Y) + \gamma_G(X \cup Y) + d_G(X,Y)) = k|X \cap Y| - (k+1) - \gamma_G(X \cap Y) + k|X \cup Y| - (k+1) - \gamma_G(X \cup Y) + d_G(X,Y) = b(X \cap Y) + b(X \cup Y) + d(X,Y)$. □

Lemma 2.4. *If $X_1, X_2, X_3 \subseteq V$ and $|X_j \cap X_l| = 1$ for all possible pairs and $|X_1 \cap X_2 \cap X_3| = 0$, then*

$$b(\bigcup_{j=1}^{3} X_j) \leq \sum_{j=1}^{3} b(X_j) - k + 2.$$

Proof. $b(\bigcup_{j=1}^{3} X_j) = k|\bigcup_{j=1}^{3} X_j| - (k+1) - \gamma_G(\bigcup_{j=1}^{3} X_j) \leq k(\sum_{j=1}^{3} |X_j| - 3) - (k+1) - \sum_{j=1}^{3} \gamma_G(X_j) = \sum_{j=1}^{3}(k|X_j| - (k+1) - \gamma_G(X_j)) - k + 2 = \sum_{j=1}^{3} b(X_j) - k + 2$. □

Lemma 2.5. *$X, Y \subseteq V$, $|X \cap Y| = 1$. Then*

$$b(X) + b(Y) = b(X \cup Y) - 1 + d(X,Y).$$

Proof. $b(X) + b(Y) = k|X| - (k+1) - \gamma_G(X) + k|Y| - (k+1) - \gamma_G(Y) = k(|X| + |Y| - 1) - (k+1) - 1 - (\gamma_G(X) + \gamma_G(Y)) = k|X \cup Y| - (k+1) - 1 - (\gamma_G(X \cup Y) - d_G(X,Y)) = b(X \cup Y) - 1 + d(X,Y)$.

□

From now on G is a graph which satisfies (2) in our theorem, that is, G is a Laman-graph, and not K_2^{k-1}. It is easy to see that there exists a node s with degree $d(s)$ such that $k \leq d(s) \leq 2k - 1$. It is also clear that the multiplicity of edge uv is at most $k - 1$ (by (2): $\gamma_G(\{u,v\}) \leq k|\{u,v\}| - (k+1) = k - 1$).

Observation 2.6. The edges su and sv cannot be split off (that is, adding the edge uv to the induced subgraph of G on $V - s$ does not result in an admissible graph) if and only if there exists a tight set in G which does not contain s but u and v.

Observation 2.7. By Lemma 2.4 a splitting off at node s cannot be kept on if and only if the remaining neighbours of s are in a tight set which does not contain s or there is only one remaining neighbour of s.

Theorem 2.8. *Let G be a Laman-graph. If $s \in V$ has degree k or $k + 1$, then a complete splitting off is applicable at it.*

Proof. If s has degree k, then a complete splitting off means deleting it with all its adjacent edges. This results obviously in a Laman-graph.

If s has degree $k+1$, then we should find a pair of edges su and sv with $u \neq v$ such that $G - s + uv$ is an admissible graph.

There is no tight set X not containing s which contains all the neighbours of s because, if there was one, then $b_G(X + s) < 0$ which contradicts to (2). Because of the fact that there are no edges with multiplicity greater than $k - 1$, the neighbour-set of s in G has at least two elements, so by Lemma 2.4 and Observation 2.6 there is an admissible splitting off. □

If $k = 2$, then there is no other case, so we proved the theorem of Henneberg and Laman: every node with degree 2 or 3 admits a complete splitting off. If $k \geq 3$, then life is much more complicated, as was observed by Z. Király [6]. He found a graph for $k = 3$ in which there is no splitting off 2 pairs of edges (that we would need) at a node with degree 5.

Here we give a necessary and sufficient condition for a node with degree $k + i$ ($2 \leq i \leq k-1$) (let us call a node like this a **small node**) which admits a complete splitting off. Let $\Gamma_G(v)$ denote the number of the nodes in graph G that are connected to node v by an edge.

Theorem 2.9. *At node s with $d_G(s) = k + i$ ($2 \leq i \leq k - 1$) there exists a complete splitting off if and only if there do not exist the following subsets $X_1, X_2, \ldots, X_m \subset V - s$ such that the following holds:*

a) *$X_j \cap X_l = \{t\}$ with a fix node $t \in V - s$ for all possible pairs X_j, X_l,*

b) *$b_G(X) < d_G(s, X_j - t)$ for all possible j,*

c) *$d_G(s,t) > (k - i) + d_G(s, V - s - \cup_{j=1}^{m} X_j) + \sum_{j=1}^{m} b_G(X_j)$.*

Proof. Let us consider a small node s. The necessity of the condition is obvious, because the sets X_j give that the maximum number of edges between s and t which can be split off with other edges is at most $d_G(s, V - s - \cup_{j=1}^{m} X_j) + \sum_{j=1}^{m} b_G(X_j)$, but by **c)**, for a complete splitting off, we would need more (i).

Sufficiency. Let us consider a maximal splitting off with respect to the number of split edges, moreover in the resulting graph G'_s the number of neighbours of s is maximal, that is, $|\Gamma_{G'_s}(s)|$ is maximal, moreover, if $|\Gamma_{G'_s}(s)| \geq 2$, then the tight set containing $|\Gamma_{G'_s}(s)|$ is maximal. By Observation 2.6 this tight set gives the fact that there is no more splitting off at s.

If we managed to split off i pairs of edges at s, then it is the inverse operation of (3), so there exists a complete splitting off. If not, then we will find the sets X_j.

Lemma 2.10. *G'_s is obtained by a maximal but not complete splitting off at s. If s has only one neighbour t in G'_s, then there exists a split edge which is disjoint from t.*

Proof. Let us suppose that we split maximum number l pairs of edges and one endnode of every split edge is t. Since this splitting off is not complete, $l < i$. Then in the original graph G:

$$d_G(s,t) = d_G(s) - l = k + i - l > k,$$

which contradicts the condition in (2) for the set $\{s,t\}$. □

Lemma 2.11. *G'_s is obtained by a maximal but not complete splitting off at s. If s has at least 2 neighbours in G'_s, then let P_{max} denote the maximal tight set which covers all the neighbours of s in G'_s.*

Then there exists a split edge which is disjoint from the nodes of P_{max}.

Proof. Let us consider a maximal splitting off at node s, and let j denote the number of split edges in P_{max} and let l denote the number of edges with exactly one endnode in P_{max}. Let us suppose that there are no other split edges. Then:

$$\gamma_G(P_{max}+s) = \gamma_{G'_s}(P_{max}) + j + l + (k+i-2(j+l)) = \gamma_{G'_s}(P_{max}) + k + (i-(j+l))$$
$$> k|P_{max}| - (k+1) + k = k|P_{max}+s| - (k+1).$$

□

Lemma 2.12. *Let G be an admissible graph and X is a maximal tight set in it which contains the distinct nodes c_1, c_2. Let d be a node in $V - X$. Then there is no tight set which contains c_i and d for $i = 1$ or 2.*

Proof. We may suppose that there is a tight set P containing c_1 and d. According to Lemma 2.3 $P \cap X = \{c_1\}$ because X is maximal. By Lemmas 2.3 and 2.4 we can see that there is no tight set containing c_2 and d. □

Lemma 2.13. *Let G'_s denote the graph obtained by splitting off some edges at s. Let $as, bs \in E(G'_s)$ $(a \neq b)$ and uv be a split edge in G'_s such that the maximal tight set P does not contain s, u, v but a, b.*

If sa and sb are both multiple edges, then instead of spliting off su, sv we can split off (sa, su and sa, sv) or (sb, su and sb, sv).

If there is a third distinct node c in P with edge sc in G'_s such that su, sc is not splittable, then instead of spliting off su, sv we can split off (sa, su and sb, sv) or (sb, su and sa, sv).

Proof. According to Lemma 2.12, we can see that there are no tight sets which would be obstacles to the 'splitting off's in every case, it is remained to see, that we can apply the corresponding two 'splitting off's at the same time. If not, then there is a set with too many induced edges, which contains the two new split edges. But it means, that, before this, there is a tight set containing a, b, u, v but not s, which contradicts the maximality of P according to Lemma 2.3.

□

Case 1. Let us suppose that $|\Gamma_{G'_s}(s)| \geq 3$, let a_1, a_2, a_3 denote three of these nodes.

According to Observation 2.7, there exists a maximal tight set P containing $\Gamma_{G'_s}(s)$. By Lemma 2.11, there is a split edge uv disjoint from P. By Lemma 2.13, it follows that the splitting off we consider is not maximal, a contradiction.

Case 2. Let us suppose that $|\Gamma_{G'_s}(s)| = |\{t, z\}| = 2$. If $d_{G'_s}(s,t) \geq 2$ and $d_{G'_s}(s,z) \geq 2$, then, as above:

According to Observation 2.7 there exists a maximal tight set P containing $\Gamma_{G'_s}(s)$. By Lemma 2.11 there is a split edge uv disjoint from P. By Lemma 2.13, it follows that the splitting off we consider is not maximal, a contradiction.

We may suppose, that the multiplicity of edge sz is in G'_s exactly one, that is, it is not a multiple edge. We have: $d_{G'_s}(s,t) \geq k + i - 2(i-1) - 1 = k - i + 1 \geq 2$.

Let $u \in V$ be an arbitrary node which is incident to some split edge which is disjoint from t. Let P_u be the maximal tight set which does not contain s but u and t and contains the minimal number of split edges that are disjoint from t. Let P_z be the maximal tight set which does not contain s but z and t and contains the minimal number of split edges that are disjoint from t. We will see that these sets give the setsystem in Theorem 2.9.

Case 3. Let us suppose that $|\Gamma_{G'_s}(s)| = |\{t\}| = 1$. We have: $d_{G'_s}(s,t) \geq k+i-2(i-1) = k-i+2 \geq 3$.

Here, let us define the sets P_u as in the above case.

In the above two cases the setsystem of sets P_u is called the **flower** of node s. A set P_u is called a **petal** of the flower or node s.

Proposition 2.14. *There is no split edge in an arbitrary petal which is disjoint from t.*

Proof. Let us suppose on the contrary that there is a split edge ab.

Let us consider P_z. First let us suppose that a, b, z are three distinct nodes. Since splitting off st, sz intead of sa, sb would result in a maximal splitting off with three neighbours of s, there is a tight set X which is an obstacle to it, that is, it contains t, z and exactly one of a and b. $X \cap P_z$ contains a smaller number of split edges, which gives a contradiction. Now let us suppose that $a = z$. The justification is just the same as above.

Let us consider P_u. Now we have a split edge uv such that $v \notin P_u$ (if not, then split off st, su instead of su, sv results in a maximal splitting off with one more remaining neighbour of s). $P_v \cap P_u = \{t\}$ because of Lemma 2.3. Splitting off su, st and sv, sa instead of sa, sb and su, sv would result in a maximal splitting off with one more neighbour of s, so there exists an obstacle to it, that is, the set X containing a, u, v, t, not s which is tight in G'_s. But then $X \cap P_z$ contains a smaller number of split edges, which gives a contradiction.

□

Proposition 2.15. *Let us suppose we defined P_u and P_v for nodes u, v. Then $P_u = P_v$, or $P_u \cap P_v = \{t\}$.*

Proof. By Lemma 2.14 $P_u \subseteq P_v$ can not be. If $P_u \neq P_v$ and $|P_u \cap P_v| \geq 2$, then by Lemma 2.3 $d_{G'_s}(P_u, P_v) = 0$ and $P_u \cup P_v$ is tight. Since it does not contain any split edge disjoint from t, it contradicts the choice of P_u by maximality.

□

We state that the sets P_u satisfy the condition of the theorem. Now **a)** and **b)** follows. **c)** is implied by the fact that the maximal splitting off that we consider is not complete and the number of the neighbours of s after the maximal splitting off is $\Gamma_{G'_s}(s) \leq 2$. This is the end of the proof of Theorem 2.9.

From now on, if s is a small node which does not admit a complete splitting off, then we have a flower with it, and it can be the type of Case 2 (that is, which comes from a maximal splitting off with two remaining neighbours of s) then we will refer to it as a first type flower, or it can be the type of Case 3 (that is, s has one remaining neighbour), then we refer to it as a second type flower.

Let us fix the set-system $\mathcal{P}_s$ that we defined to every small node without a complete splitting off. (That is, we consider a special flower whose number of its petals is minimal, moreover the petals are maximal sets.) Let $T(s)$ denote the node t for every s, and it is called the **blocking node** of s, that is, it is the center node of the flower of s.

It follows from Theorem 2.9 that every small node which does not admit a complete splitting off has a unique blocking node. But this is not important for our proof, let us fix one flower, and it has a unique centre.

We have the following.

Lemma 2.16 (number of petals). *For any small node s which does not admit a complete splitting off: $|\mathcal{P}_s| \geq 3$.*

Proof. If the flower of s is of first type, then there exists a maximal tight set P_{max} in graph G'_s which was obtained after a maximal splitting off which contains P_z as a subset. By Proposition 2.14, there exists a split edge ab disjoint from P_{max}. P_z together with P_a and P_b are three different petals.

If the flower of s is of second type, then it is enough to see, that there are at least two split edges not incident to t (the centre of the flower). By Lemma 2.10, there exists one split edge like this. Let us suppose that there is no other split edge disjoint from t. Then: let m be the number of split edges incident to t in set $P_u \cup P_v$, moreover let l be the number of split edges incident to t with the other endnode not in $P_u \cup P_v$. Since $P_u \cup P_v$ is a tight set in G'_s, $b_G(P_u \cup P_v) = m + 1$. As we have a maximal but not complete splitting off, $m + l + 1 < i$. So, $b_G(P_u \cup P_v + s) = b_G(P_u \cup P_v) + k - d_G(s, P_u \cup P_v) = m + 1 + k - (k + i - l) = m + l + 1 - i < 0$, which is a contradiction. $\square$

Lemma 2.17 (flower-lemma). *If petal P of s contains the small node s' such that $T(s) = T(s')$ and P' is a petal of s' and $P' - P \neq \emptyset$, then $s \in P'$.*

Proof. Let us suppose that $s \notin P'$.

First case. $|P \cap P'| \geq 2$.

Let n be the number of split edges in $P \cap P'$ in graph $G'_{s'}$ (since this set is a subset of a petal, there cannot be split edges not incident to t), moreover let m be the number of split edges in $P' - P$ in the same graph. By **a)**, $b_G(P') = n + m$ and $b_G(P \cap P') \geq n$, $d_G(P, P') \geq m$.

Now we have $b_G(P) \leq b_G(P \cup P') = b_G(P) + b_G(P') - b_G(P \cap P') - d_G(P, P') \leq b_G(P) + n + m - n - m = b_G(P)$. Which means that $P \cup P'$ contradicts to the maximality of petal P of s.

Second case. $|P \cap P'| = 1$.

Let m be the number of split edges in P' in graph $G'_{s'}$ (these are all incident to t since P' is a petal). This gives $b_G(P') = m$, $d_G(P, P') \geq m + 1$ (+1 follows from the fact that every petal P_0 of s' contains a node which is incident to a split edge disjoint from t in graph $G'_{s'}$ which is obtained by a maximal splitting off. That is, there is at least one more edge between s' and $P_0 - t$ in G.)

So, $b_G(P) \leq b_G(P \cup P') = b_G(P) + b_G(P') + 1 - d_G(P, P') \leq b_G(P) + m + 1 - (m+1) = b_G(P)$. Which means that $P \cup P'$ contradicts to the maximality of petal P of s.

□

Let us consider a petal P of the flower of a small node s. If there is a small node $s' \in P$ whose blocking node is also $T(s) = t$, then let us define the following (the flower of s' is denoted by $X_1, X_2, \dots X_m$):

$$\tau(s') = \min_{l=1,2,\dots,m} |\{s'' \in (\cup_{j=1}^{m} X_j - X_l) \cap P : T(s'') = t\}|.$$

According to the Flower lemma, there is a small node s_0 in P such that $\tau(s_0) = 0$. This means that s_0 has at least two petals that are entirely in P and do not contain any small node with blocking node $T(s) = t$.

It is clear that a tight set X either has two nodes with $k-1$ parallel edges, or $d(v, X - v) \geq k$ for an arbitrary $v \in X$.

Proposition 2.18. *Let X_1 and X_2 be two petals of s_0 that do not contain any small node with blocking node t. Then $d_G(t, X_1 - t) + d_G(t, X_2 - t) \geq k$*

Proof. $d_G(t, X_1 - t) + d_G(t, X_2 - t) \geq d_{G'_s}(t, X_1 - t) + d_{G'_s}(t, X_2 - t) - (i-1) \geq d_{G'_s}(t, X_1 - t) + d_{G'_s}(t, X_2 - t) - (k-2) \geq 2(k-1) - (k-2) = k$. □

Let us give a lower bound on the edges that are incident to some blocking node $T(s) = t$ for some s and whose other endnode is not a small node (that is, has degree at least $2k$) or a small node whose blocking node is also t. Let $\Delta(t)$ denote this number for blocking node t.

Let us consider an arbitrary blocking node t that is a blocking node of some small nodes. Let s be a small node like that. Let us consider three petals of s P_1, P_2, P_3 (they exist by lemma 'number of petals'). We may exchange some indices to get one of the following four cases.

Case A. There are no small nodes in the above petals blocking by t.

Then $\Delta(t) \geq \sum_{j=1}^{3} d_G(t, P_j - t) \geq \sum_{j=1}^{3} d_{G'_s}(t, P_j - t) - (i-1) \geq 3(k-1) - (k-2) = 2k - 1$.

Case B. P_1 contains at least one small node blocking by t, P_2, P_3 do not.

By Proposition 2.18: $\Delta(t) \geq k + d_G(t, P_2 - t) + d_G(t, P_3 - t) \geq k + d_{G'_s}(t, P_2 - t) + d_{G'_s}(t, P_3 - t) - (i-1) \geq k + (k-1) + (k-1) - (k-2) = 2k$.

Case C. P_1, P_2 contains at least one small node blocking by t, P_3 does not.

By Proposition 2.18: $\Delta(t) \geq 2k + d_G(t, P_3 - t) \geq 2k + d_{G'_s}(t, P_3 - t) - (k-2) \geq 2k + (k-1) - (k-2) = 2k + 1$.

Case D. P_1, P_2, P_3 contains at least one small node blocking by t.

Then: $\Delta(t) \geq 3k$.

We have in every case: $\Delta(t) \geq 2k - 1$.

We saw that, for a small node s, if $d(s) = k+i$ and $T(s) = t$, then $d_G(s, t) \geq k-i+1$. Let n_{k+i} denote the number of nodes with degree $(k+i)$, $2 \leq i \leq k-1$.

If there is a node with degree less than $k+2$, than a complete splitting off is applicable at it by Theorem 2.8. To prove Theorem 2.1, let us suppose that every node has degree at least $k+2$. Let $T \subseteq V$ be the set of the blocking nodes for every small node. Let us suppose that every small node has a blocking node, that is, it does not admit a complete splitting off.

Now we have:

$$2|E| = 2(k|V| - (k+1)) = 2k|V| - 2k - 2 = 2|E| \geq$$

$$\geq \sum_{t \in T} d_G(t) + \sum_{i=2}^{k-1} (k+i) n_{k+i} + 2k(|V| - |T| - \sum_{i=2}^{k-1} n_{k+i}) \geq$$

$$\geq (2k-1)|T| + \sum_{i=2}^{k-1} (k-i+1) n_{k+i} + \sum_{i=2}^{k-1} (k+i) n_{k+i} + 2k(|V| - |T| - \sum_{i=2}^{k-1} n_{k+i}) =$$

$$= 2k|V| - |T| + \sum_{i=2}^{k-1} n_{k+i} \geq 2k|V|.$$

$\sum_{i=2}^{k-1} n_{k+i} \geq |T|$ holds because we fixed one blocking node to every small node. So we arrive at a contradiction which means there exists a small node at which a complete splitting off is applicable.

The following theorem can be proved by a slight modification of the above computation.

Theorem 2.19. *If G is k-stiff and is not K_2^{k-1}, then there are at least three nodes such that a complete splitting off is applicable at them.*

An open question is how we can quickly give the k edge-disjoint spanning trees after adding an arbitrary new edge if we know how the graph is built up by the operations.

The following theorem characterizes the connected graphs that are the union of k forests after adding an arbitrary edge.

Theorem 2.20. *Graph G is the union of k spanning trees after adding an arbitrary edge if and only if it is a connected subgraph of a k-stiff graph.*

Proof. It is straightforward that any connected subgraph of a k-stiff graph has this property.

By the theorem of Nash-Williams [12], $G = (V, E)$ is the union of k (not necessarily edge-disjoint) spanning trees after adding an arbitrary edge if and only if it is connected and $\gamma_G(X) \leq k|X| - (k+1)$ for all $X \subseteq V$. We claim that if $|E| < k|V| - (k+1)$, then we can add an edge e such that $G + e$ is also the union of k forests after adding an arbitrary edge. This will prove the theorem.

Let us consider a maximal tight set X and node $u \in X$, other node $v \notin X$. If we cannot add edge uv, then there exists a tight set Y containing u and v. According to Lemma 2.4, there is a node a in $X - Y$ and a node b in $Y - X$ such that we may add edge ab to G. □

We remark about constructive characterizations in general that the fact that one obtains the required type of graphs by the operations is always much easier to prove than the other direction: in a graph which has the required property we can always find a node where we can perform an inverse operation such that after it we get a smaller graph which also has the required properity.

In the proof of the theorem of Lovász or that of Mader one finds the following: an inverse operation can always be performed at a node with suitable degree such that it gives a smaller graph of the same kind.

This was also the case for $k = 2$ with k-stiff graphs, but not if k is greater than 2. Here we can easily find a node with the degree we are looking for (this is not the case in the theorems of Lovász and Mader) but in general there is no way to perform the inverse operation at that node.

(We remark that there are other constructive characterizations where the inverse operation is not only considering one single node and performing some operation with the edges but something different as in the case of Theorem 1.1.)

We finish this section by putting the question if there is a similar constructive characterization of graphs that are the union of k edge-disjoint spanning trees after adding arbitrary l number of edges. We mention that the basic lemmas we used in our proof are valid if and only if $2l \leq k$, furthermore there is no graph on three nodes that are the union of k edge-disjoint spanning trees after adding l number of arbitrary edges if $2l > k$.

3 Construction of $(k, 1)$-edge-connected digraphs

In a directed graph by **splitting off** a pair of edges $e = uz, f = zv$ we mean the operation of replacing e and f by a new directed edge from u to v. Suppose that the in-degree and the out-degree of z is the same, that is, $\varrho(z) = \delta(z)$. By a **complete splitting** at z we mean the following operation: pair the edges entering and leaving z and split off all these pairs.

For non-negative integers $l \leq k$, we call a digraph D **(k,l)-edge-connected** (in short, (k,l)-ec) if D has a node s so that there are k (resp., l) edge-disjoint paths from s to every other node (there are l edge-disjoint paths from every node to s). If there is an exceptional node z for which the existence of these edge-disjoint paths is not required, we say that D is **(k,l)-edge-connected apart from** z. When the role of s is emphasized, we say that D is (k,l)-ec with respect to root-node s. (k,k)-edge-connectivity is abbreviated by k-edge-connectivity and $(k,0)$-edge-connectivity is sometimes called rooted k-edge-connectivity. Note that by Menger's theorem a digraph is (k,l)-ec if and only if

$$\varrho(X) \geq k \text{ for every subset } \emptyset \subset X \subseteq V - s \tag{1}$$

and

$$\delta(X) \geq l \text{ for every subset } \emptyset \subset X \subseteq V - s \tag{2}$$

where $\varrho(X) := \varrho_D(X)$ and $\delta(X) := \delta_D(X)$ denote the number of edges entering and leaving the subset X, respectively.

We say an undirected graph $G=(V,E)$ is (k,l)**-partition-connected** if there are at least $k(t-1)+l$ edges connecting distinct classes of every partition of V into t $(t\geq 2)$ non-empty subsets.

The following result exhibits a link between the two concepts. It is a special case of a general orientation theorem appeared in [1].

Theorem 3.1. *Let $0\leq l\leq k$ be integers. An undirected graph $G=(V,E)$ has a (k,l)-edge-connected orientation if and only if G is (k,l)-partition-connected.*

Mader's directed splitting off theorem [10] is as follows.

Theorem 3.2. *Let $D=(U+z,E)$ be a digraph which is k-edge-connected apart from z. If $\varrho(z)=\delta(z)$, then there is a complete splitting at z resulting in a k-ec digraph on node-set U.*

This result has been extended in [2] as follows.

Theorem 3.3. *Let $D=(U+z,E)$ be a digraph which is (k,l)-edge-connected apart from z. If $\varrho(z)=\delta(z)$, then there is a complete splitting at z resulting in a (k,l)-ec digraph on node-set U.*

We need the following corollary of Theorem 3.2.

Theorem 3.4. *Let $D=(U+z,E)$ be a digraph which is*

$$(k,0)\text{ -ec apart from } z\ (k\geq 1) \text{ with respect to a root node } s\in V. \tag{3}$$

If $\varrho(z)>\delta(z)$, then there are $\varrho(z)-\delta(z)$ edges entering z so that (3) continues to hold after discarding these edges. If $\varrho(z)=\delta(z)$, then there is a complete splitting at z preserving (3).

Proof. For every node $v\in U+z$ for which $\varrho(v)>\delta(v)$, add $\varrho(v)-\delta(v)$ parallel edges from v to s. In the resulting digraph D' clearly $\varrho'(v)\leq\delta'(v)$ holds for every node $v\in U-s$. Hence $\delta'(X)\geq\varrho'(X)=\varrho(X)\geq k$ holds for every subset $X\subseteq V-s, X\neq\{z\}$, that is, D' is k-ec apart from z.

By Theorem 3.2 there is a complete splitting at z resulting in a k-ec digraph. It follows that in case $\varrho(z)=\delta(z)$ this complete splitting, when applied to D, preserves (3). If $\varrho(z)>\delta(z)$, then there are $\varrho(z)-\delta(z)$ edges entering z such that their pairs at the complete splitting are necessarily newly added edges from z to s. Therefore these edges can be deleted from D without destroying (3). □

W. Mader used Theorem 3.2 to derive Theorem 1.3 on the constructive characterization of k-ec digraphs. Analogously, Theorem 3.4 may be used to derive the following.

Theorem 3.5. *A directed graph $D=(V,E)$ is $(k,0)$-edge-connected if and only if D can be obtained from a single node by the following two operations: (i) Add a new edge, (ii) pinch j $(0\leq j\leq k-1)$ existing edges with a new node z, and add $k-j$ new edges entering z.*

Given these constructive characterizations of (k,k)-ec and $(k,0)$-ec digraphs, one may formulate the following general conjecture.

Conjecture 3.6. *A directed graph D is (k,l)-edge-connected ($0 \leq l \leq k-1$) if and only if it can be built up from a node by the following two operations: (j) add a new edge, (jj) pinch i $(l \leq i \leq k-1)$ existing edges with a new node z, and add $k-i$ new edges entering z and leaving existing nodes.*

Conjecture 3.7. *An undirected graph G is (k,l)-partition-connected if and only if it can be built up from a node by the following two operations: (j) add a new edge, (jj) pinch i $(l \leq i \leq k-1)$ existing edges with a new node z, and add $k-i$ new edges connecting z with existing nodes.*

By Theorem 3.1 the second conjecture follows from the first one. Theorem 3.5 asserts the truth of this conjecture for $l=0$. The conjecture was proved for $l=k-1$ in [4]. Here we verify the conjecture for $l=1$. The proof relies on the following lemma.

Lemma 3.8. *Let $D=(V,E)$ be a $(k,1)$-edge-connected digraph which is minimal in the sense that the deletion of any edge destroys $(k,1)$-edge-connectivity ($k \geq 2, |V| \geq 2$). Then D has a node z with $k=\varrho(z)>\delta(z)$ for which there is a set F of $\varrho(z)-\delta(z)$ edges entering z so that $D-F$ is $(k,1)$-edge-connected apart from z.*

Proof. By (2), there is an edge e entering s. Since (1) cannot break down by deleting e, it follows from the minimality of D that e leaves a subset $X \subset V-s$ for which $\delta(X)=1$. Since $\varrho(X) \geq k \geq 2$, there must be a node z in X for which $\varrho(z)>\delta(z)$. Let us choose such a node z so that the distance of s from z is as large as possible.

Proposition 3.9. *Let F be a subset of at most $k-1$ edges entering z. Then $D' := D-F$ satisfies (2).*

Proof. Assume indirectly that there is a subset $X \subseteq V-s$ for which $\delta_{D'}(X)=0$. As $\delta(X) \geq 1$, the elements of set of edges of D leaving X are all in F. Therefore $\delta(X) \leq |F| < k$ and, by $\varrho(X) \geq k$, X must contain a node z' for which $\varrho(z')>\delta(z')$. Since the head of each edge leaving X is z, we obtain that each path from z' to s must go through z contradicting the maximal-distance choice of z. □

Proposition 3.10. *$\varrho(z)=k$.*

Proof. By Proposition 3.9 property (2) cannot break down when an edge entering z is left out. Hence the minimality of D implies that every edge entering z enters a subset $X \subseteq V-s$ for which $\varrho(X)=k$. If X and Y are two subsets of V containing z for which $k=\varrho(X)=\varrho(Y)$, then $\varrho(X)+\varrho(Y) \geq \varrho(X \cap Y)+\varrho(X \cup Y) \geq k+k$ from which $\varrho(X \cap Y)=k$ follows. This implies that there is a unique smallest subset Z containing z for which $\varrho(Z)=k$ such that every edge entering z enters Z as well. But then the in-degree of z cannot exceed k and hence $\varrho(z)=k$ as D is $(k,1)$-ec. □

□

By Theorem 3.4 there is a subset F of edges of D entering z for which $|F| = \varrho(z) - \delta(z) < k$ and the digraph $D - F$ is $(k, 0)$-ec. Now Proposition 3.9 implies that $D - F$ is actually $(k, 1)$-ec, completing the proof of the lemma.

Theorem 3.11. *A digraph $D_0 = (V, E)$ is $(k, 1)$-edge-connected if and only if D_0 can be built up from a node by the following two operations: (j) add a new edge, (jj) pinch i $(1 \leq i \leq k-1)$ existing edges with a new node z, and add $k - i$ new edges entering z and leaving existing nodes.*

Proof. It is straightforward to see that the two operations preserve $(k, 1)$-edge-connectivity. To prove the reverse direction we use induction on the number of edges. If there is an edge e whose deletion preserves $(k, 1)$-edge-connectivity, then $D_0 - e$ has a required construction by the inductive hypothesis from which the construction of D_0 can be obtained by giving back e (operation (j)).

Therefore we may assume that D_0 is minimally $(k, 1)$-edge-connected with respect to edge deletion. We are done if $|V| = 1$ so assume that $|V| \geq 2$.

By Lemma 3.8 there is a node z with $k = \varrho(z) > \delta(z)$ for which there is a subset F of $\varrho(z) - \delta(z)$ edges entering z so that the digraph $D_0 - F$ is $(k, 1)$-ec apart from z By Theorem 3.3 there is a complete splitting at z so that the resulting digraph $D_1 = (V - z, E_1)$ is $(k, 1)$-ec. By the inductive hypothesis D_1 can be constructed from a node by the two given operations. But then D_0 is also constructible this way as D_0 arises from D_1 by operation (ii). $\square$

By combining this result with Theorem 3.1 we obtain the the following special case of Conjecture 3.7.

Theorem 3.12. *An undirected graph G is $(k, 1)$-partition-connected if and only if it can be built up from a node by the following two operations: (j) add a new edge, (jj) pinch i $(1 \leq i \leq k-1)$ existing edges with a new node z, and add $k - i$ new edges connecting z with existing nodes.*

Acknowledgement. The authors thank Zoltán Király for fruitful discussions on the topic.

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