Combined Connectivity Augmentation and Orientation Problems

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Abstract. Two important branches of graph connectivity problems are connectivity augmentation, which consists of augmenting a graph by adding new edges so as to meet a specified target connectivity, and connectivity orientation, where the goal is to find an orientation of an undirected or mixed graph that satisfies some specified edge-connection property. In the present work an attempt is made to link the above two branches, by considering degree-specified and minimum cardinality augmentation of graphs so that the resulting graph has an orientation satisfying a prescribed edge-connection requirement, such as (k,l)-edge-connectivity. Our proof technique involves a combination of the supermodular polyhedral methods used in connectivity orientation, and the splitting off operation, which is a standard tool in solving augmentation problems.

1 Introduction

In a connectivity augmentation problem the goal is to augment a graph or digraph by adding a cardinality- or degree-constrained new graph so as to meet a specified target connectivity. Initial deep results of the area are due to Lovász [6] and to Watanabe and Nakamura [10] on augmenting a graph to make it k-edge-connected. Since then, augmentation results for many different connectivity properties of graphs and digraphs have been proved, employing various versions of the splitting off technique, which was originally introduced by Lovász [6] and subsequently developed by Mader [7] and others.

In a connectivity orientation problem one is interested in the existence of an orientation of an undirected graph that satisfies some specified edge-connection properties. For example, classical results of Nash-Williams [8] and of Tutte [9] characterize graphs having k-edge-connected and rooted k-edge-connected orientations. For a common generalization of their results, call a digraph $D = (V, A)$

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(k,l)-edge-connected for non-negative integers $k \geq l$ if there is a node $s \in V$ such that there are k edge-disjoint paths from s to any other node, and there are l edge-disjoint paths to s from any other node. Then (k,k)-edge-connectivity is equivalent to k-edge-connectivity, and $(k,0)$-edge-connectivity is equivalent to rooted k-edge-connectivity from some node s. Good characterizations of undirected and mixed graphs having a (k,l)-edge-connected orientation were given in [1] and [3], with the help of submodular flows and related polyhedral methods.

In this paper an attempt is made to link these two branches of connectivity problems by studying combined augmentation and orientation problems. For example we characterize undirected and mixed graphs that can be augmented by an appropriate degree-specified undirected graph so as to have a (k,l)-edge-connected orientation. Another new result concerns the minimum number of new edges whose addition to an initial undirected graph results in a graph admitting a (k,l)-edge-connected orientation. Our proof methods for these characterizations combine the splitting off technique used in connectivity augmentation with extensions of the supermodular polyhedral techniques used in [3] to solve connectivity orientation problems. Since these methods are constructive from an algorithmic point of view, the proofs presented here give rise to polynomial algorithms for finding a feasible augmentation.

The results are presented in the customary framework for connectivity orientations. We consider graphs with no loops, but possibly with multiple edges. Given a graph $G = (V,E)$ and a set function $h : 2^V \to \mathbb{Z}$, an orientation $\vec{G}$ of G is said to *cover* h if $\varrho_{\vec{G}}(X) \geq h(X)$ for every set $X \subseteq V$, where $\varrho_{\vec{G}}(X)$ denotes the number of edges of the digraph $\vec{G}$ entering the set X. Throughout the paper we assume that $h(\emptyset) = h(V) = 0$. The h-orientation problem is to find an orientation of G that covers h. For general h this includes **NP**-complete problems, so special classes of set functions must be considered. A set function h is called *crossing G-supermodular* with respect to a given graph $G = (V,E)$ if

$$h(X) + h(Y) \leq h(X \cap Y) + h(X \cup Y) + d_G(X,Y) \tag{1}$$

for every crossing pair (X,Y) (where the sets $X, Y \subseteq V$ are *crossing* if none of $X - Y$, $Y - X$, $X \cap Y$ and $V - (X \cup Y)$ are empty), and $d_G(X,Y)$ is the number of edges in E connecting $X - Y$ and $Y - X$. As in [3], we restrict our attention to crossing G–supermodular set functions. The *augmentation problem* corresponding to h-orientation is the following: given a graph G, find a graph G' (either with specified degrees, or with minimum number of edges), so that $G + G'$ has an orientation covering h.

It was shown in [1] that for a graph G and a non-negative crossing G–supermodular set function h the h-orientation problem can be solved in polynomial time. In Sect. 3 we solve the corresponding degree-specified and minimum cardinality augmentation problem, as well as minimum cost augmentation for node-induced cost functions.

These results are used in Sect. 4 to augment a graph to obtain one admitting a (k,l)-edge-connected orientation, and we show that in this special case the characterizations can be further simplified. The theorems obtained can also be interpreted independently of orientations. A graph G is called *(k,l)-tree-connected* if any graph obtained by deleting l edges from G contains k edge-disjoint spanning trees. It is known that if $k \geq l$, then (k,l)-tree-connected graphs are exactly those that have a (k,l)-edge-connected orientation; thus we can solve the (k,l)-tree-connectivity augmentation problem.

In [3], submodular flows were used to solve the h-orientation problem when h is a crossing G–supermodular set function that can have negative values; this implies for example that we can find a (k,l)-edge-connected orientation of a mixed graph M. In Sect. 5 we generalize this result by considering the h-orientation problem for positively crossing G–supermodular functions, and by solving the corresponding degree-specified augmentation problem. The proof exploits the TDI-ness of a system closely related to the intersection of two base polyhedra.

2 Preliminaries

A *family of sets* is a collection of subsets of the ground set V, with possible repetition. If every member of a family $\mathcal{F}$ is replaced by its complement, the resulting family is denoted by $\overline{\mathcal{F}}$. For an element $v \in V$, $d_{\mathcal{F}}(v)$ denotes the number of members of $\mathcal{F}$ containing v. A *composition* of a set $X \subseteq V$ is a family $\mathcal{F}$ for which $d_{\mathcal{F}} - d_{\{X\}}$ is constant. A composition of V is called a *regular family*. The *covering number* of a family $\mathcal{F}$ is $\min_{v \in V} d_{\mathcal{F}}(v)$.

For a function $x : V \to \mathbb{R}$ and a set $Z \subseteq V$, and analogously for a set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ and a family $\mathcal{F}$, we use the notations $x(Z) = \sum_{v \in Z} x(v)$ and $p(\mathcal{F}) = \sum_{X \in \mathcal{F}} p(X)$. The *upper truncation* of p is

$$p^{\wedge}(Z) := \max\{p(\mathcal{F}) \mid \mathcal{F} \text{ is a partition of } Z\} . \tag{2}$$

If p is intersecting supermodular, then $p^{\wedge}$ is fully supermodular. If p is crossing supermodular, then so is $p^{\wedge}$. To the set function p we associate the polyhedra

$$C(p) := \{x : V \to \mathbb{R} \mid x(Z) \geq p(Z)\ \forall Z \subseteq V\} , \tag{3}$$

$$B(p) := \{x : V \to \mathbb{R} \mid x(V) = p(V);\ x(Z) \geq p(Z)\ \forall Z \subseteq V\} . \tag{4}$$

Clearly, $C(p) = C(p^{\wedge})$. A polyhedron is a *contra-polymatroid* if it equals $C(p)$ for some monotone increasing fully supermodular function p; it is a *base polyhedron* if it can be represented as $B(p)$ for some fully supermodular function p. The following theorem of Fujishige [5] deals with base polyhedra given by crossing supermodular set functions.

Theorem 1 (Fujishige [5]). *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a crossing supermodular function. Then $B(p)$ is non-empty if and only if*

$$\sum_{i=1}^{t} p(X_i) \leq p(V) ,$$

$$\sum_{i=1}^{t} p(\overline{X_i}) \leq (t-1)p(V)$$

both hold for every partition $\{X_1, \ldots, X_t\}$. Furthermore, if $B(p)$ is non-empty, then it is a base polyhedron. □

Let $G = (V,E)$ be a graph. For a set $X \subseteq V$, $i_G(X)$ denotes the number of edges $uv \in E$ with $u, v \in X$. An important property of i_G is that if a set function h is crossing G–supermodular, then $h + i_G$ is crossing supermodular. For a family $\mathcal{F}$ of sets we define

$$e_G(\mathcal{F}) := \max\left\{\varrho_{\vec{G}}(\mathcal{F}) \mid \vec{G} \text{ is an orientation of } G\right\} .$$

Note that $e_G(\mathcal{F})$ can be easily computed since we can orient the edges independently. For partitions it equals the number of cross-edges; more generally, if $\mathcal{F}$ is a regular family with covering number α, then

$$e_G(\mathcal{F}) = \alpha|E| - \sum_{X \in \mathcal{F}} i_G(X) . \tag{5}$$

A family $\mathcal{F}$ is *cross-free* if it has no crossing members. Simple examples are partitions and co-partitions; in fact, it is easy to show that these are the only minimal regular cross-free families:

Proposition 1. *Every regular cross-free family decomposes into partitions and co-partitions.* □

3 Non-negative Crossing G–Supermodular Set Functions

The first result is a theorem on the degree-specified augmentation problem. The characterizations given are good in the sense that they provide an easily verifiable certificate if the augmentation is impossible. Moreover, the proof is constructive and gives rise to a polynomial algorithm, since it refers to polyhedral and splitting off problems that can be solved in polynomial time.

Theorem 2. *Let $G = (V,E)$ be a graph, $h : 2^V \to \mathbb{Z}_+$ a non-negative crossing G–supermodular set function on V, and $m : V \to \mathbb{Z}_+$ a degree specification with $m(V)$ even. There exists an undirected graph $G' = (V, E')$ such that $G + G'$ has an orientation covering h and $d_{G'}(v) = m(v)$ for all $v \in V$, if and only if the following hold for every partition $\mathcal{F}$:*

$$\frac{m(V)}{2} \geq h(\mathcal{F}) - e_G(\mathcal{F}) , \tag{6}$$

$$\min_{X \in \mathcal{F}} m(\overline{X}) \geq h(\mathcal{F}) - e_G(\mathcal{F}) , \tag{7}$$

$$\frac{m(V)}{2} \geq h(\overline{\mathcal{F}}) - e_G(\overline{\mathcal{F}}) , \tag{8}$$

$$\min_{X \in \mathcal{F}} m(\overline{X}) \geq h(\overline{\mathcal{F}}) - e_G(\overline{\mathcal{F}}) . \tag{9}$$

Proof. To see the necessity of these conditions, observe that $m(V)/2$ is the number of new edges, while $h(\mathcal{F}) - e_G(\mathcal{F})$ measures the deficiency of a partition $\mathcal{F}$, hence (6) simply requires that the deficiency of a partition should not exceed the number of new edges. The necessity of (7) is also straightforward since each new cross-edge must have an endnode in $\overline{X}$, so the number of new cross-edges, which should be at least the deficiency of $\mathcal{F}$, is at most $m(\overline{X})$. The necessity of (8) and (9) can be seen analogously.

To prove sufficiency, add a new node z to the set of nodes, and for every $v \in V$ add $m(v)$ parallel edges between v and z; the resulting graph is denoted by $G_0 = (V_0, E_0)$. Define the following extension of the set function h:

$$h_0(z) = h_0(V) := \frac{m(V)}{2}\,,$$
$$h_0(X+z) = h_0(X) := h(X) \qquad \text{if } \emptyset \neq X \subset V\,.$$

The proof consists of finding an orientation of G_0 that covers h_0, and then splitting off the directed edges at z so that the resulting digraph on the ground set V covers h. To find an orientation covering h_0, we resort to a lemma that is a standard tool for orientation problems:

Lemma 1. *For a given vector $x : V_0 \to \mathbb{Z}_+$, there is an orientation $\vec{G_0}$ of G_0 such that $\varrho_{\vec{G_0}}(v) = x(v)$ for every $v \in V_0$, if and only if $x(V_0) = |E_0|$ and $x(Z) \geq i_{G_0}(Z)$ for every $Z \subseteq V_0$.*

Proof. The necessity is obvious. We prove the sufficiency by induction on the number of edges. Call a set Z *tight* if $x(Z) = i_{G_0}(Z)$. Let $uv \in E_0$ be an arbitrary edge. If there are no tight $\overline{u}v$-sets and $x(v) > 0$, then we can remove the edge uv, decrease $x(v)$ by one, find a feasible orientation of the resulting graph by induction, and add the directed edge uv. If $x(v) = 0$, then $x(u) > 0$ and there is no tight $\overline{v}u$-set X for otherwise $X + v$ would violate the condition. So we can assume that $x(u), x(v) > 0$, there is a tight $v\overline{u}$-set X, and similarly that there is a tight $u\overline{v}$-set Y. But then $d_{G_0}(X,Y) > 0$, thus $i_{G_0}(X) + i_{G_0}(Y) \leq i_{G_0}(X \cap Y) + i_{G_0}(X \cup Y) - d_{G_0}(X,Y)$ implies that either $x(X \cap Y) < i_{G_0}(X \cap Y)$ or $x(X \cup Y) < i_{G_0}(X \cup Y)$, contradicting the conditions. □

Lemma 1 and the non-negativity of h imply that if we can find a vector $x : V_0 \to \mathbb{Z}_+$ that satisfies $x(V_0) = |E_0|$ and $x(X) \geq h_0(X) + i_{G_0}(X)$ for every $X \subseteq V_0$, then there is an orientation $\vec{G_0}$ of G_0 such that $\varrho_{\vec{G_0}}(v) = x(v)$ for every $v \in V_0$, and $\vec{G_0}$ covers h_0 since $\varrho_{\vec{G_0}}(X) = x(X) - i_{G_0}(X) \geq h_0(X)$. Such a vector x is called *feasible*. By the definition of h_0, $x(z)$ must be equal to $m(V)/2$; let $x' : V \to \mathbb{Z}_+$ denote the projection of x to V. Let

$$p_m(X) := h(X) + i_G(X) + \max\left\{0, m(X) - \frac{m(V)}{2}\right\} \qquad (X \subseteq V)\,.$$

Then it easily follows from the definition of h_0 that the vector x is feasible if and only if x' is an element of the polyhedron $B(p_m)$ as defined in (4).

Claim. The set function p_m is crossing supermodular.

Proof. The G-supermodularity of h implies that $h + i_G$ is crossing supermodular. Let $m^*(X) := \max\{0, m(X) - m(V)/2\}$; we show that this set function is fully supermodular. Indeed, if $m^*(Y) = 0$, then $m^*(X) + m^*(Y) = m^*(X) \leq m^*(X \cup Y) = m^*(X \cap Y) + m^*(X \cup Y)$. If $m^*(X), m^*(Y) > 0$, then $m^*(X) + m^*(Y) = m(X \cap Y) + m(X \cup Y) - m(V) \leq m^*(X \cap Y) + m^*(X \cup Y)$. The sum of a crossing supermodular and a fully supermodular function is crossing supermodular. □

Claim. Suppose that (6)–(9) are true. Then $B(p_m)$ is non-empty.

Proof. By Theorem 1 it suffices to show that $p_m(\mathcal{F}) \leq |E| + m(V)/2$ and $p_m(\overline{\mathcal{F}}) \leq (t-1)(|E| + m(V)/2)$ for every partition $\mathcal{F}$ with t members. Observe that a partition has at most one member X with $m(X) > m(V)/2$. If there is no such member then (6) and the identity (5) imply that $p_m(\mathcal{F}) \leq |E| + m(V)/2$; if there is one such member, then (7) and (5) imply the same. Similarly, a co-partition has at most one member X with $m(X) < m(V)/2$, so (8) or (9) (depending on the existence of such a member) and (5) for the co-partition $\overline{\mathcal{F}}$ imply $p_m(\overline{\mathcal{F}}) \leq (t-1)(|E| + m(V)/2)$. □

By Theorem 1, $B(p_m)$ is a base polyhedron, therefore it has an integral point x'; as we have seen, this and Lemma 1 implies that G_0 has an orientation $\vec{G_0} = (V_0, \vec{E_0})$ covering h_0.

Let $m_i(v)$ be the multiplicity of the edge zv in $\vec{G_0}$, and $m_o(v)$ the multiplicity of the edge vz in $\vec{G_0}$; let $\vec{G}$ denote the edges of $\vec{G_0}$ not incident with z. Then $m_i(X) \geq h(X) - \varrho_{\vec{G}}(X)$ and $m_o(V - X) \geq h(X) - \varrho_{\vec{G}}(X)$ for every $X \subseteq V$, since $\vec{G_0}$ covers h_0. By the crossing G-supermodularity of h, the set function $p(X) := h(X) - \varrho_{\vec{G}}(X)$ is crossing supermodular. Thus we can use the following result in [2], which generalizes Mader's directed splitting theorem:

Lemma 2 ([2]). *Let p be a positively crossing supermodular set function on V. Let m_i, m_o be non-negative integer-valued functions on V for which $m_i(V) = m_o(V)$. There exists a digraph $D = (V, A)$ such that $\varrho_D(v) = m_i(v)$, $\rho_D(V - v) = m_o(v)$ for every $v \in V$, and $\varrho_D(X) \geq p(X)$ for every $X \subseteq V$, if and only if*

$$m_i(X) \geq p(X) \textit{ for every } X \subseteq V$$

and

$$m_o(V - X) \geq p(X) \textit{ for every } X \subseteq V\,.$$

□

To complete the proof of Theorem 2, observe that if G' is the underlying undirected graph of the digraph D given by Lemma 2, then G' satisfies the degree specification, and $G + G'$ has a feasible orientation, namely $\vec{G} + D$. □

This theorem can be used to derive the following min-max theorem for minimum cardinality augmentation:

Theorem 3. *Let $G = (V, E)$ be a graph, and $h : 2^V \to \mathbb{Z}_+$ a non-negative crossing G-supermodular set function. There is an undirected graph $G' = (V, E')$ with γ edges such that $G + G'$ has an orientation covering h if and only if*

$$\gamma \geq h(\mathcal{F}) - e_G(\mathcal{F}) \tag{10}$$

holds for every partition and co-partition $\mathcal{F}$, and

$$2\gamma \geq h(\mathcal{F}) - e_G(\mathcal{F}) \tag{11}$$

holds for every cross-free regular family $\mathcal{F}$ that decomposes into a partition of some $X \subseteq V$ and a co-partition of $\overline{X}$.

Proof. Again, $h(\mathcal{F}) - e_G(\mathcal{F})$ measures the deficiency of the family $\mathcal{F}$, so the necessity follows easily by observing that an oriented new edge can cover at most one member of a (sub)partition or a (sub)-copartition.

Sufficiency can be proved by showing that if (10) and (11) hold, then there exists a vector $m : V \to \mathbb{Z}_+$ with $m(V) = 2\gamma$ satisfying (6)–(9); thus by Theorem 2 we can find a feasible augmentation with degree-specification m. The essential result in the proof is that the polyhedron

$$C := \{m : V \to \mathbb{Z}_+ \mid m \text{ satisfies (6)–(9)}\}$$

is a contra-polymatroid. Define the set functions

$$p_1(X) := h(X) + i_G(X) ,$$

$$p_2(X) := h(\overline{X}) + i_G(\overline{X}) - |E| .$$

By the crossing G–supermodularity of h, the set functions p_1 and p_2 are crossing supermodular, therefore the set functions $p_1^\wedge$ and $p_2^\wedge$ (as defined in (2)) are also crossing supermodular. By the identity (5), a non-negative vector m satisfies (6)–(9) if and only if the following hold:

$$m(V) \geq 2\left(\max_{X \subset V} p_1^\wedge(X) + p_2(X)\right) , \tag{12}$$

$$m(X) \geq p_1^\wedge(X) + p_2(X) \text{ for every } X \subset V , \tag{13}$$

$$m(V) \geq 2\left(\max_{X \subset V} p_1(X) + p_2^\wedge(X)\right) , \tag{14}$$

$$m(X) \geq p_1(X) + p_2^\wedge(X) \text{ for every } X \subset V . \tag{15}$$

For a set $X \subset V$, define

$$p(X) := \max\{p_1^\wedge(X) + p_2(X),\ p_1(X) + p_2^\wedge(X),\ 0\} , \tag{16}$$

and let

$$p(V) := 2 \max_{X \subset V} p(X) . \tag{17}$$

Then the polyhedron C can be characterized as

$$C = \{m : V \to \mathbb{Z} \mid m(X) \geq p(X)\ \forall X \subseteq V\} .$$

To prove that C is a contra-polymatroid, we will show that the set function $p^\wedge$ is fully supermodular. First we establish some other properties of $p^\wedge$:

Proposition 2. *For every proper subset X of V, the value of $p^\wedge(X)$ is given by*

$$p^\wedge(X) = \max_{Y \subseteq X} (p_1^\wedge(Y) + p_2^\wedge(Y)) .$$

Proof. By definition $p^\wedge$ is less or equal to the maximum on the right side. For the other direction, suppose indirectly that there exists an $Y \subseteq X$ and partitions $\mathcal{F}_1$ and $\mathcal{F}_2$ of Y such that

$$p^\wedge(X) < p_1(\mathcal{F}_1) + p_2(\mathcal{F}_2) .$$

Repeat the following step as many times as possible:

- If $X \in \mathcal{F}_1$ and $Y \in \mathcal{F}_2$ are crossing, then replace X in $\mathcal{F}_1$ by $X - Y$, and replace Y in $\mathcal{F}_2$ by $Y - X$.

Observe that the resulting families are partitions of some proper subset of Y, so the procedure terminates after a finite number of steps. Furthermore, X and $\overline{Y}$ are crossing, so $h(X) + h(\overline{Y}) \leq h(X \cap \overline{Y}) + h(X \cup \overline{Y}) + d_G(X, \overline{Y})$; this implies that $p_1(X) + p_2(Y) \leq p_1(X - Y) + p_2(Y - X)$. Let $\mathcal{F}_1'$ and $\mathcal{F}_2'$ denote the families obtained at the end of the procedure; then $\mathcal{F}_1'$ and $\mathcal{F}_2'$ are partitions of some $Y' \subseteq Y$, and $p^\wedge(X) < p_1(\mathcal{F}_1') + p_2(\mathcal{F}_2')$. Moreover, $\mathcal{F}_1' + \mathcal{F}_2'$ is cross-free, which means that there is a partition $Y_1', \ldots, Y_t'$ of Y', such that for every i either $\mathcal{F}_1'$ contains Y_i' and $\mathcal{F}_2'$ contains a partition of Y_i', or vice versa. But then $p_1(\mathcal{F}_1') + p_2(\mathcal{F}_2') \leq p^\wedge(Y') \leq p^\wedge(X)$, a contradiction. □

Proposition 3. *The set function p satisfies*

$$p(X) + p(Y) \leq p^\wedge(X \cap Y) + p^\wedge(X \cup Y) \tag{18}$$

for every pair (X, Y).

Proof. The inequality is obvious if one of $p(X)$ and $p(Y)$ is zero, or X and Y are not intersecting. If $X \cup Y = V$, then $p(X) + p(Y) \leq 2\max\{p(X), p(Y)\} \leq p(V) = p(X \cup Y) \leq p^\wedge(X \cap Y) + p^\wedge(X \cup Y)$.

By Proposition 2 it suffices to prove that if $p(X), p(Y) > 0$ and X and Y are crossing, then

$$p(X) + p(Y) \leq (p_1^\wedge + p_2^\wedge)(X \cap Y) + (p_1^\wedge + p_2^\wedge)(X \cup Y) .$$

Using the definition of p and the supermodularity of $p_1^\wedge$ and $p_2^\wedge$,

$$\begin{aligned} p(X) + p(Y) &\leq p_1^\wedge(X) + p_2^\wedge(X) + p_1^\wedge(Y) + p_2^\wedge(Y) \\ &\leq p_1^\wedge(X \cap Y) + p_2^\wedge(X \cap Y) + p_1^\wedge(X \cup Y) + p_2^\wedge(X \cup Y) . \end{aligned}$$

□

This property is sufficient for the supermodularity of $p^\wedge$, as the following lemma states:

Lemma 3. *If a set function p (with $p(\emptyset) = 0$) satisfies (18) for every pair (X, Y), then $p^\wedge$ is fully supermodular.*

Proof. For a set $X \subseteq V$, let $\mathcal{F}_X$ denote a partition of X for which $p^\wedge(X) = p(\mathcal{F}_X)$. Let $X, Y \subseteq V$ be an arbitrary pair. Starting from the family $\mathcal{F} = \mathcal{F}_X + \mathcal{F}_Y$, repeat the following operation as many times as possible:

- If there is an intersecting pair X' and Y' in the family, remove both of them, and add the sets of $\mathcal{F}_{X' \cap Y'}$ and of $\mathcal{F}_{X' \cup Y'}$.

The operation doesn't change $d_{\mathcal{F}}$, and doesn't decrease $p(\mathcal{F})$, since p has the property (18). Since the operation either increases the cardinality of the family, or increases $\sum_{X \in \mathcal{F}} |X|^2$ without changing the cardinality, after a finite number of steps we get a laminar family $\mathcal{F}'$ for which $p(\mathcal{F}') \geq p(\mathcal{F})$. Such a family decomposes into a partition of $X \cap Y$ and a partition of $X \cup Y$, hence $p^\wedge(X) + p^\wedge(Y) \leq p^\wedge(X \cap Y) + p^\wedge(X \cup Y)$. □

Lemma 3 and Proposition 3 implies that $p^\wedge$ is fully supermodular, and it is obviously monotone increasing, hence the polyhedron C is a contra-polymatroid defined by $p^\wedge$. It is known that in this case the minimum cardinality of an integral element of the contra-polymatroid C is $p^\wedge(V)$. Thus, for a fix γ, there exists an element m of C with $m(V) = 2\gamma$ if and only if $p^\wedge(V) \leq 2\gamma$. This exactly gives conditions (10) and (11) of the theorem. □

Remark 1. The following example shows that (10) is not sufficient in Theorem 3. Let $V = \{v_1, v_2, v_3, v_4\}$, $E = \{v_1v_2, v_1v_3, v_1v_4\}$. Let $h = 1$ on the sets $\{v_2\}, \{v_3\}, \{v_4\}$ and on their complement; $h = 0$ on all other sets. We need at least 2 new edges for a feasible orientation, but (10) gives only $\gamma \geq 1$.

Remark 2. A cost function $c : E \to \mathbb{R}$ is called *node induced* if $c(uv) = c'(u) + c'(v)$ where $c' : V \to \mathbb{R}$ is a linear cost function on the nodes. To solve the minimum cost augmentation for node induced cost functions, one can find a minimum cost element m of the contra-polymatroid C according to the cost function c', using the greedy algorithm. Then this m can be used as a degree specification to find a minimum cost augmentation.

For general edge costs the problem is **NP**-complete: let G be the empty graph, and let $c(e) = 1$ on the edges of a fix graph G^*, $c(e) = 2$ on the other edges. Let $h(X) = 1$ if $X \neq \emptyset, V$; thus h is crossing supermodular. Now the minimum cost of the augmentation is $|V|$ if and only if G^* contains a Hamiltonian cycle.

4 (k, l)-Edge-Connected Orientations

In the introduction we defined (k, l)-edge-connectivity for non-negative integers $k \geq l$, and mentioned that the (k, l)-edge-connectivity orientation problem is a common generalization of k-edge-connectivity orientation (with $l = k$) and rooted k-edge-connectivity orientation (with $l = 0$). Recently, it was shown in [4] that the case $l = k - 1$ has an important role in orientations with parity constraints. As for the corresponding augmentation problems, the degree-specified and minimum cardinality augmentation of a graph to have a k-edge-connected orientation is already solved, but the minimum cost augmentation is **NP**-complete even for $k = 1$. Conversely, for rooted k-edge-connected orientations, the minimum cost augmentation is easily solvable by matroid techniques, but the degree constrained augmentation was hitherto unsolved.

To show how the results of the previous section can be used to solve degree-specified and minimum cardinality augmentation of a graph so that the new graph has a (k, l)-edge-connected orientation, fix a node $s \in V$, and introduce the following family of set functions:

$$h_{kl}(X) := \begin{cases} k & \text{if } s \notin X\,, \\ l & \text{if } s \in X\,. \end{cases} \tag{19}$$

Menger's Theorem implies that an orientation is (k, l)-edge-connected from root s if and only if it covers h_{kl}. The set function h_{kl} is crossing G–supermodular for any G. Note that if a digraph is (k, l)-edge-connected from root s, and for some $s' \in V - s$ we reverse the orientation of the edges of $k - l$ edge-disjoint paths from s to s', then we get a digraph that is (k, l)-edge-connected from root s'. Thus the root can be selected arbitrarily in orientation problems.

Theorem 4. *Let $G = (V, E)$ be a graph, and $m : V \to \mathbb{Z}_+$ a degree specification with $m(V)$ even. There exists an undirected graph $G' = (V, E')$ such that $G + G'$ has a (k, l)-edge-connected orientation and $d_{G'}(v) = m(v)$ for all $v \in V$, if and only if the following hold for every partition $\mathcal{F} = \{X_1, \ldots, X_t\}$ of V:*

$$\frac{m(V)}{2} \geq (t-1)k + l - e_G(\mathcal{F})\,, \tag{20}$$

$$\min_i m(\overline{X_i}) \geq (t-1)k + l - e_G(\mathcal{F})\,. \tag{21}$$

Proof. The necessity can be shown as in Theorem 2. As for the sufficiency, we can fix a node $s \in V$ and use Theorem 2 with h_{kl}. In this case the inequalities (8) and (9) in Theorem 2 are consequences of (6) and (7), since $h_{kl}(\mathcal{F}) \geq h_{kl}(\overline{\mathcal{F}})$ and $e_G(\mathcal{F}) = e_G(\overline{\mathcal{F}})$ for every partition $\mathcal{F}$. □

Theorem 5. *Let $G = (V, E)$ be a graph. There is a graph G' with γ edges such that $G + G'$ has a (k, l)-edge-connected orientation, if and only if the following two conditions are met:*

1. $\gamma \geq (t-1)k + l - e_G(\mathcal{F})$ for every partition $\mathcal{F}$ with t members.

2. *$2\gamma \geq t_1 k + t_2 l - e_G(\mathcal{F})$ for every $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$ where $\mathcal{F}_1$ is a partition of some X with t_1 members, $\mathcal{F}_2$ is a co-partition of $\overline{X}$ with t_2 members, and every member of $\mathcal{F}_2$ is the complement of the union of some members of $\mathcal{F}_1$.*

Proof. As in the proof of Theorem 4, we demand that $G + G'$ should have an orientation covering h_{kl}. Going back to the proof of Theorem 3, the set function p defined in (16) can be defined in this case as

$$p(X) := \begin{cases} \max\{p_1^{\wedge}(X) + p_2(X),\ 0\} & \text{if } X \subset V\,, \\ 2\max_{Y \subset V}(p_1^{\wedge}(Y) + p_2(Y)) & \text{if } X = V\,. \end{cases} \tag{22}$$

As it was proved in Theorem 3, a feasible augmentation with γ edges exists if and only if $p^{\wedge}(V) \leq 2\gamma$; by the above characterization of p, this is equivalent to the conditions of the theorem. □

There are other equivalent characterizations of graphs that have a (k,l)-edge-connected orientation. For given non-negative integers k and l, a graph $G = (V,E)$ is called *(k,l)-tree-connected* if any graph obtained by deleting l edges from G contains k edge-disjoint spanning trees; it is called *(k,l)-partition-connected* if $e_G(\mathcal{F}) \geq k(t-1) + l$ for every partition $\mathcal{F}$ with t members. Tutte [9] proved that a graph is $(k,0)$-tree-connected if and only if it is $(k,0)$-partition-connected. This immediately implies that a graph is (k,l)-tree-connected if and only if it is (k,l)-partition-connected.

Simple calculation shows that for $k \leq l$, a graph G is (k,l)-tree-connected if and only if it is $(k+l)$-edge-connected; hence the (k,l)-tree-connectivity augmentation problem is interesting only for $k \geq l$.

Proposition 4. *For $k \geq l$, a graph $G = (V,E)$ is (k,l)-tree-connected if and only if it has a (k,l)-edge-connected orientation.*

Proof. It follows from the orientation theorem in [1] that for $k \geq l$, a graph has a (k,l)-edge-connected orientation if and only if it is (k,l)-partition-connected. □

Thus Theorems 4 and 5 solve the degree-specified and minimum cardinality (k,l)-tree-connectivity augmentation problem.

5 Positively Crossing G–Supermodular Set Functions

A set function h is *positively crossing G–supermodular* if (1) holds for every crossing pair (X,Y) for which $h(X), h(Y) > 0$.

Let $M = (V;E,A)$ be a mixed graph, where E is the set of undirected edges and A is the set of directed edges. Then the task of finding a (k,l)-edge-connected orientation of M for a fix root s is equivalent to finding an orientation of the edges in E that covers the set function $\max\{h_{kl} - \varrho_A, 0\}$, where h_{kl} is defined in (19). This set function isn't crossing G–supermodular anymore, but it is positively crossing G–supermodular for any G. This motivates the study of the h-orientation problem for positively crossing G–supermodular set functions, and the corresponding augmentation problems.

The characterizations in this section involve some complicated set families. Every cross-free family $\mathcal{F}$ has a *tree-representation* (T,φ), where $T = (W,B)$ is a directed tree, and $\varphi : V \to B$ is a mapping such that $\{\varphi^{-1}(W_e) \mid e \in B\} = \mathcal{F}$, where W_e is the component of $T - e$ entered by e. *A tree-composition* of $\emptyset \neq X \subset V$ is a cross-free composition of X which has a tree-representation $(T = (W,B), \varphi)$ such that $\varphi^{-1}(w) \neq \emptyset$ for every $w \in W$. Equivalently, a tree-composition of X is a cross-free composition of X that contains no partitions and co-partitions of V. A partition or a co-partition of V will be regarded as a tree-composition of $\emptyset$.

In this section we solve the degree-specified augmentation problem, by mainly the same methods as in Sect. 3, but instead of relying on the properties of base polyhedra, we use the following extension of the classical result on the TDI-ness of the intersection of base polyhedra:

Lemma 4. *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a fully supermodular set function, and let $q : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a set function that is supermodular on the crossing pairs (X,Y) for which $p(X) < q(X)$ and $p(Y) < q(Y)$. Then the system*

$$\{x \in \mathbb{R}^V \mid x(V) = p(V);\ x(Z) \geq p(Z),\ x(Z) \geq q(Z)\ \forall Z \subseteq V\} \tag{23}$$

is TDI; it has a feasible solution if and only if

$$p(\overline{X}) + q(\mathcal{F}) \leq (\alpha + 1)p(V) \tag{24}$$

for every $X \subset V$ (including the empty set) and every tree-composition $\mathcal{F}$ of X with covering number α.

Proof. To prove TDI-ness, we have to show that the dual system

$$\max\{y_1 p + y_2 q - \beta p(V) :\ (y_1 + y_2)A - \beta\mathbf{1} = c,\ y_1, y_2, \beta \geq 0\}$$

has an integral optimal solution for every integral c, where $y_1, y_2 : 2^V \to \mathbb{Q}_+$ are dual variables on the sets, y_1 corresponding to the inequalities featuring p, y_2 corresponding to those featuring q, $\beta \in \mathbb{Q}_+$ is the dual variable for the inequality $x(V) \leq p(V)$, and A is the incidence matrix of all subsets of V. The main observation is that we can assume that y_1 is positive on a chain and y_2 is positive on a cross-free family: this can be achieved by a slight modification of the usual uncrossing technique. Consider the following operations:

- If $y_1(X), y_1(Y) > 0$ and neither $X \subseteq Y$, nor $Y \subseteq X$, decrease y_1 on X and Y by $\min\{y_1(X), y_1(Y)\}$, and increase y_1 by the same amount on $X \cap Y$ and $X \cup Y$.
- If $y_2(X), y_2(Y) > 0$, $p(X) < q(X)$, $p(Y) < q(Y)$ and X, Y are crossing, then decrease y_2 on X and Y by $\min\{y_2(X), y_2(Y)\}$, and increase y_2 by the same amount on $X \cap Y$ and $X \cup Y$.
- If $y_2(X) > 0$ and $p(X) \geq q(X)$, then decrease y_2 on X to 0 and increase y_1 on X by the same amount.

These operations do not decrease $y_1 p + y_2 q - \beta p(V)$, and they maintain $(y_1 + y_2)A - \beta 1 = c$. We show that by repeatedly applying these operations (in any order), in a finite number of steps we get an optimal dual solution (y_1', y_2', β) such that y_1' is positive on a chain and y_2' is positive on a cross-free family.

Since $y_1, y_2 \in \mathbb{Q}$, there is a positive integer ν such that νy_1 and νy_2 are integral. The sum

$$\nu \left(2 \sum_{y_1(X)>0} y_1(X)|X|^2 + \sum_{y_2(X)>0} y_2(X)|X|^2 \right)$$

increases by at least 1 with any of the above operations, and it is bounded from above by $2\nu|V|^2(\beta + \max_{v\in V} c(v))$. Thus the procedure terminates after a finite number of steps.

We proved that there is an optimal dual solution (y_1', y_2', β) where y_1' is positive on a chain and y_2' is positive on a cross-free family; but this means that this is also an optimal solution of the dual of the system we get if we restrict p to the sets where y_1' is positive, and restrict q to the sets where y_2' is positive (changing their value to $-\infty$ on all other sets). This system is the intersection of two base polyhedra, so it has an integral optimal dual solution, which is in turn optimal for the dual of the system (23); therefore the system (23) is TDI.

The proof of the non-emptiness condition (24) is similar: the infeasibility of the system is equivalent to the feasibility of its dual by the Farkas Lemma; a feasible dual solution can be uncrossed as above, so dual feasibility implies the emptiness of the intersection of the two base polyhedra given by p and q restricted to the sets where y_1' and y_2' are positive. Thus the emptiness condition for the intersection of base polyhedra (which is of the form (24)) is sufficient for the infeasibility of the original system. □

Theorem 6. *Let $G = (V, E)$ be a graph, $h : 2^V \to \mathbb{Z}_+$ a positively crossing G-supermodular set function on V, and $m : V \to \mathbb{Z}_+$ a degree specification with $m(V)$ even; let*

$$h_m(X) := h(X) + \max\left\{0, m(X) - \frac{m(V)}{2}\right\} .$$

There exists an undirected graph $G' = (V, E')$ such that $G+G'$ has an orientation covering h and $d_{G'}(v) = m(v)$ for all $v \in V$ if and only if

$$h_m(\mathcal{F}) + \max\left\{0, m(\overline{X}) - \frac{m(V)}{2}\right\} \le e_G(\mathcal{F}) + (\alpha+1)\frac{m(V)}{2}$$

for every $X \subset V$ and for every tree-composition $\mathcal{F}$ of X with covering number α.

Proof. The necessity follows from the fact that if $\mathcal{F}'$ is a regular family with covering number $\alpha + 1$, then $\varrho_{\vec{G}}(\mathcal{F}') \le e_G(\mathcal{F}')$ for any orientation of G, and

$$\varrho_{\vec{G}'}(\mathcal{F}') \le (\alpha+1)\frac{m(V)}{2} - \sum_{X\in\mathcal{F}'} \max\left\{0, m(X) - \frac{m(V)}{2}\right\}$$

for any orientation $\vec{G}'$ of a graph G' satisfying the degree specification.

The sufficiency can be proved in essentially the same way as in Theorem 2: define G_0 and h_0 similarly, and for $X \subseteq V$, let

$$p(X) := i_G(X) + \max\left\{0, m(X) - \frac{m(V)}{2}\right\} ,$$

$$q(X) := h(X) + i_G(X) + \max\left\{0, m(X) - \frac{m(V)}{2}\right\} .$$

In this case Lemma 1 implies that an orientation of G_0 covering h_0 exists if and only if the polyhedron

$$\{x : V \to \mathbb{R} \mid x(V) = p(V);\ x(Z) \ge p(Z),\ x(Z) \ge q(Z)\ \forall Z \subseteq V\}$$

has an integral point.

Claim. The set function p is fully supermodular, and the set function q is supermodular on the crossing pairs (X, Y) for which $p(X) < q(X)$ and $p(Y) < q(Y)$.

Proof. The set function p is the sum of two fully supermodular functions, so it is fully supermodular. Since h is positively crossing G-supermodular, q is supermodular on the crossing pairs (X, Y) for which $h(X), h(Y) > 0$, and these are exactly the crossing pairs for which $p(X) < q(X)$ and $p(Y) < q(Y)$. □

Lemma 4 implies that an orientation of G_0 covering h_0 exists if and only if

$$p(\overline{X}) + q(\mathcal{F}) \le (\alpha+1)p(V)$$

for every $X \subset V$ and every tree-composition $\mathcal{F}$ of X with covering number α. Using (5) this is equivalent to the condition of the theorem.

From here we can follow the line of the proof of Theorem 2. Let $\vec{G_0}$ be the orientation of G_0 covering h_0, and let $\vec{G}$ denote the edges of $\vec{G_0}$ not incident with z. Let $m_i(v)$ be the multiplicity of the edge zv in $\vec{G_0}$, and $m_o(v)$ the multiplicity of the edge vz in $\vec{G_0}$. Define the set function $h'(X) = h(X) - \varrho_{\vec{G}}(X)$; h' is positively crossing supermodular. As in the proof of Theorem 2, we can apply Lemma 2 (with the m_i, m_o and h' defined above) to obtain a directed graph D whose underlying undirected graph is a feasible augmentation. □

Remark 3. We can use the ellipsoid method to prove that the above theorem gives rise to a polynomial algorithm. To prove that the optimization for (23) can be solved in polynomial time, we show that the separation can be solved for a vector x. We know that the separation algorithm works for supermodular functions. Thus we can determine if there is a set X with $x(X) < p(X)$. If not, then for the set function $q^*(X) = \max(x(X), q(X))$, $B(q^*)$ is a base polyhedron. Therefore we can solve the corresponding optimization problem, which implies the solvability of the separation problem; this is equivalent to the separation problem for q concerning x.

Remark 4. The condition involving tree-compositions may seem unfriendly, but it is unavoidable, even in the special case when the problem is to find an orientation of the undirected edges of a mixed graph such that the resulting digraph is k-edge-connected. This orientation problem was already considered in [3], where crossing G–supermodular set functions with possible negative values were studied. The following example shows that the positively G–supermodular case is more general, i. e. not every positively crossing G–supermodular set function h can be made crossing G–supermodular by decreasing the value of h on some of the sets where it is 0.

Let X_1, X_2, X_3 be three subsets of a ground set V in general situation. Let $h(X_i) = 1$, $h(X_i \cup X_j) = 2$ $(i \neq j)$, $h(X_1 \cup X_2 \cup X_3) = 4$, and $h(X) = 0$ on the remaining sets; this is a positively crossing supermodular function. The value of $X_1 \cap X_2$ cannot be decreased since

$$h(X_1 \cap X_2) \geq h(X_1) + h(X_2) - h(X_1 \cup X_2) = 0 .$$

Therefore it is impossible to correctly modify h so as to satisfy

$$h(X_1 \cap X_2) \leq h(X_1 \cap X_2 \cap X_3) + h(X_1 \cap X_2 \cup X_3) - h(X_3) \leq -1 .$$

References

1. Frank, A.: On the orientation of graphs. J. Combinatorial Theory B **28** No. 3 (1980) 251–261
2. Frank, A.: Connectivity augmentation problems in network design. In: Birge, J.R., Murty, K.G. (eds.): Mathematical Programming: State of the Art. Univ. of Michigan (1994) 34–63
3. Frank, A.: Orientations of graphs and submodular flows. Congressus Numerantium **113** (1996) 111–142
4. Frank, A., Király, Z.: Parity constrained k-edge-connected orientations. In: Cornuejols, G., Burkard, R., Woeginger, G.J. (eds.): Lecture Notes in Computer Science, Vol. 1610. Springer-Verlag, Berlin Heidelberg New York (1999) 191–201
5. Fujishige,S.: Structures of polyhedra determined by submodular functions on crossing families. Mathematical Programming **29** (1984) 125–141
6. Lovász, L.: Combinatorial Problems and Exercices. North-Holland (1979)
7. Mader,W.: Konstruktion aller n-fach kantenzusammenhängenden Digraphen. Europ. J. Combinatorics **3** (1982) 63–67
8. Nash-Williams, C.St.J.A.: On orientations, connectivity and odd vertex pairings in finite graphs. Canad. J. Math. **12** (1960) 555–567
9. Tutte,W.T.: On the problem of decomposing a graph into n connected factors. J. London Math. Soc. **36** (1961) 221–230
10. Watanabe, T., Nakamura, A.: Edge-connectivity augmentation problems. Computer and System Sciences **35** No. 1 (1987) 96–144