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Vertex-Disjoint Simple Paths of Given Homotopy in a Planar Graph

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ABSTRACT. We characterize the existence of pairwise vertex-disjoint simple paths $P_1, \ldots, P_k$ of prescribed homotopy in a given planar graph when all end points of the paths are at the "holes" in the plane. Moreover, we give a polynomial-time algorithm for finding these paths, if they exist. Our methods are polyhedral and make use of the ellipsoid method and of considering a fractional solution to the packing problem.

1. The theorem

We prove the following theorem , conjectured by L. Lovász and P. D. Seymour.

THEOREM. *Let $G = (V, E)$ be a planar graph, embedded in $\mathbb{R}^2$, let $I_1, \cdots, I_p$ be (the interiors of) some of its faces (including the unbounded face), and let $P_1, \ldots, P_k$ be paths in G, each with end points on the boundary of $I_1 \cup \cdots \cup I_p$. Then there exist pairwise vertex-disjoint simple paths $\tilde{P}_1, \ldots, \tilde{P}_k$ in G so that $\tilde{P}_i$ is homotopic to P_i in $\mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p)$ for $i = 1, \ldots, k$ if and only if*

(1) (i) *there are pairwise disjoint simple curves $C_1, \ldots, C_k$ in $\mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p)$ such that C_i is homotopic to P_i in $\mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p)$ for $i = 1, \ldots, k$;*
(ii) *for each curve $D: [0, 1] \to \mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p)$ with $D(0), D(1) \in \mathrm{bd}(I_1 \cup \cdots \cup I_p)$ we have $\mathrm{cr}(G, D) \geq \sum_{i=1}^{k} \min \mathrm{cr}(P_i, D)$;*

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(iii) *if* $D_1, D_2\colon S_1 \to \mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p)$ *is a pair of closed curves with the properties that* (a) $D_1(1) = D_2(1) \notin G$, (b) *if* D_1 *or* D_2 *passes any vertex* v *of* G, *then for each* $i = 1, \ldots, k$ *there exists a curve homotopic to* P_i *in* $\mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p)$ *not passing* v, (c) *for* $j = 1, 2$: $\mathrm{cr}(G, D_j) \not\equiv \sum_{i=1}^{k} \min \mathrm{cr}(P_i, D_j)$ (mod 2), *then we have*

$$\mathrm{cr}(G, D_1 \cdot D_2) \geq 2 + \sum_{i=1}^{k} \min \mathrm{cr}(P_i, D_1 \cdot D_2).$$

Note that the case $k = 1$ amounts to the existence of one simple path of given homotopy.

In the theorem and in the sequel we use the following conventions and terminology.

Graphs and their embeddings. We identify a planar graph $G = (V, E)$ embedded in $\mathbb{R}^2$ with its embedding. We consider faces as *open* regions in $\mathbb{R}^2$ and edges as *open* curves (so without end points). The boundary of .. is denoted by bd(..).

Curves. A *curve* is a continuous function $C\colon [0, 1] \to \mathbb{R}^2$. A *closed curve* is a continuous function $C\colon S_1 \to \mathbb{R}^2$ (where S_1 denotes the unit circle in $\mathbb{C}$). For closed curves $D_1, D_2\colon S_1 \to \mathbb{R}^2$ with $D_1(1) = D_2(1)$, the closed curve $D_1 \cdot D_2$ is defined by: $D_1 \cdot D_2(z) = D_1(z^2)$ if $\mathrm{Im}(z) \geq 0$ and $D_1 \cdot D_2(z) = D_2(z^2)$ if $\mathrm{Im}(z) < 0$.

Homotopy. Two curves $C, D\colon [0, 1] \to X \subseteq \mathbb{R}^2$ are called *homotopic* (in X), in notation $C \sim D$, if there exists a continuous function $\Phi\colon [0, 1] \times [0, 1] \to X$ so that $\Phi(0, x) = C(x)$, $\Phi(1, x) = D(x)$, $\Phi(x, 0) = C(0)$, and $\Phi(x, 1) = C(1)$ for all $x \in [0, 1]$. (Note that this implies $C(0) = D(0)$ and $C(1) = D(1)$.) Two closed curves $C, D\colon S_1 \to X \subseteq \mathbb{R}^2$ are called (*freely*) *homotopic* (in X), in notation $C \sim D$, if there exists a continuous function $\Phi\colon [0, 1] \times S_1 \to X$ so that $\Phi(0, x) = C(x)$ and $\Phi(1, x) = D(x)$ for all $x \in S_1$. (Note that not necessarily $C(1) = D(1)$.)

Paths. A *path* in graph $G = (V, E)$ is a sequence

(2) $$(v_0, e_1, v_1, \ldots, e_l, v_l),$$

where $v_0, \ldots, v_l$ are vertices and $e_1, \ldots, e_l$ are edges, so that e_i connects v_{i-1} and v_i $(i = 1, \ldots, l)$. The path is *simple* if $v_0, \ldots, v_l$ are all distinct. Two paths are *vertex-disjoint* if they do not have a vertex in common. When G is embedded in $\mathbb{R}^2$, we identify a path in the obvious way with any curve following this path in the embedding. (That is, we identify (2) with any curve $P\colon [0, 1] \to \mathbb{R}^2$ so that $P(i/l) = v_i$ for $i = 0, \ldots, l$ and $P(x) \in e_i$ if $(i-1)/l < x < i/l$.)

Counting intersections. If $C, D\colon [0, 1] \to \mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p)$ are curves, where $I_1, \ldots, I_p$ are faces of a graph $G = (V, E)$ embedded in $\mathbb{R}^2$, then

(3) $$\begin{aligned} \mathrm{cr}(C, D) &:= |\{(x, y) \in [0, 1] \times [0, 1] \mid C(x) = D(y)\}|, \\ \min \mathrm{cr}(C, D) &:= \min\{\mathrm{cr}(\tilde{C}, \tilde{D}) | \tilde{C} \sim C, \tilde{D} \sim D \text{ (in } \mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p))\}, \\ \mathrm{cr}(G, D) &:= |\{y \in [0, 1] | D(y) \in G\}|, \\ &\qquad \text{if } D \text{ is not a constant function}, \\ &:= 1, \quad \text{if } D \text{ is a constant function}. \end{aligned}$$

If $C\colon [0, 1] \to \mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p)$ is a curve and $D\colon S_1 \to \mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p)$ is a closed curve, then

(4) $$\begin{aligned} \mathrm{cr}(C, D) &:= |\{(x, y) \in [0, 1] \times S_1 \mid C(x) = D(y)\}|, \\ \min \mathrm{cr}(C, D) &:= \min\{\mathrm{cr}(\tilde{C}, \tilde{D}) | \tilde{C} \sim C, \tilde{D} \sim D \text{ (in } \mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p))\}, \\ \mathrm{cr}(G, D) &:= |\{y \in S_1 \mid D(y) \in G\}|. \end{aligned}$$

Crossings. Two (closed) curves C, D are said to *cross* if there exist x, y so that $C(x) = D(y)$ and there exists a homeomorphism $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ so that the functions $\phi \circ C$ and $\phi \circ D$ are linear functions in neighbourhoods of x and y, respectively, with different angles. In that case, (x, y) is said to *give a crossing*. If C and D do not cross, they are called *noncrossing*.

The greater part of this paper consists of proving sufficiency of the conditions (1), which is based on Lemmas 1 and 2 proved in Sections 3 and 4. Lemma 1 is shown with the help of an auxiliary theorem proved in Section 2.

2. An auxiliary theorem on edge-disjoint paths

One ingredient for our proof is the following "homotopic flow-cut theorem" ([6]).

HOMOTOPIC FLOW-CUT THEOREM. *Let* $G = (V, E)$ *be a planar graph embedded in* $\mathbb{R}^2$, *let* $I_1, \ldots, I_p$ *be some of the faces of* G *(including the unbounded face), and let* $C_1, \ldots, C_k$ *be curves in* $\mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p)$ *with end points in* $V \cap \mathrm{bd}(I_1 \cup \cdots \cup I_p)$. *Then there exist paths* $P_1^1, \ldots, P_1^{t_1}, P_2^1, \ldots, P_2^{t_2}, \ldots, P_k^1, \ldots, P_k^{t_k}$ *in* G *and rational numbers* $\lambda_1^1, \ldots, \lambda_1^{t_1}, \lambda_2^1, \ldots, \lambda_2^{t_2}$,

$\ldots, \lambda_k^1, \ldots, \lambda_k^{t_k} > 0$ *so that*

(10) (i) $P_i^j \sim C_i$ *in* $\mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p)$ $\quad (i = 1, \ldots, k; j = 1, \ldots, t_i)$,

(ii) $\sum_{j=1}^{t_i} \lambda_i^j = 1 \quad (i = 1, \ldots, k)$,

(iii) $\sum_{i=1}^{k} \sum_{j=1}^{t_i} \lambda_i^j \chi^{P_i^j}(e) \leq 1 \quad (e \in E)$,

if and only if for each curve $D: [0, 1] \to \mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p \cup V)$ *with* $D(0)$, $D(1) \in \mathrm{bd}(I_1 \cup \cdots \cup I_p)$ *we have*

$$\text{(11)} \qquad \mathrm{cr}(G, D) \geq \sum_{i=1}^{k} \min \mathrm{cr}(C_i, D).$$

Here, for any path P in G and any edge e of G, $\chi^P(e)$ denotes the number of times P passes e.

As our "auxiliary theorem" we derive that under certain circumstances the λ_i^j can be taken to be integral.

AUXILIARY THEOREM. *Let* $G = (V, E)$ *be a planar graph embedded in* $\mathbb{R}^2$, *let* $I_1, \ldots, I_p$ *be some of the faces of* G *(including the unbounded face), and let* $C_1, \ldots, C_k$ *be curves in* $\mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p)$ *with end points in* $V \cap \mathrm{bd}(I_1 \cup \cdots \cup I_p)$, *so that*

(12) (i) *each* C_i *has only a finite number of self-intersections and no self-crossings;*
(ii) *each two of the* C_i *have only a finite number of intersections and no crossings;*
(iii) *each vertex of* G *either has degree even and is no end point of any* C_i, *or has degree* 1 *and is an end point of exactly one* C_i.

Then there exist pairwise edge-disjoint and pairwise noncrossing paths $P_1, \ldots, P_k$ *in* G, *without self-crossings and not passing the same edge more than once, so that* $P_i \sim C_i$ *for* $i = 1, \ldots, k$, *if and only if for each curve* $D: [0, 1] \to \mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p \cup V)$ *with* $D(0), D(1) \in \mathrm{bd}(I_1 \cup \cdots \cup I_p)$ *we have* (11).

[Here paths are called *edge-disjoint* if they do not have any edge in common.]

PROOF. The "if" part is trivial, since for any D in question we have

$$\text{(13)} \qquad \mathrm{cr}(G, D) \geq \sum_{i=1}^{k} \mathrm{cr}(P_i, D) \geq \sum_{i=1}^{k} \min \mathrm{cr}(C_i, D).$$

To see the "only if" part, suppose that (11) is satisfied for each curve D in question. By the homotopic flow-cut theorem, there exist paths P_i^j in G and rationals $\lambda_i^j > 0$ (for $i = 1, \ldots, k; j = 1, \ldots, t_i$) satisfying (10). In fact, as the λ_i^j can be written with one common denominator, say K, we may assume that $t_1 = \cdots = t_k = K$ and that each λ_i^j is equal to $1/K$ (this is achieved by replacing each P_i^j by $K \cdot \lambda_i^j$ copies of P_i^j). Replacing each edge of G by K parallel edges, we obtain a graph $G' = (V, E')$ and pairwise edge-disjoint paths $P_1^1, \ldots, P_1^K, \ldots, P_k^1, \ldots, P_k^K$ in G'. Clearly each face of G corresponds to a face of G', and we will use the same name for both of them. In particular, $I_1, \ldots, I_p$ are again faces of G'.

CLAIM 1. We may assume that P_i^j and $P_{i'}^{j'}$ are noncrossing, if $i \neq i'$.

PROOF. Suppose we have chosen the paths $P_1^1, \ldots, P_k^K$ so that

$$\text{(14)} \qquad \sum_{i=1}^{k} \sum_{i'=i+1}^{k} \sum_{j=1}^{k} \sum_{j'=1}^{K} (\text{number of crossings of } P_i^j \text{ and } P_{i'}^{j'})$$

is as small as possible. We must show that this sum is 0. Indeed, suppose P_i^j and $P_{i'}^{j'}$ have a crossing, where $i \neq i'$. As C_i and $C_{i'}$ have no crossings, there exist $x, x', y, y' \in [0, 1]$ so that $(x, x') \neq (y, y')$, $P_i^j(x) = P_{i'}^{j'}(x')$ and $P_i^j(y) = P_{i'}^{j'}(y')$, so that both (x, x') and (y, y') give crossings, and so that the x-y part of P_i^j is homotopic to the x'-y' part of $P_{i'}^{j'}$ (cf. [6]). Exchanging these two parts decreases sum (14), contradicting its minimality. □

Clearly, we may assume moreover that no P_i^j has null-homotopic parts.

Now in order to prove our auxiliary theorem we apply induction on the number of edges of G plus the number of faces of G not in $\{I_1, \ldots, I_p\}$.

If all $C_1, \ldots, C_k$ are homotopic trivial, the theorem is trivial. So assume without loss of generality that C_1 is not homotopic trivial. Let e, e' be the first two edges of G passed by P_1^1. That is, $P_1^1 = (v_0, e, v_1, e', v_2, \alpha)$ for some string α. We consider two cases.

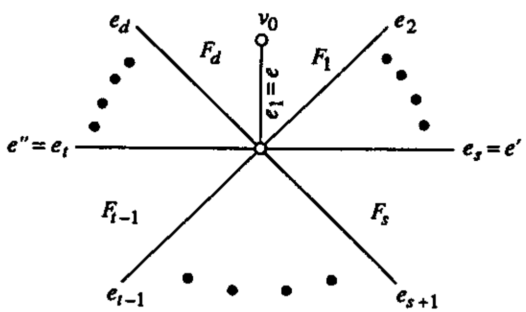
CASE 1. Each of $P_1^1, \ldots, P_1^K$ passes e' as second edge. In this case no other P_i^j can pass edge e' (by (11)(iii)). Now delete edges e and e' from G, add a new vertex w in the new face $F' \cup e' \cup F''$ (where F' and F are the faces incident to e' (possibly $F = F'$)), and add a new edge e'' connecting w and v_2. Replace C_1 by (w, e'', v_2, α). Replace $C_2, \ldots, C_k$ by $P_2^1, \ldots, P_k^1$, respectively. Replace $\{I_1, \ldots, I_p\}$ by

$$(\{I_1, \ldots, I_p\} \setminus \{F, F'\}) \cup \{(F \cup e' \cup F') \setminus (e'' \cup w)\}.$$

We claim that condition (11) is maintained in the new situation. This follows from the fact that in the new situation there exists a "fractional" packing of paths as in the homotopic flow-cut theorem: for each $i = 2, \ldots, k$, all P_i^j are homotopic to P_i^1 in $\mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p \cup F \cup e' \cup F')$; moreover, for $j = 1, \ldots, K$, we can write $P_1^j = (v_0, e, v_1, e', v_2, \alpha')$ so that $(w, e'', v_2, \alpha') \sim (w, e'', v_2, \alpha)$.

CASE 2. Not each of $P_1^1, \ldots, P_1^K$ passes e' as second edge. Without loss of generality, path P_1^2 passes edge $e'' \neq e'$ as second edge. Consider the neighbourhood of v_1, with edges $e_1, \ldots, e_d$ and faces $F_1, \ldots, F_d$ as indicated:

(15)



So $e = e_1$, $e' = e_s$, $e'' = e_t$, and $F_1 = F_d \in \{I_1, \ldots, I_p\}$. As $P_1^1 \cdot (P_2^1)^{-1}$ is a homotopic trivial cycle, we know $F_s, \ldots, F_{t-1} \notin \{I_1, \ldots, I_p\}$. Now let $I_{p+1} := F_s$. We claim

CLAIM 2. For each curve $D\colon [0, 1] \to \mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_{p+1} \cup V)$ with $D(0), D(1) \in \text{bd}(I_1 \cup \cdots \cup I_{p+1})$ we have

$$\text{(16)} \qquad \text{cr}(G, D) \geq \sum_{i=1}^{k} \min \text{cr}'(P_i^1, D),$$

where

$$\min \text{cr}'(P_i^1, D) := \min\{\text{cr}(\widetilde{P}, \widetilde{D}) | \widetilde{P} \sim P_i^1, \widetilde{D} \sim D \text{ in } \mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_{p+1})\}.$$

PROOF. Let Q be the path from v_1 to v_1 following the boundary of face F_s clockwise (cf. (15); so Q starts with e_s and ends with e_{s+1}). For $j = 1, \ldots, K$, let

(17)
$R_1^j := P_1^j$ if P_1^j uses one of the edges $e_1, \ldots, e_s$ as second edge;
$R_1^j := (v_0, e, v_1, Q, v_1, \beta)$ if P_i^j uses one of the edges $e_{s+1}, \ldots, e_d$ as second edge, and $P_1^j = (v_0, e, v_1, \beta)$.

So $R_1^j \sim P_1^1$ in $\mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_{p+1})$ for $j = 1, \ldots, K$. Moreover, for $i = 2, \ldots, k$ and $j = 1, \ldots, K$ let $R_i^j := P_i^j$. By Claim 1, $R_i^j \sim P_i^1$ in $\mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_{p+1})$, for $i = 2, \ldots, k$ and $j = 1, \ldots, K$. Now for each edge e of G:

$$\text{(18)} \qquad \text{(i)} \sum_{i=1}^{k} \sum_{j=1}^{K} \lambda_i^j \chi^{R_i^j}(e) \leq 1 \qquad \text{if } e \text{ is not incident to } F_{s'};$$

$$\text{(ii)} \sum_{i=1}^{k} \sum_{j=1}^{K} \lambda_i^j \chi^{R_i^j}(e) < 2 \qquad \text{if } e \text{ is incident to } F_s.$$

(The strict inequalities follow from the fact that the sum of those λ_1^j for which P_1^j uses one of the edges $e_{s+1}, \ldots, e_d$ as second edge is strictly less than 1 (since P_1^1 uses e_s as second edge).)

Now choose $D\colon [0, 1] \to \mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_{p+1} \cup V)$ with $D(0), D(1) \in \text{bd}(I_1 \cup \cdots \cup I_{p+1})$. We may assume that if $D(z) \in G$, then D has a crossing with G at z. This implies that $D \cap \text{bd}(F_s) = \{D(0), D(1)\} \cap \text{bd}(F_s)$.

If not both $D(0)$ and $D(1)$ belong to $\text{bd}(F_s)$ we have by (18):

$$\text{(19)} \qquad \begin{aligned} \text{cr}(G, D) &= \sum_{e \in E} \chi^D(e) > -1 + \sum_{e \in E} \chi^D(e) \sum_{i=1}^{k} \sum_{j=1}^{K} \lambda_i^j \chi^{R_i^j}(e) \\ &= -1 + \sum_{i=1}^{k} \sum_{j=1}^{K} \lambda_i^j \sum_{e \in E} \chi^{R_i^j}(e) \chi^D(e) \\ &= -1 + \sum_{i=1}^{k} \sum_{j=1}^{K} \lambda_i^j \cdot \text{cr}(R_i^j, D) \\ &\geq -1 + \sum_{i=1}^{k} \min \text{cr}'(P_i^1, D). \end{aligned}$$

[Here $\chi^D(e)$ denotes the number of times D intersects e.] Note that (19) implies (16). If both $D(0)$ and $D(1)$ belong to $\text{bd}(F_s)$, then using (18) one similarly shows

$$\text{(20)} \qquad \text{cr}(G, D) > -2 + \sum_{i=1}^{k} \min \text{cr}'(P_i^1, D).$$

Now by condition (12)(iii),

$$\text{(21)} \qquad \text{cr}(G, D) \equiv \sum_{i=1}^{k} \text{cr}(P_1^1, D) \equiv \sum_{i=1}^{k} \min \text{cr}(P_1^1, D) \pmod 2,$$

since $D(0)$ and $D(1)$ belong to the boundary of the same face F_s. Now (21) and (20) imply (16). □

So by induction there exist pairwise edge-disjoint and pairwise noncrossing paths $\widetilde{P}_1, \ldots, \widetilde{P}_k$ (without self-crossing and not using the same edge more than once), so that $\widetilde{P}_i \sim P_i^1$ in $\mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_{p+1})$, for $i = 1, \ldots, k$. This implies $P_i \sim P_i^1 \sim C_i$ in $\mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p)$. □

3. Lemma 1

The first part of the proof of our theorem consists of showing that conditions (1)(i) and (ii) are equivalent to the existence of a certain "graph-disjoint" system of curves. This is the content of Lemma 1.

Let $G = (V, E)$ be a planar graph embedded in the plane $\mathbb{R}^2$, and let $I_1, \dots, I_p$ be some of its faces, including the unbounded face. With any curve $C: [0, 1] \to \mathbb{R}^2$ we can associate its *face sequence*

(22) $$(\varphi_0, \dots, \varphi_t)$$

where each φ_j is a vertex, edge, or face of G, so that C starts in φ_0, next passes φ_1, next φ_2, and so on, until it terminates in φ_t. (We consider vertices also as a singleton set.) So φ_{j-1} and φ_j are incident for $j = 1, \dots, t$. (φ and φ' are called *incident* if $\varphi \neq \varphi'$ and $\varphi \cup \varphi'$ is connected.)

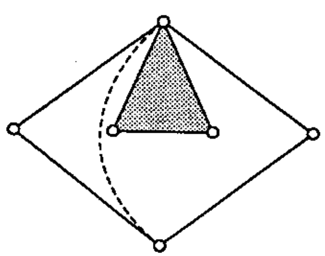
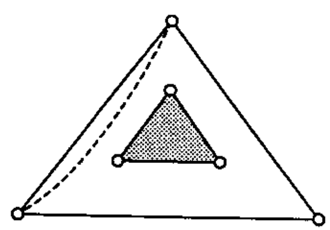
Now let $C_1, \dots, C_k: [0, 1] \to \mathbb{R}^2 \setminus (I_1 \cup \dots \cup I_p)$ be curves, where C_i has face sequence

(23) $$(\varphi_{i,0}, \dots, \varphi_{i,t_i}),$$

for $i = 1, \dots, k$. We call $C_1, \dots, C_k$ *graph-disjoint* (with respect to $G; I_1, \dots, I_p$) if for all $i, i' = 1, \dots, k$; $j = 1, \dots, t_i$; $j' = 1, \dots, t_{i'}$:

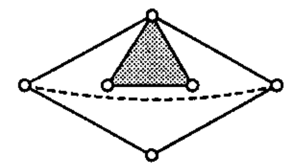
(24)
- (i) $\varphi_{i,j} = \varphi_{i',j'}$ if and only if $i = i'$ and $j = j'$;
- (ii) $\varphi_{i,j}$ and $\varphi_{i',j'}$ are incident if and only if $i = i'$ and $|j - j'| = 1$;
- (iii) if $i = i'$ and $|j - j'| = 1$ then each closed curve in $\varphi_{i,j} \cup \varphi_{i,j'}$ is homotopically trivial in $\mathbb{R}^2 \setminus (I_1 \cup \dots \cup I_p)$.

Condition (24)(iii) is meant to exclude, e.g., the following situations:

(25)  and 

where the interrupted curve indicates a curve C_i and where the shaded region

indicates one of the faces $I_1, \dots, I_p$. However, the following is allowed:

(26) 

It is easy to see that if each C_i is a curve in G (i.e., no element in (23) is a face of G), then the conditions (24) amount to the C_i forming a collection of pairwise vertex-disjoint simple paths in G.

We show:

Lemma 1. *Let $G = (V, E)$ be a planar graph, embedded in $\mathbb{R}^2$, let $I_1, \dots, I_p$ be some of its faces (including the unbounded face), and let $P_1, \dots, P_k$ be paths in G, each with end points on* $\mathrm{bd}(I_1 \cup \dots \cup I_p)$. *Then the following are equivalent:*

(27)
- (a) *conditions* (1)(i) *and* (ii) *hold;*
- (b) *there exists a graph-disjoint collection of curves $C_1, \dots, C_k$ where $C_i \sim P_i$ in $\mathbb{R}^2 \setminus (I_1 \cup \dots \cup I_p)$ for $i = 1, \dots, k$.*

Proof. I. To see (b) $\Rightarrow$ (a) in (27), let $C_1 \sim P_1, \dots, C_k \sim P_k$ form a graph-disjoint collection of curves. Then clearly, by (24)(i), there are *simple* curves $\tilde{C}_1 \sim C_1, \dots, \tilde{C}_k \sim C_k$ which are again graph-disjoint, and hence they are disjoint. This shows (1)(i).

To derive (1)(ii), let $D: [0, 1] \to \mathbb{R}^2 \setminus (I_1 \cup \dots \cup I_p)$ be any curve, with $D(0), D(1) \in \mathrm{bd}(I_1 \cup \dots \cup I_p)$, and with face-sequence say $(\psi_0, \dots, \psi_t)$, so that $\mathrm{cr}(G, D)$ is finite. Then $\mathrm{cr}(G, D) = \frac{1}{2}(t+1)$. Moreover, we can draw D so that, leaving its face-sequence and homotopy invariant, it only intersects any C_i if it is necessary; that is, D does not intersect any C_i both in ψ_{j-1} and in ψ_j (as ψ_{j-1} and ψ_j are incident, one of them being a face of G). So

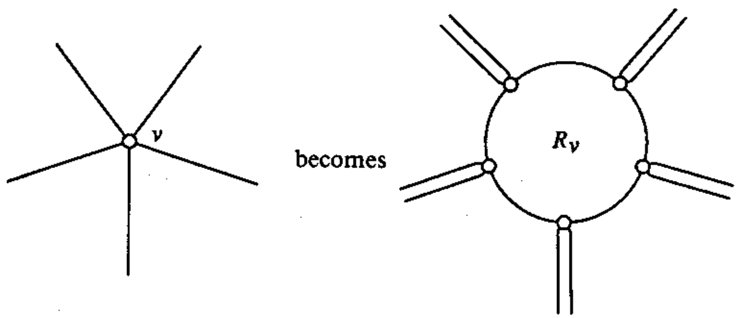
(28) $$\mathrm{cr}(G, D) = \frac{1}{2}(t+1) \geq \sum_{i=1}^{k} \min \mathrm{cr}(C_i, D),$$

and therefore (1)(ii) holds.

II. We next show the implication (a) $\Rightarrow$ (b) in (27). To this end, we construct from G an auxiliary graph $G' = (V', E')$ as follows. Let $\varepsilon > 0$ be so that $\varepsilon < \frac{1}{2}\|v - w\|$ for each two vertices v and w of G. For each vertex v of G, remove from G all points in $R_v := \{p \in \mathbb{R}^2 \mid \|p - v\| < \varepsilon\}$ and add the circle $S_v := \{p \in \mathbb{R}^2 \mid \|p - v\| = \varepsilon\}$. Each vertex v thus gives $\deg(v)$ new vertices, on S_v. Next, for each edge e of G, replace the part of e left by two parallel edges. So, altogether, the neighbourhood of any vertex

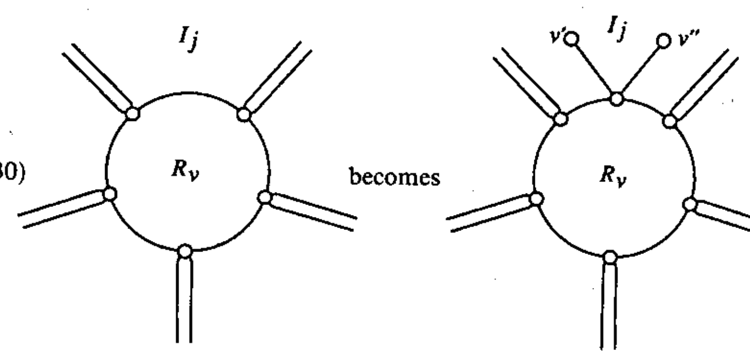
v (for example):

(29)



Now if path P_i starts in $v \in \text{bd}(I_1 \cup \cdots \cup I_p)$, choose $j = 1, \ldots, p$ so that $v \in \text{bd}(I_j)$. Next add, in the face of the new graph corresponding to I_j, two new vertices, v' and v'' say, and connect them by edges, say $e_{v'}$ and $e_{v''}$, to some point on $S_v \setminus G$. So

(30)



We proceed similarly at the end point w of P_i, yielding the vertices w' and w'' and edges $e_{w'}$ and $e_{w''}$. Now replace P_i by the curves P_i' and P_i'' as follows. P_i' is obtained from P_i by adding, at the beginning, a curve from v' to v, first following $e_{v'}$ and next passing R_v, and at the end, a curve from w to w', first passing R_w and next following $e_{w'}$. Curve P_i'' is obtained similarly from P_i using v'', $e_{v''}$, w'', and $e_{w''}$.

We do this for each $i = 1, \ldots, k$. This defines the graph $G' = (V', E')$, together with the curves $P_1', P_1'', \ldots, P_k', P_k''$. Let F' denote the face of G' corresponding to any face F of G. By condition (1)(i) we know that there exist curves $C_1' \sim P_1'$, $C_1'' \sim P_1''$, $\ldots$, $C_k' \sim P_k'$, $C_k'' \sim P_k''$ (in $\mathbb{R}^2 \setminus (I_1' \cup \cdots \cup I_p')$) so that $C_1', C_1'', \ldots, C_k', C_k''$ satisfy conditions (12)(i) and (ii) (possibly by flipping v' and v'' in (30)). Clearly, also condition (12)(iii) holds for G

and $C_1', C_1'', \ldots, C_k', C_k''$. Moreover, condition (1)(ii) implies

$$\text{(31)} \qquad \text{cr}(G', D') \geq \sum_{i=1}^{k} \min \text{cr}(C_i', D') + \sum_{i=1}^{k} \min \text{cr}(C_i'', D')$$

for each curve $D': [0, 1] \to \mathbb{R}^2 \setminus (I_1' \cup \cdots \cup I_p' \cup V')$ with $D'(0), D'(1) \in \text{bd}(I_1' \cup \cdots \cup I_p')$. Indeed, for each such D', we can "construct" the parts of D' in $\overline{R_v}$. We obtain a curve D with $\text{cr}(G', D') = 2\,\text{cr}(G, D)$ and $\min \text{cr}(C_i', D') \leq \min \text{cr}(P_i, D)$, $\min \text{cr}(C_i'', D') \leq \min \text{cr}(P_i, D)$ for $i = 1, \ldots, k$. Hence by (1) (ii) we have (31).

So our auxiliary theorem gives us pairwise edge-disjoint and pairwise non-crossing paths $Q_1' \sim C_1'$, $Q_1'' \sim C_1''$, $\ldots$, $Q_k' \sim C_k'$, $Q_k'' \sim C_k''$. Let R_i' and R_i'' be the paths in G obtained from P_i' and P_i'' by contracting G' to G, for $i = 1, \ldots, k$. Then for each $i = 1, \ldots, k$, the cycle $R_i' \cdot (R_i'')^{-1}$ follows the boundary of a simply-connected subset S_i of $\mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p)$ with $P_i(0), P_i(1) \in S_i$, where S_i is a union of faces, edges, and vertices. Let C_i be a "free-est possible" curve in S_i connecting $P_i(0)$ and $P_i(1)$. This means that if $C_i(z)$ is a vertex for some z, then there exist $z', z'' \in [0, 1]$ so that $C_i(z) = R_i'(z') = R_i''(z'')$ and so that the 0-z' part of R_i' is homotopic to the 0-z'' part of R_i''.

Obviously, the curves $C_1, \ldots, C_k$ form the graph-disjoint system of curves, with $C_i \sim P_i$ for $i = 1, \ldots, k$. This finishes the proof of Lemma 1. □

4. Lemma 2

The second part of our proof consists of showing that the existence of a graph-disjoint system of curves together with condition (1)(iii) implies the existence of a packing of paths as required by the theorem, which is the content of Lemma 2. So together with Lemma 1 this implies our theorem.

A basic ingredient for the proof of Lemma 2 is the following well-known observation. (For sharpenings, see Deming [2], Sterboul [7], and Korach [5].)

PROPOSITION. *Let $G = (V, E)$ be an undirected graph (loops allowed), and let $M \subseteq E$ be a perfect matching. Then G has a coclique K with $|K| = \frac{1}{2}|V|$ if and only if G contains no cycle*

$$\text{(32)} \qquad (v_0, e_1, v_1, \ldots, e_l, v_l)$$

where

(33) (i) *$v_0 = v_l$, e_i is an edge connecting the vertices v_{i-1} and v_i $(i = 1, \ldots, l)$ and l is even;*
(ii) *$e_1, e_3, e_5, \ldots, e_{l-1} \in M$ and $e_2, e_4, \ldots, e_l \notin M$;*
(iii) *$v_t = v_0$ and $v_{t-1} = v_1$ for some odd t.*

[Here a *loop* is considered as a singleton. A *perfect matching* is a set of $\frac{1}{2}|V|$ edges covering V (so they are pairwise disjoint and nonloops). A *coclique* is a set of vertices not containing any edge as subset.]

PROOF. I. To show the "only if" part, suppose G has a coclique K of size $\frac{1}{2}|V|$ and G contains a cycle (32) satisfying (33). Then for each edge in M exactly one of its end points belongs to K. As $v_0 = v_l$ it follows that either $v_0, v_2, \ldots, v_l \in K$ or $v_1, v_3, \ldots, v_{l-1} \in K$ Since $v_0 = v_t$ and $v_1 = v_{t-1}$ for some odd t, in both cases it follows that $v_0, v_1 \in K$ — a contradiction as e_1 connects v_0 and v_1.

II. The "if" part is shown by induction on $|V|$. Suppose G does not contain any cycle (32) satisfying (33). Then no edge in M has at both of its vertices a loop attached.

If for each edge in M, exactly one of its vertices has a loop attached, we can choose for K the set of all vertices at which no loop is attached.

If there exists an edge $e_0 \in M$ so that at none of its vertices is there a loop attached, let e_0 connect v and w, and define

(34)
$$\begin{aligned} V' &:= V \setminus \{v, w\}, \\ \delta(v) &:= \{v' \in V' | \{v, v'\} \in E\}, \\ \delta(w) &:= \{w' \in V' | \{w, w'\} \in E\}, \\ E' &:= \{e \in E | e \subseteq V'\} \cup \{\{v', w'\} | v' \in \delta(v), w' \in \delta(w)\}, \\ M' &:= M \setminus \{e_0\}. \end{aligned}$$

One easily checks that graph $G' = (V', E')$, with perfect matching M', again has no cycle (32) satisfying (33). Hence, by induction, G' contains a coclique K' of size $\frac{1}{2}|V'|$. Then $\delta(v) \cap K' = \emptyset$ or $\delta(w) \cap K' = \emptyset$ (as $\{v', w'\} \in E'$ for each $v' \in \delta(v)$ and $w' \in \delta(w)$). So $K' \cup \{v\}$ or $K' \cup \{w\}$ is a coclique of size $\frac{1}{2}|V|$ in G. □

We derive from this:

LEMMA 2. *Let $G = (V, E)$ be a planar graph embedded in $\mathbb{R}^2$, let $I_1, \ldots, I_p$ be some of its faces (including the unbounded face), and let $P_1, \ldots, P_k$ be paths in G, each with end points on $\mathrm{bd}(I_1 \cup \cdots \cup I_p)$. Suppose there exists a graph-disjoint system of curves $C_1 \sim P_1, \ldots, C_k \sim P_k$. If (1)(iii) holds, then there exist pairwise vertex-disjoint simple paths $\tilde{P}_1 \sim P_1, \ldots, \tilde{P}_k \sim P_k$ in G.*

PROOF. From $C_1, \ldots, C_k$ and G we construct an auxiliary graph $G' = (V', E')$, with a perfect matching M, as follows. For $i = 1, \ldots, k$, let C_i have face sequence

(35)
$$(\varphi_{i,0}, \ldots, \varphi_{i,t_i}).$$

If $\varphi_{i,j}$ is a face of G, it is divided by curve C_i into two open parts, say $\varphi'_{i,j}$ and $\varphi''_{i,j}$. Place in each of these parts a point, called $v'_{i,j}$ and $v''_{i,j}$. All these points (for all $i = 1, \ldots, k$, $j = 1, \ldots, t_i$ with $\varphi_{i,j}$ a face of G) form the vertex set V' of G'.

Let each pair $v'_{i,j}, v''_{i,j}$ be connected by an edge, drawn in $\varphi_{i,j}$ intersecting C_i once. These edges form the perfect matching M in G'. Moreover, vertices $v^\alpha_{i,j}$ and $v^\beta_{i',j'}$ of G' are connected by an edge if:

(36)
(i) $i \neq i'$ and there exists a vertex ψ of G contained both in $\overline{\psi^\alpha_{i,j}}$ and in $\overline{\psi^\beta_{i',j'}}$; or
(ii) $i = i'$, and there exists a vertex ψ of G contained both in $\overline{\varphi^\alpha_{i,j}}$ and in $\overline{\varphi^\beta_{i',j'}}$, so that there is a closed curve K, not homotopic trivial in a $\mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p)$, with face sequence $(\psi, \varphi_{i,j}, \ldots, \varphi_{i,j'}, \psi)$.

Note that (ii) yields a loop in G' if $i = i'$, $j = j'$, $\alpha = \beta$, and $\overline{\varphi^\alpha_{i,j}}$ contains a closed curve not being homotopic trivial in $\mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p)$.

In fact, each edge e of G' can be represented by a curve in $\mathbb{R}^2$ connecting $v^\alpha_{i,j}$ and $v^\beta_{i',j'}$. If $i \neq i'$, it starts in $v^\alpha_{i,j}$, moves in $\varphi^\alpha_{i,j}$ to vertex ψ as in (36)(i), and next moves in $\varphi^\beta_{i',j'}$ to $v^\beta_{i',j'}$. If $i = i'$, we can make the curve e so that

(37) the closed curve formed by
- the curve e,
- the edge in M connecting $v'_{i,j}$ and $v''_{i,j}$, until its crossing with C_i,
- the edge in M connecting $v'_{i,j'}$ and $v''_{i,j'}$ until its crossing with C_i,
- the part of C_i between these two edges in M

is not homotopic trivial in $\mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p)$.

This defines graph $G' = (V', E')$, with perfect matching M, embedded in $\mathbb{R}^2$, possibly with crossings. Each edge in M intersects the union of the curves C_i exactly once and does not intersect G. The edges not in M do not cross any C_i and intersect G exactly once.

Now if G' has a coclique K of size $\frac{1}{2}|V'|$, then for each i, j exactly one of the two vertices $v'_{i,j}$ and $v''_{i,j}$ belongs to K. Hence for each $i = 1, \ldots, k$:

(38)
- the part of the boundary of $\varphi'_{i,j}$ not in C_i, if $v'_{i,j} \in K$;
- the part of the boundary of $\varphi''_{i,j}$ not in C_i, if $v''_{i,j} \in K$;
- the edges and vertices among $\varphi_{i,0}, \ldots, \varphi_{i,t_i}$

contain a simple path $\tilde{P}_i \sim P_i$ — in such a way that $\tilde{P}_1, \ldots, \tilde{P}_k$ are pairwise vertex-disjoint. So in this case we are at the required conclusion.

Therefore, assume G' has no coclique of size $\frac{1}{2}|V'|$. By the proposition above, it follows that G' has a cycle (32) satisfying (33). As G' is drawn in $\mathbb{R}^2$, we can represent this cycle as a closed curve $D: S_1 \to \mathbb{R}^2$. Let D_1 and D_2 be the closed curves corresponding to parts

(39) $\quad (v_0, e_1, v_1, \ldots, e_t, v_t)$ and $(v_t, e_{t+1}, v_{t+1}, \ldots, e_l, v_l)$

of (32). So D can be written as $D_1 \cdot D_2$.

We show that D_1 and D_2 give a contradiction to condition (1)(iii). First note that $D_1(1) = D_2(1) = v_0$ does not belong to G, as v_0 is a vertex of G'. If D_1 or D_2 passes vertex v of G, then no curve C_i passes v. So conditions (a) and (b) in (1)(iii) are fulfilled.

Now, $\mathrm{cr}(G, D)$ is equal to the number of h for which e_h in (32) does not belong to M, while $\sum_{i=1}^k \mathrm{cr}(C_i, D)$ is equal to the number of h for which e_h belongs to M. Hence

(40) $$\mathrm{cr}(G, D) = \frac{1}{2}l \quad \text{and} \quad \sum_{i=1}^k \mathrm{cr}(C_i, D) = \frac{1}{2}l.$$

Similarly,

(41) (i) $\mathrm{cr}(G, D_1) = \frac{1}{2}(t-1)$ and $\sum_{i=1}^k \mathrm{cr}(C_i, D) = \frac{1}{2}(t+1)$;

(ii) $\mathrm{cr}(G, D_2) = \frac{1}{2}(l-t+1)$ and $\sum_{i=1}^k \mathrm{cr}(C_i, D_2) = \frac{1}{2}(l-t-1)$.

In particular, by (7), $\mathrm{cr}(G, D_1) \not\equiv \sum_{i=1}^k \min \mathrm{cr}(P_i, D_1) \pmod 2$ and $\mathrm{cr}(G, D_2) \not\equiv \sum_{i=1}^k \min \mathrm{cr}(P_i, D_2) \pmod 2$. So also condition (c) in (1)(iii) is fulfilled.

Now (40) contradicts (1)(iii) when we have proved

(42) $$\mathrm{cr}(C_i, D) = \min \mathrm{cr}(P_i, D) \quad \text{for } i = 1, \ldots, k.$$

Now $\min \mathrm{cr}(P_i, D) = \min \mathrm{cr}(C_i, D)$ as $P_i \sim C_i$. By the results of [6], if $\min \mathrm{cr}(C_i, D) < \mathrm{cr}(C_i, D)$, there exist $g, h \in \mathbb{Z}$ so that (taking indices mod l):

(43)
(i) $g < h$,
(ii) C_i intersects e_g and e_h,
(iii) the part of C_i between e_g and e_h is homotopic to the part $(e_g, v_g, e_{g+1}, v_{g+1}, \ldots, e_{h-1}, v_{h-1}, e_h)$ of D^{h-g}

(where D^{h-g} is the closed curve going $h-g$ times around D). Actually, we begin and end the parts mentioned in (iii) at the crossing points of e_g and e_h with C_i.

We may assume that we have chosen i, g, h so that $h-g$ is as small as possible. Note that $h-g$ is even. If $h-g=2$ we are in contradiction with (37). If $h-g>2$, consider the edge e_{g+2}. As $e_{g+2} \in M$, there exists an i' so that e_{g+1} crosses $C_{i'}$ (possibly $i = i'$). Since the e_g-e_h part of C_i together with $(e_g, v_g, e_{g+1}, v_{g+1}, \ldots, e_{h-1}, v_{h-1}, e_h)$ forms a homotopic trivial cycle, since $C_{i'}$ does not cross C_i and since both end points of $C_{i'}$ are at one of the faces $I_1, \ldots, I_p$, there exists an h' so that $g+2 < h' < h$, so that $C_{i'}$ crosses $e_{h'}$ and so that the e_{g+2}-$e_{h'}$ part of $C_{i'}$ is homotopic to $(e_{g+2}, v_{g+2}, \ldots, e_{h'-1}, v_{h'-1}, e_{h'})$. As $h'-(g+2) < h-g$, this contradicts the minimality of $h-g$.

This completes the proof of Lemma 2. □

This completes the proof of our theorem. □

Remark. Note that we in fact proved in Lemma 2 that (if there exists a graph-disjoint set of curves $C_1 \sim P_1, \ldots, C_k \sim P_k$) there exist pairwise vertex-disjoint simple paths $\tilde{P}_1 \sim P_1, \ldots, \tilde{P}_k \sim P_k$ in G, if and only if the graph G' constructed in the proof has a coclique of size $\frac{1}{2}|V'|$.

5. Polynomial-time solvability

It is not directly clear that our theorem gives a "good characterization" for the problem:

(44) *given*: — a planar graph $G = (V, E)$, embedded in $\mathbb{R}^2$,
— faces $I_1, \ldots, I_p$ of G (including the unbounded face),
— paths $P_1, \ldots, P_k$ in G, each with end points on $\mathrm{bd}(I_1 \cup \cdots \cup I_p)$,
find: paths $\tilde{P}_1, \ldots, \tilde{P}_k$ in G so that $\tilde{P}_i \sim P_i$ in $\mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p)$ for $i = 1, \ldots, k$

(i.e., that our theorem implies that the decision version of (44) belongs to $\mathcal{NP} \cap \text{co-}\mathcal{NP}$). We show in this section that problem (44) in fact is solvable in polynomial time (i.e., that (44) belongs to $\mathcal{P}$).

We describe a "brute force" polynomial-time method. We do not aim at designing a most efficient algorithm, but rather at giving an existence proof of a polynomial-time method.

We first show that there exists a polynomial-time algorithm for the following *shortest homotopic path problem*:

(45) *given*: — a planar graph $G = (V, E)$ embedded in $\mathbb{R}^2$,
— faces $I_1, \ldots, I_p$ of G (including the unbounded face),
— a path P in G,
— a "length" function $l: E \to \mathbb{Z}_+$;
find: a path $\tilde{P}$ in G with $\tilde{P} \sim P$ in $\mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p)$ of shortest length.

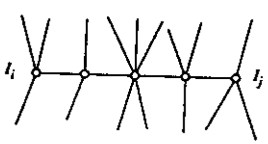
[The *length* of a path P is the sum of the lengths of the edges passed by P, counting any edge as often as it is traversed by P.]

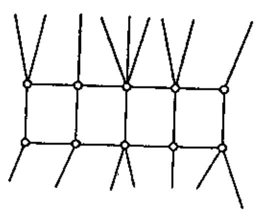
PROPOSITION 1. *Problem* (45) *is solvable in polynomial time.*

PROOF. For each pair $i, j \in \{1, \ldots, p\}$ determine a path Q_{ij} in G, connecting $\mathrm{bd}(I_i)$ and $\mathrm{bd}(I_j)$, of shortest length. Determine a spanning tree T in the complete graph on $\{1, \ldots, p\}$ of shortest length, where $\mathrm{length}(ij) := \mathrm{length}(Q_{ij})$.

We may assume that if $\{i, j\}$ and $\{i', j'\}$ belong to T, then Q_{ij} and $Q_{i'j'}$ do not cross. Otherwise we could replace $\{i, j\}, \{i', j'\}$ either by $\{i, i'\}, \{j, j'\}$ or by $\{i, j'\}, \{j, i'\}$, without increasing the length of the spanning tree.

To facilitate our description, we "double" each path Q_{ij} with $\{i, j\} \in T$ in the following way:

(46) 

becomes


Let Q'_{ij} and Q''_{ij} denote the two copies of Q_{ij}. Let M_{ij} be the matching consisting of the "new" edges connecting Q'_{ij} and Q''_{ij}. Let each edge in M_{ij} have length 0.

Without loss of generality, our original graph G is of this form. Let G' be the graph obtained by deleting all edges in all M_{ij} for $\{i, j\} \in T$. So each circuit in G' is homotopic trivial in $\mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p)$.

Now the homotopy class of any path R in G can be encoded as follows. If $R = (v_0, e_1, v_1, \ldots, e_t, v_t)$, we delete from this string the elements $v_1, \ldots, v_{t-1}$ and those e_g which do not belong to $\bigcup_{\{i,j\} \in T} M_{ij}$. If

$e_g \in M_{ij}$ for some $\{i, j\} \in T$, then

(47) we replace e_g by M_{ij} if $v_{g-1} \in Q'_{ij}$ and $v_g \in Q''_{ij}$, and
we replace e_g by M_{ij}^{-1} if $v_{g-1} \in Q''_{ij}$ and $v_g \in Q'_{ij}$.

Let us call the string thus obtained the *homotopy string* of R. An example is as follows:

(48) $$(v_0, M_{13}, M_{32}^{-1}, M_{57}^{-1}, M_{32}, M_{37}, M_{37}^{-1}, v_t).$$

Clearly, this homotopy string determines the homotopy of the path R in $\mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p)$. Moreover, deleting (repeatedly) any pair of successive symbols M_{ij}, M_{ij}^{-1} or M_{ij}^{-1}, M_{ij}, we are left with a string uniquely determined by the homotopy of R. Let us call this string the *reduced homotopy string* of R.

Let our input path P have reduced homotopy string $(v, \alpha_1, \ldots, \alpha_t, w)$, where $\alpha_1, \ldots, \alpha_t \in \{M_{ij} \mid \{i, j\} \in T\} \cup \{M_{ij}^{-1} \mid \{i, j\} \in T\}$. Now make a graph H as follows. First make $t+1$ copies of G', numbered $0, 1, \ldots, t$. Next, for $h = 1, \ldots, t$, if $\alpha_h = M_{ij}$ connect Q'_{ij} in the $(h-1)$th copy of G' by a matching (similar to M_{ij}) to Q''_{ij} in the hth copy of G'. If $\alpha_h = M_{ij}^{-1}$ connect Q''_{ij} in the $(h-1)$th copy of G' by a similar matching to Q'_{ij} in the hth copy of G'.

The length function l on G can be "lifted" to the edges of H in the obvious way. Let R be a shortest path in H from vertex v in the 0th copy of G' to vertex w in the tth copy of G'. Let $\tilde{P}$ be the "projection" of R to G. We claim that $\tilde{P}$ is a shortest path homotopic to P.

Indeed, let P' be a shortest path in G homotopic to P. Let P' have homotopy string $(v, \beta_1, \ldots, \beta_s, w)$. We may assume that no pair of successive elements in this string is equal to M_{ij}, M_{ij}^{-1}: if M_{ij}, M_{ij}^{-1} occurs, we can replace the corresponding part of P' by a subpath of Q'_{ij} without increasing the length of P' (as Q'_{ij} is a shortest path) and without changing the homotopy of P' (as circuits in G' are homotopic trivial). Similarly, we may assume that no two successive elements are equal to M_{ij}^{-1}, M_{ij}. But then P' is the projection of some path R' in H connecting v in the 0th copy of G' with w in the tth copy of G'. Hence $\mathrm{length}(P') = \mathrm{length}(R') \geq \mathrm{length}(R) = \mathrm{length}(\tilde{P})$. □

Note that the algorithm described also shows that a shortest path $\tilde{P} \sim P$ can be taken so that no edge is passed more than $p \cdot m$ times, where m is the number of edges in P (as the reduced homotopy string of P has at most $p \cdot m$ elements).

We next show that there exists a polynomial-time algorithm for the following problem (characterized in our "auxiliary theorem"):

(49) *given*: — a planar graph $G = (V, E)$, embedded in $\mathbb{R}^2$,
— faces $I_1, \dots, I_p$ of G (including the unbounded face),
— curves $C_1, \dots, C_k$, satisfying (12),
find: pairwise edge-disjoint and pairwise noncrossing paths $P_1 \sim C_1, \dots, P_k \sim C_k$ in G, without self-crossings and not using the same edge more than once.

PROPOSITION 2. *There exists a polynomial-time algorithm for problem* (49).

PROOF. I. We first show that we can decide in polynomial time if paths $P_1, \dots, P_k$ exist. In Section 2 above we saw that the existence of these paths is equivalent to the existence of a fractional packing of paths as in the "homotopic flow-cut theorem." This last is equivalent to the fact that the vector $(1, \dots, 1; 1, \dots, 1) \in \mathbb{R}^k \times \mathbb{R}^E$ belongs to the convex cone K generated by the following vectors:

(50) (i) $(\varepsilon_i; \chi^P)$ $(i = 1, \dots, k;\ P$ path in G with $P \sim C_i)$,
(ii) $(\mathbf{0}; \varepsilon_e)$ $(e \in E)$.

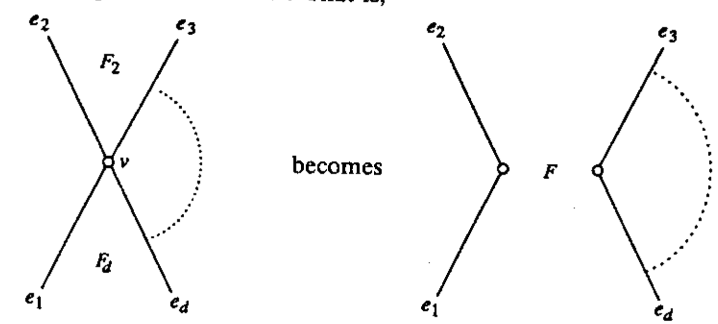
Here ε_i denotes the ith unit vector in $\mathbb{R}^k$ and ε_e denotes the eth unit vector in $\mathbb{R}^E$. Now by the ellipsoid method (see Grötschel, Lovász, and Schrijver [4]), membership of $(1, \dots, 1; 1, \dots, 1)$ to K can be tested in polynomial time, if for any vector $(d; l) \in \mathbb{Q}^k \times \mathbb{Q}^E$ we can test in polynomial time if

(51) $$(d; l)(x; y)^{\mathrm{T}} \geq 0$$

for every vector $(x; y) \in K$. This is equivalent to testing if $(d; l)(x; y)^{\mathrm{T}} \geq 0$ for every $(x; y)$ among (50). This last can be done by first testing if l is nonnegative, and if so, by testing, for each $i = 1, \dots, k$ separately, if the minimum length of a path in G homotopic to C_i is at least $-d_i$. This can be done in polynomial time by Proposition 1.

II. We next show that one actually can find the paths $P_1, \dots, P_k$ if they exist. Consider a vertex v of G of degree at least 4, and "try to" split off two adjacent edges incident to v. That is,

(52)

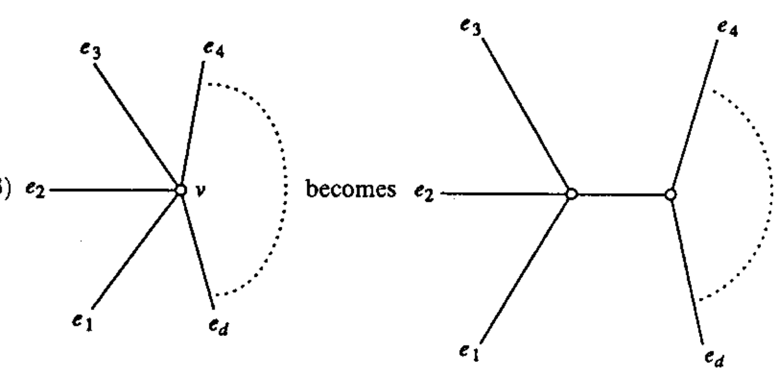


For the new situation, we test if paths $P_1, \dots, P_k$ as required exist, where if F_2 or F_d occurs in $\{I_1, \dots, I_p\}$, we replace it by F (see (52)). (If some C_i would traverse F we can reroute around the boundary of F.) This testing can be done in polynomial time by I above. If these paths exist, we replace G by the new graph. If not, we leave G unchanged.

We do this for each such pair. After at most $|E|^2$ iterations, we have a graph in which no more split-offs of such pairs can be performed.

Next for any vertex v of degree at least 6, and any *triple* of edges e_1, e_2, e_3 incident to v (where e_1 and e_2 are adjacent and e_2 and e_3 are adjacent), we try to perform a split-off as follows:

(53)



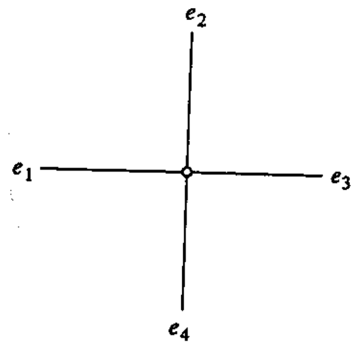
Again, for the new situation we test if paths $P_1, \dots, P_k$ as required exist (with I above). If so, we replace G by the new graph. If not, we leave G unchanged.

We do this for each such triple. After at most $|E|^3$ iterations, we have a graph in which no more split-offs of such triples can be performed.

As the final graph G contains paths $P_1, \dots, P_k$ as required, each vertex v of G has degree at most 4. For suppose vertex v has degree at least 6. If no path P_i uses vertex v, we can split off a pair as in (52). So at least one P_i uses vertex v. Suppose P_i contains $\dots, e_1, v, e_t, \dots$, using notation as in (53). Suppose, moreover, we have chosen P_i and the indices of $e_1, \dots, e_d$ so that t is as small as possible. If $t = 2$ or $t = 3$ we could split off a pair or a triple — a contradiction. If $t \geq 4$, then by the minimality of t the edges e_2 and e_3 are not used by any P_j. Hence we could have split off the pair e_2, e_3 from v — a contradiction.

This shows that each vertex of our final graph has degree 1, 2, or 4. One similarly shows that if one of the paths P_i passes a vertex v of degree 4, then it either uses e_1 and e_3 or e_2 and e_4, using notation given in

(54)



So from our final graph we uniquely determine the paths $P_1, \dots, P_k$. This directly yields paths as required in the original graph. □

We derive that the problem discussed in Section 3 is solvable in polynomial time:

(55) *given*: — a planar graph $G = (V, E)$, embedded in $\mathbb{R}^2$,
— faces $I_1, \dots, I_p$ of G (including the unbounded face),
— paths $P_1, \dots, P_k$ in G, each with end points in $\mathrm{bd}(I_1 \cup \cdots \cup I_p)$,
find: a graph-disjoint collection of curves $C_1 \sim P_1, \dots, C_k \sim P_k$.

PROPOSITION 3. *There exists a polynomial-time algorithm for problem* (55).

PROOF. We describe a polynomial-time algorithm. Given input as in (55), construct the graph G' as in the proof of Lemma 1. By Proposition 2, we can find, in polynomial time, paths $Q_1', Q_1'', \dots, Q_k', Q_k''$ as in the proof of Lemma 1. Now by contracting G' to G, we obtain paths $P_1', P_1'', \dots, P_k', P_k''$. From each pair P_i', P_i'' it is not difficult (by following the faces, edges, and vertices at one side of P_i') to identify the face sequence of the curve C_i, and hence to find C_i itself. □

Finally we show that our main problem is solvable in polynomial time.

PROPOSITION 4. *There exists a polynomial-time algorithm for problem* (44).

PROOF. We describe a polynomial-time algorithm. Let input as in (44) be given. First find output as in (55) if it exists (with the algorithm of

Proposition 3). If it does not exist, then neither does output as in (44). If it does exist, construct the graph $G' = (V', E')$ as in the proof of Lemma 2, together with the perfect matching M. Now the existence of paths as required is equivalent to the existence of a coclique of size $\frac{1}{2}|V'|$ in G' (see the Remark at the end of Section 4). Now this last can be tested in polynomial time — it is a special case of the 2-satisfiability problem (see Cook [1] and Even, Itai, and Shamir [3]).

Again by a splitting technique as in (52) we can actually find the paths P_i as required. □

Note that, although our algorithm for (44) uses the ellipsoid method as a subroutine, the final algorithm is "strongly" polynomial: since the input of (44) does not contain numbers, polynomiality and strong polynomiality coincide.

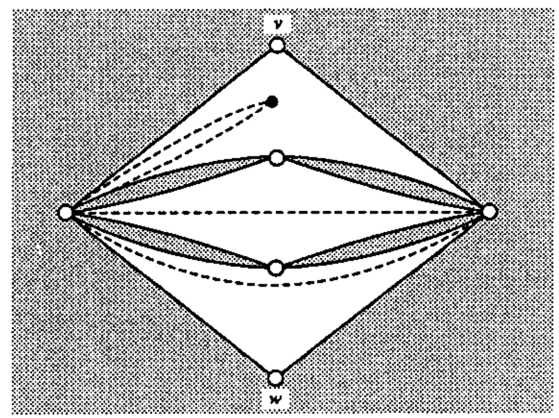
6. Two examples

It can be shown that the class of curves D in condition (1)(ii) can be restricted to curves of a simpler type. As an illustration, we close this paper with two examples showing that the closed curves D_1 and D_2 can be rather complicated.

In both examples, only one simple path of given homotopy is required, namely that of the straight vertical line connecting vertices v and w. The shaded areas indicate the faces in $I_1, \dots, I_p$.

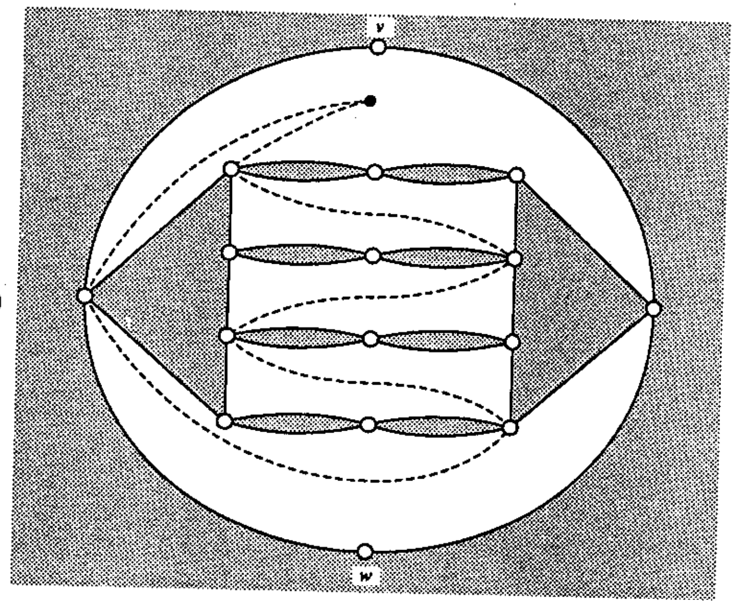
Our first example:

(56)



Our second example:

(57)



In both examples, conditions (1)(i) and (ii) are satisfied, but there exist closed curves D_1 and D_2 violating (1)(iii). Curve D_1 is indicated by an interrupted curve (where the solid point indicates $D_1(1)$), while curve D_2 arises by reflecting D_1 into the straight line segment $\overline{vw}$.

NOTE. In [7] a combinatorial polynomial-time algorithm for the problem discussed in this paper is given. Moreover, an extension to disjoint homotopic trees is described.

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