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MATROIDS FROM CROSSING FAMILIES

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ABSTRACT

Let $\mathcal{F}$ be a collection of subsets of a finite groundset S and f an integer valued function on $\mathcal{F}$ such that $X, Y \in \mathcal{F}$ and $X \cap Y \neq \phi$, $X \cup Y \neq S$ imply that $X \cap Y, X \cup Y \in \mathcal{F}$ and $f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$. For a fixed integer k, the collection $\mathcal{D} = \{D: |D \cap X| \leq f(X)$ for $X \in \mathcal{F}$, $|D| = k\}$ forms the set of bases of a matroid.

This construction, which extends an earlier one due to J. Edmonds is used to show how one may produce a minimum weight covering of directed cuts in an arrow-weighted digraph if a weighted matroid intersection algorithm is available. We also deduce an orientation theorem due to Nash-Williams from the matroid polyhedron intersection theorem.

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1. INTRODUCTION

Let $S_1, S_2, \ldots, S_m$ be a partition of a finite set S and $b_1, b_2, \ldots, b_m$ positive integers. As is well known the collection $\mathcal{I} = \{I: I \subseteq S, |I \cap S_i| \leqslant b_i\}$ satisfies the independence axioms of matroids. A matroid is said to be a *partition matroid* if it can be got this way.

As a possible generalization of this concept, one may ask whether the collection $\mathcal{I} = \{I: |I \cap F| \leqslant f(F)$ for $F \in \mathcal{F}\}$ satisfies the independence axioms where $\mathcal{F}$ consists of some subsets of S and f is an integer-value function on $\mathcal{F}$. The answer is negative, in general. For example, set $S = \{a, b, c\}$, $\mathcal{F} = \{ab, bc, ac\}$ and $f(ab) = 2$, $f(bc) = f(ac) = 1$ then both $\{a, b\}$ and $\{c\}$ are maximal members of $\mathcal{I}$ contradicting the independence axiom that maximal independent sets have the same cardinality.

However, J. Edmonds proved [2] that

Theorem 1. $\mathcal{I} = \{I: |I \cap X| \leqslant b(X)$ *for* $X \in \mathcal{B}\}$ *forms the independent sets of a matroid* $\mathcal{M}$ *if* $X, Y \in \mathcal{B}$, $X \cap Y \neq \phi$ *imply that*

(1) $X \cap Y,\ X \cup Y \in \mathcal{B}$,

(2) $b(X) + b(Y) \geqslant b(X \cap Y) + b(X \cup Y)$.

Furthermore the polyhedron P *spanned by the incidence vectors of independent sets of* $\mathcal{M}$ *is* $\{x: 0 \leqslant x \leqslant 1,\ x(B) \leqslant b(B)$ *for* $B \in \mathcal{B}\}$.

We shall need the obvious consequence that the vertices of $P_k = \{x: x \in P,\ 1x = k\}$ are the incidence vectors of the bases of a truncation of $\mathcal{M}$. Edmonds also gave the rank function of $\mathcal{M}$:

$$r(A) = \min\left\{\sum b(X_i) + |A - \cup X_i|: X_i \in \mathcal{B},\ X_i \cap X_j = \phi\right\}.$$

One purpose of this paper is to generalize this construction of Edmonds by using crossing families rather than intersecting ones. Then we shall use this new way of constructing matroids to show that a weighted matroid intersection algorithm can be used to construct a minimum weight covering of directed cuts in an edge-weighted digraph. As a second application, we deduce Nash-Williams' orientation theorem from the matroid polyhedron intersection theorem. In both special cases one matroid is a partition matroid while the other is defined by a crossing family.

2. PRELIMINARIES, NOTATION

Subsets A, B of a finite set S are *intersecting* if none of $A \cap B$, $A - B$, $B - A$ is empty. If in addition, $A \cup B \neq S$ then A, B are *crossing*. A family $\mathcal{F}$ of subsets of S is *intersecting* (*crossing*) if $A \cap B$, $A \cup B \in \mathcal{F}$ for all intersecting (crossing) members of $\mathcal{F}$ and $\phi \notin \mathcal{F}$ ($\phi, S \notin \mathcal{F}$). A set function b is *submodular* on A, B if $b(A) + b(B) \geqslant b(A \cap B) + b(A \cup B)$. A set function p is *supermodular* if $-p$ is submodular.

For a set $X \subseteq S$, denote the complement of X by $\bar{X}$. For a family $\mathcal{F}$ of subsets of S, set $\bar{\mathcal{F}} = \{X: \bar{X} \in \mathcal{F}\}$. For a vector x in $\mathbf{R}^S$, $x(F) = \sum (x(v): v \in F)$.

Let $G = (V, E)$ be a graph. We call an element e of E an *arrow* if G is directed and an *edge* if G is undirected. An arrow from u to v is denoted by uv; an edge between u and v is denoted by (u, v). In a directed graph an arrow uv enters a subset X if $u \notin X$, $v \in X$. An arrow leaves X if it enters $V - X$.

Let $\rho(v)$ denote the number of arrows entering a vertex v and $\rho(X)$ denote the number of arrows entering a subset X. Call a set X ($\phi \subset X \subset V$) a *kernel* if no arrow leaves X. The nonempty set of arrows entering a kernel X is called the *directed cut* or *dicut* defined by X. A subset C of arrows is said to be a *covering* if C meets all dicuts of G.

3. A NEW MATROID CONSTRUCTION

There were two indications that generalization of Edmonds' construction might exist. One of them comes from a general min–max theorem of Edmonds and Giles [1] where crossing families played a role instead of intersecting ones. However, if we try to weaken the hypothesis in Theorem 1 and replace the intersecting family by a crossing one then $\mathcal{I}$ will no longer be a matroid in general: the counterexample in the Introduction works again. We shall see however, that the crossing family does lend itself to defining the bases of a matroid.

The other root of our investigations is a slight generalization of the concept of partition matroids. We are given a partition $S_1, S_2, \ldots, S_m$ of S and positive integers $p_i \leq b_i$ for $i = 1, 2, \ldots, m$. Let k be a positive integer. The collection $\mathscr{D} = \{X: |X| = k,\ p_i \leq |X \cap S_i| \leq b_i$ for $i = 1, \ldots, m\}$ satisfies the basis axioms of matroids. We call such a matroid *g-partition* matroid. Note that a partition matroid can be obtained in this form by taking $p_i = 0$ for all i and $k = \sum_i \min(|S_i|, b_i)$.

Our main result is:

Theorem 2. *Let $\mathscr{F}$ be a crossing family and let f be an integer-valued function on $\mathscr{F}$ submodular on crossing pairs. Let k be a positive integer. Then the family*

$$\mathscr{D} = \{X:\ |X| = k,\ |X \cap F| \leq f(F) \ \text{for}\ F \in \mathscr{F}\}$$

satisfies the base axioms (unless $\mathscr{D}$ is empty). Furthermore, the polyhedron spanned by the incidence vectors of bases is given by

$$P_{\mathscr{F}} = \{x:\ 0 \leq x \leq 1,\ 1x = k,\ x(F) \leq f(F) \ \text{for}\ F \in \mathscr{F}\}.$$

Proof. To prove this theorem we need some lemmas. A slightly weaker form of the following lemma was proved by L. Lovász [9]. A detailed proof can be found in [4].

Lemma 1. *Let $\mathscr{K} \subset 2^S$ be a crossing family and p be a function on $\mathscr{K}$ supermodular on crossing pairs. Define $\mathscr{K}_1 = \{X:\ X = \bigcup X_i \neq S,$ $X_i \in \mathscr{K},\ X_i \cap X_j = \phi\} \cup \{\phi\}$ and for $X \in \mathscr{K}_1 - \{\phi\}$ let $p_1(X) =$ $= \max\left(\sum p(X_i):\ X = \cup X_i,\ X_i \cap X_j = \phi\right)$ and $p_1(\phi) = 0$. If $X, Y \in \mathscr{K}_1$, $X \cup Y \neq S$ then $X \cup Y$, $X \cap Y \in \mathscr{K}_1$ and p_1 is supermodular on X, Y.*

An analogous statement is true if the term "super" is replaced by "sub" and max by min.

Lemma 2. *Let $P_{\mathscr{K}} = \{x:\ x \geq 0,\ x(F) \geq p(F)$ for $F \in \mathscr{K}\}$ and $P_{\mathscr{K}_1} = \{x:\ x \geq 0,\ x(F) \geq p_1(F)$ for $F \in \mathscr{K}_1\}$. Then $P = P_{\mathscr{K}_1}$.*

Proof. Since $\mathscr{K} \subseteq \mathscr{K}_1$ and $p(X) \leq p_1(X)$ for $X \in \mathscr{K}$ we get $P_{\mathscr{K}} \supseteq P_{\mathscr{K}_1}$. On the other hand, for a vector $x \in P_{\mathscr{K}}$ and for $X \in \mathscr{K}_1$ we have $p_1(X) = \sum p(X_i) \leq \sum p_1(X_i) \leq \sum x(X_i) = x(X)$ for some disjoint X_i's in $\mathscr{K}$ where $\bigcup X_i = X$. This shows that $x \in P_{\mathscr{K}_1}$. ∎

Applying Lemma 2 to $\bar{\mathscr{F}}$ and $k - p$, we obtain

Lemma 3. *Let $\mathscr{F} \subset 2^S$ be a crossing family and let f be a function on $\mathscr{F}$ submodular on crossing pairs. Let k be a positive integer. Define $\mathscr{B} = \{X:\ X = \bigcap X_i \neq \phi,\ X_i \in \mathscr{F},\ \bar{X}_i \cap \bar{X}_j = \phi\} \cup \{S\}$. Define $b(X) =$ $= k - \max\left\{\left(\sum(k - f(X_i)\right):\ X = \bigcap X_i,\ X_i \in \mathscr{F},\ \bar{X}_i \cap \bar{X}_j = \phi\right\}$ if $X \in$ $\in \mathscr{B} - \{S\}$ and set $b(S) = k$. Then $\mathscr{B}$ is an intersecting family and b is submodular on intersecting pairs. Moreover, if $P_{\mathscr{F}} = \{x:\ x \geq 0,\ 1x = k,$ $x(F) \leq f(F)$ for $F \in \mathscr{F}\}$ and $P = \{x:\ x \geq 0,\ 1x = k,\ x(F) \leq b(F)$ for $B \in \mathscr{B}\}$ then $P_{\mathscr{F}} = P_{\mathscr{B}}$.*

Now Edmonds' theorem in the Introduction and Lemma 3 imply Theorem 2. ∎

The rank function of the matroid in Theorem 2 is

$$r(A) = \min\Big\{k,\ \sum f(X_{ij}) + |A - \bigcup X_i| - (h - m)k:$$
$$X_{ij} \in \mathscr{F},\ \bigcap_j X_{ij} = X_i,\ \bar{X}_{ij_1} \cap \bar{X}_{ij_2} = \phi,\ X_i \neq \phi\Big\},$$

where h and m denote the number of subsets X_{ij} and X_i, respectively.

A symmetric version of Theorem 1 is the following

Theorem 3. *Let $\mathscr{B}$ and $\mathscr{P}$ be intersecting families and let b and p be integer-valued functions on $\mathscr{B}$ and $\mathscr{P}$, respectively, such that b is submodular and p is supermodular on intersecting sets. Let k be a positive integer and assume that $B \in \mathscr{B}$, $P \in \mathscr{P}$, $B - P \neq \phi$, $P - B \neq \phi$ imply that $B - P \in \mathscr{B}$, $P - B \in \mathscr{P}$ and $b(B) - p(P) \geq b(B - P) - p(P - B)$. Then $\mathscr{D}_1 = \{X:\ |X| = k,\ |X \cap B| \leq b(B)$ for $B \in \mathscr{B}$, $|X \cap P| \geq p(P)$ for $P \in \mathscr{P}\}$ satisfies the base axioms (unless $\mathscr{D}_1$ is empty).*

Proof. Let $\mathscr{F} = \mathscr{B} \cup \bar{\mathscr{P}}$ and let

$$f(X) = \begin{cases} \min(b(X),\ k - p(\bar{X})) & \text{if } X \in \mathcal{B} \cap \bar{\mathcal{P}} \\ b(X) & \text{if } X \in \mathcal{B} - \bar{\mathcal{P}} \\ k - p(\bar{X}) & \text{if } X \in \bar{\mathcal{P}} - \mathcal{B}. \end{cases}$$

It is easy to see that $\mathcal{F}$ is a crossing family and f is submodular on crossing sets. Apply Theorem 2 and notice that $\mathcal{D}_1 = \{X: |X| = k, |X \cap F| \leq f(F)$ for $F \in \mathcal{F}\}$. ∎

COVERING OF DIRECTED CUTS

Let $G = (V, E)$ be a directed graph and s a nonnegative weight function on the arrow set E. Call a family $\mathcal{K}$ of not-necessarily distinct dicuts *s-independent* if no arrow e occurs in more than $s(e)$ members of $\mathcal{K}$. The following theorem is a weighted version of that of Lucchesi–Younger.

Theorem ([10], 1). *The minimum weight τ_S of a covering is equal to the maximum cardinality ν_S of an s-independent family of dicuts.*

In [5] we gave a polynomial algorithm for proving this result. Here we show that the primal problem, i.e. finding a minimum weight covering, is a weighted matroid intersection problem.

Replace each vertex $v \in V$ by as many new vertices as there are arrows incident to v and denote by $\varphi(v)$ the set of new copies of v. For $X \subset V$, let $\varphi(X) = \bigcup(\varphi(v): v \in X)$ and let $S = \varphi(V)$. The arrows of G determine a partition of S into two-element subsets. Denote by e_u and e_v the elements in S corresponding to an arrow $e = uv$ of G. Let us denote by $\mathcal{M}_1$ the partition matroid on S where a set is independent if it contains at most one of e_u and e_v for $e \in E$.

Let $\mathcal{F} = \{\varphi(X): X$ is a kernel of $G\}$ and for $F = \varphi(X) \in \mathcal{F}$ let $f(F) = \sum(\rho(v): v \in X) - 1$.

It can be shown that $\mathcal{F}$ is crossing and f is submodular on crossing pairs. Apply Theorem 2 to $\mathcal{F}, f$ and $k = |E|$ and denote the resulting matrid by $\mathcal{M}_2$.

Assign weights to the elements of S as follows. Let $s(e_u) = s(e)$ and $s(e_v) = 0$ for each $e = uv \in E$. Now the key observation is the following claim.

Claim. *There is a one-to-one correspondence between the coverings of G and the common bases of $\mathcal{M}_1$ and $\mathcal{M}_2$ and the weights of corresponding coverings and common bases are equal.*

Proof. Let a covering C and a common base B correspond to each other so that $e = uv \in C \Leftrightarrow e_u \in B$. ∎

This claim has the consequence that the polyhedron spanned by the incidence vectors of coverings in G is a projection of a matroid intersection polyhedron, namely we have to project the common base polyhedron of $\mathcal{M}_1$ and $\mathcal{M}_2$ along the components e_v for each $e = uv \in E$.

Another consequence is that a weighted matroid intersection algorithm lends itself for computing a minimum weight covering. In [6] we described such an algorithm. A version of that procedure for finding a minimum weight common base starts with an arbitrary one (which is available in our case since, for example, $B = \{e_v: e = uv \in E\}$ is a common base) and then successively improves it. This version needs an oracle which can

(*) find the fundamental circuits in $\mathcal{M}_2$ belonging to a common base.

Note that it is far from simple to make an independence oracle for $\mathcal{M}_2$. Fortunately, we do not need it. We need only (*) which is indeed available in this special case because of the following

Claim. *For a common base B and $s \notin B$, the fundamental circuit $C(B, s)$ in $\mathcal{M}_2$ consists of those elements z of B for which no kernel X exists such that $\varphi^{-1}(s) \in X$, $\varphi^{-1}(z) \notin X$ and the dicut defined by X is covered just once by the covering corresponding to B.*

The proof is left to the reader.

Remark. Observing that in the above reduction the family $\mathcal{F}$ can be made intersecting by adjoining S and the function $f(F) = \Sigma(\rho(v): v \in X) - 1$ $(F = \varphi(X) \in \mathcal{F} \cup \{S\})$ is submodular on any intersecting pair, one can ask whether the original construction of Edmonds is not enough for representing the minimum weight covering problem as a matroid intersection problem. The answer is no because this definition would yield a matroid of rank $|E| - 1$. On the other hand, if $f(S)$ were defined to equal $|E|$, the submodularity of f would be destroyed.

4. ORIENTATIONS OF UNDIRECTED GRAPHS

Let $G = (V, E)$ be an undirected graph. By an *orientation* of G we mean a directed graph on V such that each edge of G is replaced by an arrow (directed edge). A directed graph is *g-strongly connected* if $\rho(X) \geq g$ for $\phi \subset X \subset V$.

Theorem ([11]). *G has a g-strongly connected orientation if and only if each cut contain at least $2g$ edges.*

We are going to investigate the minimum weight version of this problem when the two possible orientations uv and vu of an edge have weight $s(uv)$ and $s(vu)$ and we are interested in finding a g-strongly connected orientation of minimum weight.

Again copy the vertices in V and define $\mathcal{M}_1$ on S exactly as in the proceeding section. Let $\mathcal{F} = \{\varphi(X)\colon \phi \subset X \subset V\}$ and for $F = \varphi(X) \in \mathcal{F}$ let $f(F) = h(X) - g$ where $h(X)$ denotes the number of edges having at least one endpoint in X. Observe that $\mathcal{F}$ is crossing and f is submodular on crossing pairs. Apply Theorem 2 to this $\mathcal{F}, f$ and $k = |E|$. Thus we obtain a matroid $\mathcal{M}_2$. Assign weights to the elements of S as follows: $s(e_u) = s(uv)$, $s(e_v) = s(vu)$ for $e = (u, v) \in E$.

Claim. *There is a one-to-one correspondence between the g-strongly connected orientations of G and the common bases of $\mathcal{M}_1$ and $\mathcal{M}_2$ and the weights of corresponding g-strongly connected orientations and common bases are equal.*

Proof. Let a g-strongly connected orientation and a common base B correspond to each other so that the orientation of an edge $e = (u, v)$ is

uv if and only if $e_u \in B$, $e_v \notin B$. ∎

If one would like to apply the weighted matroid intersection algorithm then oracle (∗) of the preceding section is available, but to find a starting common base (i.e. a g-strongly connected orientation) is not simple. For this see [3]; for the weighted case see [4].

Finally, we derive Nash-Williams' theorem. Denote by P_1 and P_2 the polyhedra in $\mathbf{R}^S$ spanned by the incidence vectors of the bases of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. By the claim, what we have to prove is that $P_1 \cap P_2$ has an integral element. By Edmonds' matroid polyhedron intersection theorem [2] the vertices of $P_1 \cap P_2$ are integral, so it is enough to show that $P_1 \cap P_2$ is nonempty. But the hypothesis of Nash-Williams' theorem implies that the vector x defined by $x(s) = \frac{1}{2}$ for $s \in S$ is in $P_1 \cap P_2$.

Remark. Nash-Williams actually proved a stronger orientation theorem. The authors really wonder whether this version can also be derived from matroid intersection.

Remark. In fact, we used the same idea in both applications and it can be shown by the same idea that the 0–1 Edmonds–Giles polyhedron (that is, when the variables are between 0 and 1) is also the projection of a matroid intersection polyhedron. In a forthcoming paper [7] we show the same thing for a polyhedron defined in [8].

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