BOLYAI SOCIETY
MATHEMATICAL STUDIES, 15

Conference on Finite and Infinite Sets
Budapest, pp. 93–141.

Edge-Connection of Graphs, Digraphs, and Hypergraphs

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To the memory of C. St. J. A. Nash-Williams and W. T. Tutte who contributed to the area with fundamental results.

In this work extensions and variations of the notion of edge-connectivity of undirected graphs, directed graphs, and hypergraphs will be considered. We show how classical results concerning orientations and connectivity augmentations may be formulated in this more general setting.

1. Introduction

A digraph $D = (V, E)$ is called **strongly connected** if there is a directed path from every node to every other node. By an easy exercise, this is equivalent to requiring that $\varrho_D(X) \geq 1$ for every proper non-empty subset X of V, where $\varrho_D(X)$, the **indegree** of X, denotes the number of edges entering X. An undirected graph, (in short, a graph) $G = (V, E)$ is called **2-edge-connected** if there are two edge-disjoint paths from every node to every other. It is not difficult to show that this is equivalent to requiring that $d_G(X) \geq 2$ for every proper non-empty subset X of V, where $d_G(X)$, the **degree** of X, denotes the number of edges connecting X and $V - X$.

*The work was started while the author visited the Institute for Discrete Mathematics, University of Bonn, July, 2000. Supported by the Hungarian National Foundation for Scientific Research, OTKA T037547.

The prototypes of theorems we are interested in concern strong-connectivity and 2-edge-connectivity.

1. Augmentation [K. P. Eswaran and R. E. Tarjan] [12]. *A digraph can be made strongly connected by adding at most γ new edges if and only if there are no $\gamma + 1$ disjoint sink-sets* (:strongly-connected components with no leaving edges) *and there are no $\gamma + 1$ disjoint source-sets* (:strongly-connected components with no entering edges). *A connected undirected graph can be made 2-edge-connected by adding at most γ new edges if and only if the number of 'leaves' is at most 2γ, where a leaf is a minimal subset X with $d_G(X) = 1$.*

2. Orientation [H. E. Robbins] [52]. *An undirected graph has a strongly connected orientation if and only if it is 2-edge-connected.*

3. Constructive characterization [folklore]. *A digraph is strongly connected if and only if it can be built from a node by the following two operations:* (i) add a new directed edge connecting existing nodes, (ii) subdivide an existing edge by a new node. *A graph is 2-edge-connected if and only if it can be built from a node by the following two operations:* (i) add a new edge connecting existing nodes, (ii) subdivide an existing edge by a new node. In both cases the two operations may be included into one: *add a path (directed, in case of digraphs) connecting two existing nodes (which may be equal)*, an operation called **adding an ear**. Therefore these theorems are often formulated in the form: *a graph is 2-edge-connected or a digraph is strongly connected if and only if it can be built from a node by adding ears.* The sequence of ears in such a construction is called an ear-decomposition of the (2-edge-connected) graph or (strongly connected) digraph. *Moreover, such an ear-decomposition exists if the initial (di)graph is an arbitrary 2-edge-connected (respectively, strongly connected) sub(di)graph.*

We survey these types of results concerning higher edge-connection. Here the word 'edge-connection' is used in its informal meaning to describe the intuitive notion of a graph $G = (V, E)$ or a digraph $D = (V, A)$ being 'pretty much connected by edges'. To capture this idea formally, there are (at least) two distinct approaches, and both of them admit several versions.

The first approach requires the (di)graph to be not dismantleable into smaller parts by leaving out only few edges. Here are four possible definitions to make this intuition formal.

(A1) A graph $G = (V, E)$ is **k-edge-connected** if discarding less than k edges leaves a connected graph. (This is easily seen to be equivalent to requiring $d_G(X) \geq k$ whenever $\emptyset \subset X \subset V$.)

(A2) A digraph $D = (V, A)$ is **k-edge-connected** if discarding less than k edges leaves a strongly connected digraph. (This is easily seen to be equivalent to requiring $\varrho_D(X) \geq k$ whenever $\emptyset \subset X \subset V$.) For $k = 1$, k-edge-connectivity is just strong-connectivity.

(A3) G is **k-partition-connected** if discarding less than kq edges leaves a graph with at most q connected components for every $q = 1, 2, \ldots, |V|-1$. Equivalently, there are at least kq edges connecting distinct parts for every partition of V into $q+1$ non-empty parts for every q, $1 \leq q \leq |V| - 1$. Note that for $k = 1$, partition-connectivity is equivalent to connectivity.

(A4) D is **rooted k-edge-connected** if there is a root-node s so that after discarding less than k edges every node keeps to be reachable from s. (This is easily seen to be equivalent to requiring $\varrho_D(X) \geq k$ for every non-empty subset X of $V - s$).

The second possible approach to capture the notion of high edge-connection is requiring the graph or digraph to contain several edge-disjoint 'simple' connected constituents. Here are four possibilities.

(B1) In G there are k edge-disjoint paths between every pair u, v of nodes.

(B2) In D there are k edge-disjoint directed paths from every node to every other.

(B3) G contains k edge-disjoint spanning trees (in which case G is called **k-tree-connected**).

(B4) D contains a node s so that there are k edge-disjoint spanning arborescences rooted at s.

Some basic results of graph theory asserts the equivalence of the corresponding definitions. Namely, by the edge-versions of Menger's theorem [15], the definitions **(A1)** and **(B1)** [resp., **(A2)** and **(B2)**] are equivalent:

Theorem 1.1 (Menger). *An undirected graph is k-edge-connected if and only if there are k edge-disjoint paths between every pair of nodes. A digraph is k-edge-connected if and only if there are k edge-disjoint paths from every node to every other.*

The equivalence of **(A3)** and **(B3)** was proved by W. T. Tutte [56].

Theorem 1.2 (Tutte). *A graph contains k edge-disjoint spanning trees if and only if, for every partition $\{V_1, \dots, V_t\}$ of V, the number of edges connecting distinct parts is at least $k(t-1)$.*

Finally, the equivalence of definitions **(A4)** and **(B4)** was proved by J. Edmonds [9].

Theorem 1.3 (Edmonds). *A digraph D contains k edge-disjoint spanning arborescences rooted at s if and only if $\varrho_D(X) \geq k$ for every non-empty subset X of $V - s$.*

We extend these notions even further. For non-negative integers $l \leq k$, a digraph D is **(k,l)-edge-connected** if D has a node s so that there are k edge-disjoint paths from s to every other node and there are l edge-disjoint paths from every node to s. Equivalently, the digraph is l-edge-connected and rooted k-edge-connected. Note that D is (k,k)-edge-connected exactly if D is k-edge-connected, and $(k,0)$-edge-connectivity is equivalent to rooted k-edge-connectivity. We also remark that, by relying on max-flow min-cut computations, it is possible to decide in polynomial time if a digraph is (k,l)-edge-connected or not.

Another general notion is as follows. For two subsets S, T of nodes, D is said to be **k-edge-connected from S to T** if there are k edge-disjoint paths from every element of S to every element of T. In the special case $S = T$ we briefly say that D is **k-edge-connected in S**. If $S = T = V$ we are back at k-edge-connectivity. If $S = \{s\}$ and $T = V$ we arrive at rooted k-edge-connectivity. Also, for an undirected graph $G = (V,E)$ we say that G is **k-edge-connected in $S \subseteq V$** if there are k edge-disjoint paths in G between any two elements of S. A directed edge st with $s \in S$, $t \in T$ will be called an **ST-edge.**

We say that a partition of V into t non-empty parts is a **t-partition.** For a given partition $\mathcal{P}$ of V, the set of edges in a graph $G = (V,E)$ connecting distinct parts of $\mathcal{P}$ is called the **border** of $\mathcal{P}$. An element of the border is called a **cross-edge** of the partition. The border of a 2-partition is traditionally called a **cut.** For an integer l (which may be negative), we call an undirected graph $G = (V,E)$ **(k,l)-partition-connected** if the border of every t-partition of V ($t \geq 2$) has at least $k(t-1) + l$ elements. For $l \geq 0$, this definition attempts to capture the intuitive notion for higher edge-connection which requires that leaving out only few edges does not result in too many components.

A very first question concerning this notion is whether there exists a polynomially checkable certificate for a graph being (k,l)-partition-connected. The answer depends on whether $l \leq 0$, or $1 \leq l \leq k$, or $k < l$. If $l = 0$, we are back at k-partition-connectivity, and then the certificate (by Tutte's theorem) is a set of k disjoint spanning trees. When $l = -\gamma$ is negative, we will prove (Theorem 2.10) that *a graph is (k,l)-partition-connected if and only if it is possible to add γ new edges so that the resulting graph contains k disjoint spanning trees.* That is, in this case the certificate for (k,l)-partition-connectivity is k disjoint spanning trees whose union may contain γ new edges.

For $l \geq k$, we claim that (k,l)-partition-connectivity is equivalent to $(k+l)$-edge-connectivity. Indeed, if G is (k,l)-partition-connected, then the definition for $t = 2$ implies that every cut contains at least $k(t-1)+l = k+l$ edges, that is, G is $(k+l)$-edge-connected. Conversely, let G be $(k+l)$-edge-connected and let $\mathcal{P} := \{V_1, \dots, V_t\}$ be a partition. By letting $e_G(\mathcal{P})$ denote the number of cross-edges of $\mathcal{P}$, we have $e_G(\mathcal{P}) = \sum_i^t d_G(V_i)/2 \geq (k+l)t/2 = tk + t(l-k)/2 \geq tk + (l-k) = k(t-1) + l$, and hence we conclude that G is (k,l)-partition-connected. Therefore we will be interested in (k,l)-partition-connectivity only if $l < k$.

Finally, for $0 < l < k$ one has the following characterization (Theorem 4.5): *a graph is (k,l)-partition-connected if and only if it has a (k,l)-edge-connected orientation.* Such an orientation may indeed serve as a certificate for (k,l)-partition-connectivity since a digraph can be tested for (k,l)-edge-connectivity by relying on Menger's theorem.

Given a groundset V, by a **co-partition** (of V) we mean a family of subsets consisting of the complementary sets of a partition of V. A family $\mathcal{F}$ of subsets of V is called a **sub-partition** of V if $\mathcal{F}$ is a partition of a subset of V. For a partition $\mathcal{F}$ of a non-empty proper subset Z of V, the family $\{V - X : X \in \mathcal{F}\}$ is called a **co-partition** of $V - Z$. For a subset X and for two elements x and y, we say that X is an $x\bar{y}$-set if $x \in X$, $y \notin X$.

For non-negative integers k, l, we call an undirected graph G **(k,l)-tree-connected** if deleting any subset of at most l edges leaves a k-tree-connected graph. By Tutte's theorem, G is (k,l)-tree-connected if and only if G is (k,l)-partition-connected.

In a graph $G = (V,E)$ the **local edge-connectivity** $\lambda(x,y;G)$ of nodes x and y is the minimum cardinality of a cut separating x and y. By Menger's theorem, this is equal to the maximum number of edge-disjoint paths connecting x and y. $e_G(X)$ denotes the number of edges with at least one endnode in X.

In a digraph $D = (V, E)$ the **local edge-connectivity** $\lambda(x, y; D)$ from node x to node y is the minimum number of edges entering a $y\bar{x}$-set. By Menger's theorem, this is equal to the maximum number of edge-disjoint paths from x to y. $\varrho(X)$ denotes the number of edges entering X and $\delta(X) := \varrho(V - X)$. For a graph or digraph H, $i_H(X)$ denotes the number of edges induced by X.

Typically we will work with directed or undirected graphs and write (di)graph when either of them is meant. Sometimes mixed graphs are also considered which may contain both directed and undirected edges.

2. Relations between old results

The three motivating theorems mentioned at the beginning of the introduction represent, respectively, the following general problem classes.

1. In a **connectivity augmentation** problem we want to add some new edges to a graph or digraph so that the resulting graph or digraph satisfies a prescribed connectivity property. In a **minimization problem** the number (or, more generally, the total cost) of new edges is to be minimized. In a **degree-specified** problem, in addition to the connectivity requirement, the (di)graph of the newly added edges must meet some (in)degree specification. Another aspect of augmentation problems distinguishes between the type of graphs of usable new edges. In a **restricted** augmentation the new edges must be chosen from a specified graph. We speak of a **free** augmentation if any possible edge is allowed to be added in any number of parallel copies. In the directed case, ST**-free** augmentations will also be considered when the new edges must be ST-edges.

2. In a **connectivity orientation** problem we want to orient the edges of an undirected graph so that the resulting digraph satisfies a prescribed connectivity property. The proof of Robbins' theorem is fairly easy (say, by ear-decomposition) but there are even easier orientation results: (A) *a graph G has a root-connected orientation* (:every node is reachable from a root-node) *if and only if G is connected*, and (B) *G has an orientation in which a specified node t is reachable from s if and only if s and t belong to the same component of G*. These are indeed so trivial that they deserve mentioning only because they serve as a good ground for possible generalizations.

3. In a **constructive characterization** problem we are interested in finding simple operations for a given connectivity property by which every (di)graph with the property may be obtained from a small initial (di)graph. It will turn out that this type of results often help proving connectivity orientation results.

In earlier survey type works ([21] [22], [23]) I endeavored to overview some aspects of connectivity orientations and augmentations with special emphasis on their relationship to sub- and supermodular functions. Therefore in the present paper those results are mentioned only when the overview of the developments of the past decade requires them. Exhibiting this progress is our main goal, with a special emphasis on some known and some newly discovered links connecting the different problems. Some new observations will also be outlined.

By comparing older results, this section is offered to demonstrate how closely the orientation, augmentation, and characterization problems are related to each other. But first a small remark is in order. The augmentation problem may be considered as one of finding a supergraph of a (di)graph with certain connectivity properties. This is naturally related to the subgraph problem which consists of finding an optimal subgraph of a (di)graph satisfying connectivity requirements (sometimes called generalized Steiner network problem). The minimum cost versions of these problems are actually equivalent, and to explain this we invoke a specific subgraph versus supergraph problem-pair. Subgraph problem: given a digraph $D = (V, A)$ with specified nodes s and t endowed with a cost function c on A, find a minimum cost subdigraph D' of D which is k-edge-connected from s to t. Supergraph (=augmentation) problem: given a digraph $D = (V, A)$ with specified nodes s and t, moreover another digraph $H = (V, F)$ endowed with a cost function c_F on F, find a minimum cost augmentation of D which is k-edge-connected from s to t. Now if the subgraph problem is tractable, then so is the supergraph problem: Let $D_1 = (V, A \cup F)$ be the union of G and H and define a cost function c_1 on $A \cup F$ by $c_1(e) := 0$ if $e \in A$ and $c_1(e) := c_F(e)$ if $e \in F$. Obviously, an optimal solution to the subgraph problem on D_1 determines an optimal solution to the augmentation problem. Conversely, the subgraph problem can be viewed as an augmentation problem because it is equivalent to augment, at a minimum cost, of the empty digraph $(V, \emptyset)$ by using edges of D, (or wording differently, by using arbitrary edges but the ones not in D have cost $+\infty$). Typically we use this equivalence in one direction: when the minimum cost subgraph problem is tractable then so is the augmentation problem. In our concrete case the subgraph problem is indeed solvable with the help of a minimum cost flow algorithm. On the same ground, as the minimum cost connected subgraph problem is solvable with the greedy algorithm, the minimum cost augmentation problem, to make a given graph connected, is also solvable.

We hasten to emphasize however that in several cases the subgraph problem is NP-complete while the corresponding (free) augmentation problem is nicely solvable. A prime example for this phenomenon is the problem of finding a minimum cardinality 2-edge-connected subgraph of a graph G which is known to be NP-complete as it includes the Hamiltonian circuit problem (:the minimum is equal to $|V|$ if and only if G is Hamiltonian). On the other hand, the second introductory problem on the corresponding connectivity augmentation is solvable.

2.1. Splitting and augmentation

The following two splitting lemmas are central to several results. By **splitting off** a pair of undirected edges $e = zu, f = zv$ we mean the operation of replacing e and f by a new edge connecting u and v. In the directed case directed edges uz and zv are replaced by a directed edge uv.

Theorem 2.1 (Lovász's undirected splitting lemma [42]). *Let $k \geq 2$ be an integer and $G = (V + z, E)$ an undirected graph with a special node z of even degree. If G is k-edge-connected in V, then there is a pair of edges $e = zu, f = zv$ which can be split off without destroying k-edge-connectivity in V.*

Theorem 2.2 (Mader's directed splitting lemma [46]). *Let $k \geq 1$ be an integer and $D = (V + z, E)$ a directed graph with a special node z having the same in- and out-degree. If D is k-edge-connected in V, then there is a pair of edges $e = zu$, $f = vz$ which can be split off without destroying k-edge-connectivity in V.*

Both lemmas may be used repeatedly, as long as there are edges incident to z, and in this case we speak of a **complete splitting.** Sometimes by the splitting lemma this complete version is meant: *Under the same hypotheses, there is a complete splitting at z so that the resulting (di)graph on node set V is k-edge-connected.*

An easy observation shows that the existence of a complete undirected splitting that preserves k-edge-connectivity is equivalent to the following degree-specified augmentation result [19]. Here and throughout the paper, we use the notation $m(X) := \sum \left[m(v) : v \in X\right]$.

Theorem 2.3. *We are given an undirected graph $G = (V, E)$, a degree-specification $m : V \to \mathbf{Z}_+$ with $m(V)$ even, and an integer $k \geq 2$. There is a graph $H = (V, F)$ so that $d_H(v) = m(v)$ for every node $v \in V$ and $G + H$ is k-edge-connected if and only if $m(X) \geq k - d_G(X)$ for every non-empty subset $X \subset V$.*

This result was used in [19] to exhibit a short derivation of T. Watanabe and A. Nakamura's [57] earlier solution to the minimization form of the undirected edge-connectivity augmentation problem:

Theorem 2.4 (Watanabe and Nakamura). *An undirected graph G can be made k-edge-connected ($k \geq 2$) by adding at most γ new edges if and only if $\sum_i \left[k - d_G(X_i)\right] \leq 2\gamma$ for every subpartition $\{X_1, \ldots, X_t\}$ of V.*

Note that the last theorem fails to hold for $k = 1$. On the other hand, for this case, even the minimum cost version is solvable by the greedy algorithm since it is equivalent to the min-cost spanning tree problem (while for $k \geq 2$ the min-cost version is NP-complete.)

Mader's directed splitting lemma is also easily seen to be equivalent to the degree-specified directed edge-connectivity augmentation problem:

Theorem 2.5. *We are given a directed graph $D = (V, E)$, in- and out-degree specifications $m_i : V \to \mathbf{Z}_+$ and $m_o : V \to \mathbf{Z}_+$ so that $m_i(V) = m_o(V)$. Let $k \geq 1$ be an integer. There is a digraph $H = (V, F)$ so that $\delta_H(v) = m_o(v)$, $\varrho_H(v) = m_i(v)$ for every node $v \in V$ and so that $D + H$ is k-edge-connected if and only if $m_i(X) \geq k - \varrho_D(X)$ and $m_o(X) \geq k - \delta_D(X)$ holds for every non-empty subset $X \subset V$.*

This implies the minimization form of directed edge-connectivity augmentation [19]:

Theorem 2.6. *A digraph $D = (V, E)$ can be made k-edge-connected ($k \geq 1$) by adding at most γ directed edges if and only if $\sum_i \left[k - \varrho_D(X_i)\right] \leq \gamma$ and $\sum_i \left[k - \delta_D(X_i)\right] \leq \gamma$ hold for every subpartition $\{X_1, \ldots, X_t\}$ of V.*

2.2. Connectivity orientation and augmentation

The easy orientation results mentioned above concerning strong-connectivity, connectivity from s to t, and s-rooted 1-edge-connectivity naturally raise questions on higher connection: when does a graph G have an orientation which is (a) k-edge-connected from s to t, (b) rooted k-edge-connected, (c) k-edge-connected? Among these, the first one is easy (given Menger's theorem).

Theorem 2.7. *For integers $k_1, k_2 \geq 0$ and specified nodes $s, t \in V$, an undirected graph $G = (V, E)$ has an orientation which is k_1-edge-connected from s to t and k_2-edge-connected from t to s if and only if every cut of G separating s and t has at least $k_1 + k_2$ edges.*

Proof. The necessity of the condition is straightforward. The sufficiency follows by observing that the condition implies, by Menger's theorem, the existence of $k_1 + k_2$ edge-disjoint paths between s and t. One can orient the edges of k_1 paths toward t, the edges of the remaining k_2 paths toward s, and the remaining edges arbitrarily. ■

The first non-trivial result concerning orientation is due to C. St. J. A. Nash-Williams [47]. He proved the following extension of Robbins' theorem (actually in a much stronger form).

Theorem 2.8 (Nash-Williams: weak form). *An undirected graph G has a k-edge-connected orientation if and only if G is $2k$-edge-connected.*

By a straightforward induction, Lovász's undirected splitting lemma implies Nash-Williams' theorem. When rooted k-edge-connectivity is the target in the orientation problem, one has the following result.

Theorem 2.9. *An undirected graph $G = (V, E)$ has a rooted k-edge-connected (that is, $(k, 0)$-edge-connected) orientation if and only if G is k-partition-connected.*

The non-trivial 'if' part is an easy consequence of Theorem 1.2 on disjoint trees since Tutte's theorem implies that a k-partition-connected graph contains k disjoint spanning trees and, by orienting each of these trees away from the root (to become a spanning arborescence) while the remaining edges arbitrarily, one obtains a rooted k-edge-connected orientation of G.

On the other hand, Theorem 2.9, when combined with Edmonds Theorem 1.3, gives rise to Tutte's Theorem 1.2. At this point the question naturally emerges: if the required orientations do not exist, then how many new undirected edges have to be added so that the augmented graph admits an orientation?

The answer is evident when the goal is to augment a graph so as to become k-edge-connected orientable. Namely, by Nash-Williams' theorem this is equivalent to augmenting the graph to make it $2k$-edge-connected, a problem solved in Theorems 2.4 and 2.3. Suppose now we want to augment G to become k-tree-connected (= k-partition-connected). For the special case of free augmentation one has the following:

Theorem 2.10. *Let $G = (V, E)$ be an undirected graph, $s \in V$ a specified node, and γ a nonnegative integer. It is possible to add at most γ new edges to G so that the enlarged graph has an s-rooted k-edge-connected orientation if and only if G is $(k, -\gamma)$-partition-connected. Moreover, all the newly added edges may be chosen to be incident to s.*

Proof. Recall that by definition G is $(k, -\gamma)$-partition-connected if

$$e(\mathcal{F}) \geq k(t-1) - \gamma \tag{1}$$

holds for every partition $\mathcal{F} := \{V_1, \dots, V_t\}$ of V, where $e(\mathcal{F})$ denotes the number of cross edges of $\mathcal{F}$. For brevity we call an orientation **good** if it is k-edge-connected from s. If there is a good orientation after adding γ edges, then $\varrho(V_i) \geq k$ holds for every subset $V_i \subset V$ not containing s and hence $e(\mathcal{F}) + \gamma \geq e^+(\mathcal{F}) \geq k(t-1)$, where e^+ refers to the enlarged graph, proving the necessity of the condition.

To see the sufficiency, add a minimum number of new edges to G, each incident to s so that the enlarged graph has a good orientation and let γ' denote this minimum. Our goal is to prove $\gamma' \leq \gamma$.

Let ϱ denote the in-degree function of the good orientation of the enlarged graph G^+. We may assume that $\varrho(s) = 0$. Let us call a set $X \subseteq V - s$ **tight**, if $\varrho(X) = k$. By standard submodular technique, we see that both the intersection and the union of two tight sets with non-empty intersection are tight. Let T denote the subset of nodes which can be reached from the head of at least one new edge. Clearly, $s \notin T$ and $\varrho(V - T) = 0$.

Lemma 2.11. *If Z is tight and $Z \cap T \neq \emptyset$, then $Z \subseteq T$.*

Proof. Suppose indirectly that $Z \nsubseteq T$. Then for $Y := V - T$ we have $k = \varrho(Y) + \varrho(Z) = \varrho(Y \cap Z) + \varrho(Y \cup Z) + d^+(Y, Z) \geq k + 0 + d^+(Y, Z) \geq k$, where $d^+(Y, Z)$ denotes the number of edges of G^+ connecting elements of

$Y - Z$ and $Z - Y$. Hence $\varrho(Y \cup Z) = 0$ and $d^+(Y,Z) = 0$. From the first equality there is a new edge $e = st$ for which $t \in Z$ for otherwise no element of $Z \cap T$ would be reachable from the head of any new edge. But then, by the existence of edge e, we have $d^+(Y,Z) > 0$, a contradiction. ∎

There are two cases. If there is a node v in T which does not belong to any tight set, then let st be a new edge for which there is a path P from t to v. Reorient each edge of P and discard e. Since v does not belong to any tight set the revised orientation is good, contradicting the minimality of γ'.

In the second case every element of T belongs to a tight set. Let $V_1, \dots, V_{t-1}$ be maximal tight sets intersecting T. These are pairwise disjoint and by the lemma they form a partition of T. Let $V_t := V - T$ and $\mathcal{F} := \{V_1, \dots, V_t\}$. Since $\varrho(V_t) = 0$, and every new edge enters T, we get $k(t-1) = \sum [\varrho(V_i) : i = 1, \dots, (t-1)] = \sum [\varrho(V_i) : i = 1, \dots, t] = i^+(\mathcal{F}) = e(\mathcal{F}) + \gamma'$. This and (1) give rise to $\gamma' = k(t-1) - e(\mathcal{F}) \leq \gamma$, as required. ∎

By combining Theorems 2.10 and 2.9, we obtain the following extension of Tutte's Theorem 1.2 which serves as a characterization of (k,l)-partition-connected graphs in case $l \leq 0$.

Theorem 2.12. *An undirected graph $G = (V,E)$ can be augmented by adding $\gamma \geq 0$ new edges so that the enlarged graph is k-tree-connected if and only if G is $(k,-\gamma)$-partition-connected. Moreover, the newly added edges may be chosen to be incident to any given node in V.*

The theorem shows that the free augmentation problem is tractable for k-tree-connectivity as a target. This is, however, not surprising since, by using matroid techniques, even the minimum cost version is solvable in polynomial time. To see this, let $G = (V,E)$ be an undirected graph and let $G_u = (V,E_u)$ be a graph, where E_u is the set of edges usable in the augmentation of G. Let $c_u : E_u \to \mathbf{R}_+$ be a cost function. We want to choose a subset F of edges of G_u of minimum total cost so that the increased graph $G^+ = (V, E+F)$ is k-tree-connected.

To this end, let us define a cost function c' on the edge set of the union $G' = (V, E + E_u)$ of G and G_u so that $c'(e) := 0$ if $e \in E$ and $c'(e) = c(e)$ if $e \in E_u$. Then the problem is equivalent to finding k disjoint spanning trees of G' with minimum total cost. Since the edge-sets which are the union of k disjoint spanning trees form the set of bases of a matroid, this problem is solvable in polynomial time by using Edmonds' matroid partition algorithm and the greedy algorithm. This approach also shows that Edmonds' matroid

partition theorem does provide a characterization for the existence of the required augmentation in Theorem 2.10. Our goal has simply been to show a direct, graphical proof.

One may also consider the degree-specified version of the k-tree-connected augmentation problem. This does not seem to be a matroid problem and it does not follow from the previous material either. Section 4 includes an answer even for the more general case of (k,l)-partition-connectivity.

2.3. Constructive characterization and splitting

Let $G' = (V + z, E')$ be an undirected graph with a special node z of even degree and suppose that G' is k-edge-connected in V. By the undirected splitting lemma we know that there is a complete splitting at z so that the resulting graph $G = (V,E)$ is k-edge-connected. In other words, the $d(z)$ edges incident to z can be paired so that splitting off these $j := d(z)/2$ pairs (and discarding z) we obtain a k-edge-connected graph. In a directed graph $D' = (V + z, A')$ a complete splitting at z consists of pairing the edges entering z with those leaving z and then splitting off the pairs. Both in the directed and in the undirected cases the inverse operation of a complete splitting is as follows. *Add a new node z, subdivide j existing edges by new nodes and identify the j subdividing nodes with z.* This will be called **pinching j edges (with z)**. When $j = 0$ this means adding a single new node z, while in case $j = 1$ pinching an edge requires the edge to be subdivided by a node z.

By the operation of adding a new edge to a (di)graph we always mean that the new edge connects existing nodes. Unless otherwise stated, the newly added edge may be a loop or may be parallel to existing edges.

After these definitions, we exhibit how the splitting lemmas give rise to constructive characterizations of $2k$-edge-connected graphs and k-edge-connected digraphs. By using the easy observation that a minimally (with respect to edge-deletion) K-edge-connected undirected graph (with at least two nodes) always contains a node of degree K, one can easily derive from the undirected splitting lemma the following constructive characterization of $2k$-edge-connected graphs.

Theorem 2.13 (Lovász). *An undirected graph $G = (V,E)$ is $2k$-edge-connected if and only if G can be obtained from a single node by the following two operations:* (i) *add a new edge,* (ii) *pinch k existing edges.*

By using a rather difficult theorem of Mader [44], stating that *a minimally (with respect to edge-deletion) k-edge-connected directed graph (with at least two nodes) always contains a node of in-degree and out-degree k*, one can derive from the directed splitting lemma the following constructive characterization of k-edge-connected digraphs.

Theorem 2.14 (Mader). *A directed graph $D = (V, E)$ is k-edge-connected if and only if D can be obtained from a single node by the following two operations:* (i) *add a new edge,* (ii) *pinch k existing edges.*

It is useful to observe that Mader's characterizaton in Theorem 2.14 for k-edge-connected digraphs combined with Nash-Williams' orientation result give rise to Theorem 2.13. The same phenomenon will occur later as well: with the help of an orientation result, a constructive characterization for directed graphs may be used to derive its undirected counterpart.

By an easy reduction, Theorem 2.14 provides a constructive characterization of rooted k-edge-connected digraphs.

Theorem 2.15. *A digraph $D = (V, E)$ is rooted k-edge-connected if and only if D can be built up from a root-node s by the following two operations:* (j) *add a new edge,* (jj) *pinch i $(0 \leq i \leq k-1)$ existing edges with a new node z, and add $k - i$ new edges entering z and leaving existing nodes.*

In [46] Mader showed that this characterization, in turn, can be used to derive Edmonds' Theorem 1.3 on disjoint arborescences. Combining Theorems 2.9 and 2.15, one obtains the following constructive characterization.

Theorem 2.16. *An undirected graph $G = (V, E)$ is k-tree-connected* (= k-partition-connected) *if and only if G can be built from a node by the following two operations:* (j) *add a new edge,* (jj) *pinch i $(0 \leq i \leq k-1)$ existing edges with a new node z, and add $k - i$ new edges connecting z with existing nodes.*

3. Splitting and Detachment

In this section first we exhibit extensions of the splitting lemmas of section 2 and of their applications. After that the notion of splitting will be extended to detachments.

3.1. Undirected splitting

As a significant generalization of Lovász's undirected splitting lemma, W. Mader [45] proved the following result. Recall (from the introduction) the definition of local edge-connectivity λ.

Theorem 3.1 (Mader). *Let $G = (V + z, E)$ be an undirected graph so that there is no cut-edge incident to z and the degree of z is even. Then there exists a complete splitting at z preserving the local edge-connectivities of all pairs of nodes $u, v \in V$.*

Mader originally formulated his result in a slightly weaker form: *If z is not a cut-node of $G = (V + z, E)$ and $d(z) \geq 4$, then there exists a pair of edges incident to z which can be split off without lowering any local edge-connecivity on V.* However the two forms can be shown to be equivalent. This and a relatively short proof of Mader's theorem was given in [20].

3.1.1. Constructive characterizations.

Mader [45] used his result to characterize $(2k + 1)$-edge-connected graphs.

Theorem 3.2 (Mader). *Let $K = 2k + 1 \geq 3$. An undirected graph $G = (U, E)$ is K-edge-connected if and only if G can be constructed from the initial graph of two nodes connected by K parallel edges by the following three operations:*

(i) *add an edge,*

(ii) *pinch k edges with a new node z' and add an edge connecting z' with an existing node,*

(iii) *pinch k edges with a new node z', pinch then again in the resulting graph k edges with another new node z so that not all of these k edges are incident to z', and finally connect z and z' by a new edge.*

The theorem is obviously equivalent to the first part of the following result:

Theorem 3.3. *An undirected graph G with more than two nodes is K-edge-connected (K odd) if and only if G can be obtained from a (smaller) K-edge-connected graph G' by one application of one of the operations* (i), (ii), (iii). *Moreover, for any node s of G, G' can be chosen so as to contain s.*

Proof. It is not difficult to check that each of these operations preserves K-edge-connectivity. (Note that if all the k edges to be pinched with z' in the second part of (iii) were adjacent to z, then only $K-1=2k$ edges would leave the subset $\{z,z'\}$.)

For a subset $X \subseteq V$, the set of edges connecting X and $V-X$ will be denoted by $[X, V-X]$. We call a cut $[X,V-X]$ **trivial** if $|X|=1$ or $|V-X|=1$. By a **minimum cut** we mean one with cardinality K.

Lemma 3.4. *Suppose that X is a minimal subset of nodes of a K-edge-connected graph $G=(U,E)$ for which*

$$(2) \qquad d_G(X)=K \quad \text{and} \quad |X| \geq 2.$$

Then any minimum cut B containing an edge $e=zz'$ with $z,z' \in X$ is trivial (that is, B is $[z,U-z]$ or $[z',U-z']$).

Proof. Suppose indirectly that there is a subset Y for which $z \in Y$, $z' \in U-Y$, $d(Y)=K$, $|Y| \geq 2$, $|U-Y| \geq 2$. Then by the minimal choice of X we have $Y \not\subseteq X$ and $U-Y \not\subseteq X$. But it is well-known (and an easy exercise anyway to show) that in a K-edge-connected graph with K odd there cannot exist two such crossing sets X, Y. (Indeed, we have $K+K=d(X)+d(Y)=d(X\cap Y)+d(X\cup Y)+2d(X,Y) \geq K+K+0$ from which $d(X\cap Y)=K=d(X\cup Y)$ and $d(X,Y)=0$, where $d(X,Y)$ denotes the number of edges connecting $X-Y$ and $Y-X$. Analogously, we obtain for $\bar{Y} := U-Y$ that $d(X\cap \bar{Y})=K=d(X\cup \bar{Y})$ and $d(X,\bar{Y})=0$. So if $\alpha := d(X\cap Y, Y-X)$, then $d(X\cap Y, X-Y)=K-\alpha=d(Y-X, U-(X\cup Y))$ from which $K=d(Y)=d(X\cap Y,X-Y)+d(Y-X,U-(X\cup Y))=2K-2\alpha$, that is, K is even, a contradiction.) ■

If there is an edge e so that $G' := G-e$ is K-edge-connected, then G arises from G' by (i). So we may assume that G is minimally K-edge-connected. We may assume that there is no node z which is connected only with s since otherwise, then by the minimality, $d(z)=K$ and then G arises from G' by operation (ii) where G' is a graph arising from G by deleting z and adding k loops at s. (Clearly G' is K-edge-connected.)

If every minimum cut is trivial, then let $e=zz'$ be an arbitrary edge not incident to s. If there are non-trivial minimum cuts, then there is a set X satisfying (2). Since the complement of X also satisfies (2), there exists a minimal set X satisfying (2) so that $s \notin X$.

Let $e=zz'$ be an arbitrary edge induced by X. As X induces a connected subgraph, such an e exists. Now e belongs to at most two minimum cuts, each is trivial. If e belongs to one minimum cut, than exactly one of z and z', say z, is of degree K. Then $G-e$ is K-edge-connected in $U-z$. By Lovász's splitting lemma there is a complete splitting at z resulting in a K-edge-connected digraph G'. Then G arises from G' by operation (ii).

If both z and z' are of degree K, then $G-e$ is K-edge-connected in $U-\{z,z'\}$. It follows from Mader's splitting Theorem 3.1 that there is a complete splitting of $G-e$ at z so that the resulting graph G_1 is K-edge-connected in $U-\{z,z'\}$. By applying the splitting lemma to G_1 (now Lovász's is enough), we obtain that there is a complete splitting at z' so that the resulting graph G' with node set $U-\{z,z'\}$ is K-edge-connected. This construction shows that G arises from G' by operation (iii).

Since in each case z and z' were chosen to be distinct from s, we have also proved the second half of the theorem. ■ ■

Operation (iii) may seem to be a bit too complicated and one's natural wish could be to try to simplify it. For example, a simpler, more symmetric version could be as follows: (iii)$'$ choose two disjoint subsets F and F' of edges both having k elements, pinch the elements of F with a new node z, pinch the elements of F' with another new node z', and finally connect z and z'. However, Mader in his original paper showed an example which cannot be obtained with operations (i), (ii), (iii)$'$.

Fortunately, for $K=3$, operations (iii) and (iii)$'$ coincide and it is worthwile to formulate this special case separately:

Corollary 3.5. *An undirected graph G with at least two nodes is 3-edge-connected if and only if G can be built from a node by the following operations:*

(i) *add an edge,*

(ii) *subdivide an existing edge $e=uv$ by a new node z and connect z to an existing node,*

(iii) *subdivide two existing edges by nodes z and z' and connect z and z' by a new edge.*

3.1.2. Orientation. Lovász's splitting lemma immediately implied Nash-Williams' orientation theorem (*:a $2k$-edge-connected graph always has a k-edge-connected orientation*). In [29] we observed that Mader's splitting theorem also rather easily gives rise to the following common generalization of theorems 2.8 and 2.7.

Theorem 3.6. *Let k_1, k_2, k be non-negative integers with $k_1 \geq k$, $k_2 \geq k$. An undirected graph $G = (V, E)$ with two specified nodes s and t has a k-edge-connected orientation which is k_1-edge-connected from s to t and k_2-edge-connected from t to s if and only if G is $2k$-edge-connected and G is $(k_1 + k_2)$-edge-connected in $\{s, t\}$.*

This immediately implies a characterization of $(2k + 1)$-edge-connected graphs.

Theorem 3.7. *An undirected graph G is $(2k + 1)$-edge-connected if and only if, for every pair of nodes s and t, G has a k-edge-connected orientation which is $(k + 1)$-edge-connected from s to t.*

Given the easy way how Lovász's splitting lemma implies the weak form of Nash-Williams orientation theorem, one may expect that Mader's stronger splitting result implies immediately the following stronger orientation result of Nash-Williams [47]:

Theorem 3.8 (Nash-Williams: strong form). *Every undirected graph $G = (V, E)$ has an orientation $\vec{G}$ for which $\lambda(x, y; \vec{G}) \geq \lfloor \lambda(x, y; G)/2 \rfloor$ for all $x, y \in V$.*

Mader was indeed able to derive Theorem 3.8 relying on his splitting theorem but the derivation is not at all simple (as neither is Nash-Williams' original proof).

In the introduction of his paper, Nash-Williams [48] remarks that his orientation theorems 'do not seem particularly closely related to much other existing work in graph theory'. These words are painfully true even after 40 years as far as the strong form is concerned, and it remains a major task to find a simple proof of Theorem 3.8 or at least to find some closer link to the body of edge-connectivity problems. Note that by now pretty much is known about the various connections of the weak form along with its numerous strengthenings and extensions. Nash-Williams also remarks that 'these theorems seem to have a somewhat natural character which would suggest that there must ultimately be a place for them in the overall structure of graph theory'. Since then it has turned out that wherever this place is located, it is not a lonely one.

Nash-Williams calls an orientation with the property given in the theorem **well-balanced.** He actually proved the existence of a well-balanced orientation that is, in addition, **near-Eulerian** which means by definition that $|\varrho(v) - \delta(v)| \leq 1$ for every node v of $\vec{G}$. Nash-Williams also outlined the proof of the following generalization of Theorem 3.8.

Theorem 3.9 [47]. *Let G be a graph and H a subgraph of G. Then G has a well-balanced and near-Eulerian orientation with the additional property that its restriction to H is a well-balanced and near-Eulerian orientation of H.*

Corollary 3.10. *Let $G = (V, E)$ be a $2k$-edge-connected graph and $H = (V, F)$ an Eulerian subgraph of G. For any Eulerian orientation of H, the edges in $E - F$ can be oriented so as to obtain a k-edge-connected orientation of G.*

This implies that in order to find a k-edge-connected orientation of a $2k$-edge-connected graph G one can pick up edge-disjoint circuits one after the other and orient them around. The corollary ensures that the remaining forest can always be oriented to get a k-edge-connected orientation of G. It would be interesting to see a direct constructive proof of this fact which does not rely on Theorem 3.9. We note that there is an easy alternative proof of Corollary 3.10 relying on submodular flows.

3.1.3. Augmentation. Let us turn to the effect of Mader's theorem on connectivity augmentation. The same way as Lovász's splitting lemma could be used for solving (global) connectivity augmentation, Mader's splitting theorem gives rise to a solution of the local edge-connectivity augmentation problem. Let $G = (V, E)$ be an undirected graph and r a non-negative integer-valued function on unordered pairs $\{u, v\}$ of distinct nodes of G, called a **requirement function.** In the local edge-connectivity augmentation problem we want to augment G so that the local edge-connectivity in the increased graph G^+ majorizes r. By Menger's theorem this is equivalent to requiring

(3) $$d_{G^+}(X) \geq R_r(X) \text{ for every subset } X \subset V,$$

where

(4) $$R_r(X) := \max\{r(u, v) : u \in X, v \in V - X\}.$$

The following two results appeared in [19].

Theorem 3.11. *Let $G = (V, E)$ be an undirected graph. Let $m : V \to \mathbf{Z}_+$ be an integer-valued function so that $m(V)$ is even and $m(C) \geq 2$ for each component C of G. There is a set F of new edges so that the local edge-connectivity in $G^+ = (V, E + F)$ is at least r and $d_F(v) = m(v)$ for every node v if and only if*

(5.10) $$m(X) \geq R_r(X) - d_G(X)$$

for every $X \subseteq V$.

Let $C(\neq V)$ be the node-set of a component of G and call C a **marginal component** (with respect to r) if $R_r(C) \leq 1$ and $R_r(X) \leq d_G(X)$ for every proper subset X of C. Let $q(X) := R_r(X) - d_G(X)$ for $X \subset V$.

Theorem 3.12. *Suppose that there are no marginal components. There is a set F of at most γ edges so that the local edge-connectivity in $G^+ = (V, E+F)$ is at least r if and only if*

$$\sum_i q(X_i) \leq 2\gamma \tag{5}$$

holds for every sub-partition $\{X_1, X_2, \ldots, X_t\}$ of V.

In [1], J. Bang-Jensen, H. Gabow, T. Jordán and Z. Szigeti investigated the augmentation problem when the possible set of new edges meets a partition constraint. Among their numerous results, we cite here only one:

Theorem 3.13. *Let $G = (V, E)$ be an undirected graph and $\mathcal{P} = \{P_1, \ldots, P_r\}$ a partition of V into at least two non-empty parts. Let $k \geq 2$ be an even integer. It is possible to add at most γ new edges to G each connecting distinct parts of $\mathcal{P}$ so that the resulting graph is k-edge-connected if and only if $\sum_{X\in\mathcal{F}} \left[k - d(X) \,:\, X \in \mathcal{F}\right] \leq 2\gamma$ holds for every subpartition $\mathcal{F}$ of V, and $\sum_{X\in\mathcal{F}_i} \left[k - d(X) \,:\, X \in \mathcal{F}_i\right] \leq \gamma$ holds for every subpartition $\mathcal{F}_i$ of P_i $(i = 1, \ldots, r)$.*

It is not difficult to check that the conditions in the theorem are necessary for even and odd k, as well. For odd k, however, they are not sufficient. But [1] did provide a characterization even for this more complicated case.

3.2. Directed splitting

Can one extend Mader's directed splitting lemma so as to preserve local edge-connectivities in directed graphs? No such a general result is known but some extensions of the directed splitting lemma are available. The following is a consequence of a result in [22].

Theorem 3.14. *Let $k \geq l \geq 1$ be integers and $D = (V + z, E)$ a directed graph with a special node z having the same in- and out-degree. If D is (k,l)-edge-connected in V, then there is a pair of edges $e = zu$, $f = vz$ which can be split off without destroying (k,l)-edge-connectivity in V.*

This result was proved in [22] in a more general form concerning coverings of crossing supermodular functions by digraphs. It can be used to solve the free- and the degree-specified augmentation problem for digraphs when the target is (k,l)-edge-connectivity. Let $D = (V, E)$ be a digraph with a root-node s and let $0 \leq l \leq k$ be integers. Define $p_{kl}(X) := \left(k - \varrho_D(X)\right)^+$ if $\emptyset \subset X \subset V - s$ and $p_{kl}(X) := \left(l - \varrho_D(X)\right)^+$ if $s \in X \subset V$.

Theorem 3.15. *For in- and out-degree specifications $m_i : V \to \mathbf{Z}_+$ and $m_o : V \to \mathbf{Z}_+$ with $m_i(V) = m_o(V)$, there is a digraph $H = (V, F)$ so that $\delta_H(v) = m_o(v)$, $\varrho_H(v) = m_i(v)$ for every node $v \in V$ and so that $D + H$ is (k,l)-edge-connected with respect to root s if and only if $m_i(X) \geq p_{kl}(X)$ and $m_o(V - X) \geq p_{kl}(X)$ holds for every non-empty subset $X \subset V$.*

Theorem 3.16. *There is a digraph $H = (V, F)$ of at most γ edges so that $D + H$ is (k,l)-edge-connected with respect to root s if and only if $\sum\left[p_{kl}(X) \,:\, X \in \mathcal{F}\right] \leq \gamma$ and $\sum\left[p_{kl}(V - X) \,:\, X \in \mathcal{F}\right] \leq \gamma$ hold for every partition $\mathcal{F}$ of V.*

3.3. Undirected detachment

Let $G = (V+z, E)$ be an undirected graph. We modify slightly the operation of splitting off a pair of edges $e = uz$, $f = vz$ as follows. Replace e and f by a new edge $h = uv$ and subdivide then h by a new node z'. More generally, by a **detachment** of node z into p nodes we mean the following operation. Replace z by p new nodes $z_1, \ldots, z_p$ and replace each edge uz by an edge uz_i. If the degree of each new node z_i is required to be a specified number d_i, we speak of a degree-specified detachment of z. In order for this to make sense we assume that $d_1, \ldots, d_p$ add up to $d_G(z)$.

Theorem 3.17 (Nash-Williams, [50]). *Let $G = (U, E)$ be a graph with a given positive integer $p(z)$ at every node z. It is possible to detach each node z into $p(z)$ parts so that the resulting graph is connected if and only if*

$$e(X) \geq p(X) + c_G(X) - 1 \tag{6}$$

holds for every non-empty subset $X \subseteq V$, where $p(X) := \sum\left[p(v) \,:\, v \in X\right]$, $e(X)$ is the number of edges having at least one end-node in X, and $c_G(X)$ denotes the number of components of $G - X$.

Note that Nash-Williams pointed out that this type of detachment can be handled as a matroid partition problem.

Suppose now that we are given at each node z of a graph $G = (U, E)$ a degree specification $d_1(z), \ldots, d_{p(z)}(z)$. Nash-Williams showed that it is possible to detach simultaneously all nodes so that there exists a degree-specified detachment of all nodes so that the resulting graph is connected if and only if (6) holds and $d_i(z) \geq 1$ for each i and $z \in V$.

What if we want a detachment which is k-edge-connected for $k \geq 2$? Clearly, for the existence of such detachment it is necessary that G be k-edge-connected and that each $d_i(z)$ is at least k. This is not always sufficient and we exhibit even two examples to show that. Let k be odd. First, suppose G consists of just two nodes u and v connected with $2k$ parallel edges, and $d_1(u) = d_2(u) = k = d_1(v) = d_2(v)$. Second, suppose that G has a cut node z of degree $2k$ and $d_1(z) = d_2(z) = k$. It is not difficult two check that no k-edge-connected detachment exists in either case. Quite surprisingly, there are no other bad cases:

Theorem 3.18 (Nash-Williams, [50]). *Let $G = (V, E)$ be an undirected graph with a degree specification $d_1(z), \ldots, d_{p(z)}(z)$ at each node z. It is possible to detach each node z into $p(z)$ nodes having specified degrees so that the resulting graph is k-edge-connected if and only if G is k-edge-connected, each requested degree $d_i(z)$ is at least k, except if k is odd and G is one of the two exceptional examples mentioned above.*

How is this result related to Lovász's undirected splitting lemma? They are not really comparable (in the sense that neither implies the other.) The splitting lemma detaches only one node, into nodes of degree two, and is clearly not 'interested' in preserving k-edge-connectivity at the detached nodes. But there is a very nice result of B. Fleiner [13] which is a generalization of Lovász's splitting lemma on one hand and implies easily Theorem 3.18 on the other.

The splitting lemma asserted that if G was k-edge-connected on V then a k-edge-connectivity preserving splitting always existed. If there are odd numbers in the degree-specification of the detachment, then this is not necessarily true. Let G consist of two disjoint triangles plus a node z connected to all the other six nodes. Then G is 3-edge-connected on V (even the whole G is) but it is not possible to detach z into two nodes of degree 3 so that the resulting graph keeps to be 3-edge-connected on V.

Theorem 3.19 (Fleiner). *Let $G = (V + z, E)$ be an undirected graph with a special node z and $k \geq 2$ an integer. Let $d_1, \ldots, d_p$ be integers for which $d_i \geq 2$, $\sum d_i = d_G(z)$. It is possible to detach z into p nodes of degree $d_1, \ldots, d_p$, respectively, so that the resulting graph is k-edge-connected in V if and only if G is k-edge-connected in V and $G - z$ is k'-edge-connected where*

(7)
$$k' := k - \sum_{i=1}^{p} \lfloor d_i/2 \rfloor.$$

Note that if each d_i is even, then $G - z$ is automatically k'-edge-connected so we do not have to explicitly require it, and this special case follows immediately from the undirected splitting lemma. As Lovász's splitting lemma could be used to derive Watanabe and Nakamura's Theorem 2.4 on minimum k-edge-connected augmentation of a graph, Fleiner used his result to prove the following generalization [13].

Theorem 3.20 (Fleiner). *Let $G = (V, E)$ be an undirected graph and $d_1, \ldots, d_p$ and k integers larger than one. It is possible to augment G by adding p new nodes of degree d_i, respectively, so that the enlarged graph G^+ is k-edge-connected on V if and only if*

(8)
$$\sum \left[\left(k - d_G(X) \right) \ : \ X \in \mathcal{F} \right] \leq \sum_{i=1}^{p} d_i$$

holds for every sub-partition $\mathcal{F}$ of V, and

(9)
$$\lambda(u, v; G) \geq k - \sum_{i=1}^{p} \lfloor d_i/2 \rfloor$$

holds for every pair of nodes $u, v \in V$, that is, G is k'-edge-connected, where $k' := k - \sum_{i=1}^{p} \lfloor d_i/2 \rfloor$.

So, Fleiner's Theorem 3.19 is one generalization of the undirected splitting lemma while Mader's Theorem 3.1 is another. Does there perhaps exist a common generalization of these difficult theorems? Yes, T. Jordán and Z. Szigeti proved the following theorem [34].

Theorem 3.21 (Jordán and Szigeti). *Let $G = (V + z, E)$ be a graph with a special node z so that there is no cut-edge incident to z. Let $d_1, \ldots, d_p$*

be integers for which $d_i \geq 2$, $\sum d_i = d_G(z)$. Also, we are given a symmetric function $r(u,v)$ on the pairs of nodes in V. There is a detachment of z into p nodes of degree $d_1, \dots, d_p$, respectively, so that in the resulting graph G' the local edge-connectivity $\lambda(u,v;G')$ is at least $r(u,v)$ for every $u,v \in V$ if and only if

$$(10) \qquad r(u,v) \leq \lambda(u,v;G) \text{ and } \lambda(u,v;G-z) \geq r(u,v) - \sum_{i=1}^{p} \lfloor d_i/2 \rfloor$$

for all $u,v \in V$.

In the augmentation results so far we always added edges to an existing graph $G = (V,E)$. This may be interpreted as adding new nodes of degree two so that the (local) edge-connectivity should attain a certain prescribed value. It is quite natural to investigate an extension of the problem when the newly added nodes are of prescribed degree, not necessarily two. The following result of Jordán and Szigeti [34] is a straight generalization of Theorem 3.12. As in Theorem 3.12, we are given an undirected graph $G = (V,E)$ and a symmetric non-negative integer-valued function $r(u,v)$ on the pair of nodes, called local edge-connectivity requirement. Let $R_r(X) := \max\{r(u,v) : u \in X, v \in V - X\}$ for every $X \subseteq V$ and let $q(X) := R_r(X) - d_G(X)$. Recall the definition from (4) of $R_r(X)$, $q(X)$ and a marginal component of G.

Theorem 3.22 (Jordán and Szigeti [34]). *Let $G = (V,E)$ be an undirected graph, $r(u,v)$ a local edge-connectivity requirement function so that there are no marginal components. Moreover, let $d_1, d_2, \dots, d_p$ be integers each larger than 1. It is possible to add to G p new nodes of degree d_i, respectively, so that the enlarged graph G^+ satisfies $\lambda(u,v;G^+) \geq r(u,v)$ for every pair of nodes $u,v \in V$ if and only if*

$$(11) \qquad \sum [q(X) : X \in \mathcal{F}] \leq \sum_{i=1}^{p} d_i$$

holds for every sub-partition $\mathcal{F}$ of V, and

$$(12) \qquad \lambda(u,v;G) \geq r(u,v) - \sum_{i=1}^{p} \lfloor d_i/2 \rfloor$$

holds for every pair of nodes $u,v \in V$.

3.4. Directed detachment

In Mader's directed splitting lemma, it was assumed for the specified node z to have the same in- and outdegree. Without this restriction a splitting at z preserving k-edge-connectivity in V does not necessarily exist. However, Berg, Jackson and Jordán [5] found the following interesting extension of the splitting lemma.

Theorem 3.23 (Berg, Jackson, Jordán). *Let $k \geq 1$ be an integer and $D = (V+z, E)$ a directed graph with a special node z for which $\varrho(z) \geq \delta(z)$. If D is k-edge-connected on V, then for every edge zu there are t edges $v_1z, \dots, v_tz$, where $1 \leq t \leq \varrho(z) - \delta(z) + 1$, entering z so that detaching z into two nodes z' and z_1 results in a digraph which is k-edge-connected on V, where z_1 has one outgoing edge z_1u and t entering edges $v_1z_1, \dots, v_tz_1$.*

By repeated applications of the theorem, one easily obtains a complete detachment version: *If k, D, z are the same as before, it is possible to detach the edges at z into $\delta(t)$ nodes so that each contains exactly one edge leaving it and so that the resulting digraph is k-edge-connected in V.*

A directed counter-part of Nash-Williams's detachment theorem was obtained by Berg, Jackson and Jordán [6]. Given a function $r : V \to \mathbf{Z}_+$, by an **r-detachment** of a digraph $D = (V,A)$ we mean a digraph arising from D by 'detaching' simultaneously each node v into $r(v)$ pieces so that each edge leaving or entering v would leave or enter one of the pieces.

Theorem 3.24 ([6]). *Let $D = (V,E)$ be a digraph and let $r : V \to \mathbf{Z}_+$. Then D has a k-edge-connected r-detachment if and only if*

(a) *D is k-edge-connected,*

(b) *$\varrho(v) \geq kr(v)$ and $\delta(v) \geq kr(v)$ for every $v \in V$.*

In addition, Berg, Jackson and Jordán proved that the in- and outdegrees of every detached node $v \in V$ can be arbitrarily specified provided that at each node v of D all the values in the indegree specifications are at least k and add up to the indegree of v and similarly for the outdegree specifications.

4. Uncrossing-based results

In the previous two sections we overviewed results evolving from the splitting lemmas. Here some fruits of another fundamental technique, the uncrossing procedure, will be surveyed. The rough idea of this approach is that for a given family of sets with certain properties or parameters one can replace two uncomparable (or intersecting, or crossing) sets by their intersection and union so as to preserve the properties or parameters of the family. By repeating this uncrossing step as long as possible, one arrives in a finite number of steps at a nicer family (chain of sets, laminar, or cross-free), preserving the essential properties or parameters of the initial one. To my best knowledge, the first appearance of this approach that appeared in print [39] was a solution of L. Lovász (a third-grade university student at that time) to Problem 11 (posed by A. Rényi) of the Memorial Mathematical Contest Miklós Schweitzer of the year 1968.

Later Lovász used the technique to provide a simple proof of the Lucchesi-Younger theorem [41] and to prove his theorem on minimum T-joins [40]. Since then the uncrossing method has proved to be an extremely powerful proof technique. In this section we briefly overview some recent results that were obtained this way.

4.0.1. A detour to the origin of uncrossing. Rényi's Problem 11 was to verify an inequality concerning the probabilities of some events in a finite probability space. In his solution, Lovász first observed that the logarithm of the probability of events is a submodular function (where product and sum of events correspond to intersection and union, respectively), and he then applied the uncrossing technique to derive the requested inequality. Actually, Lovász's proof uses nothing but the submodular property and hence it provides the corresponding inequality for *any* submodular function: we exhibit Lovász's proof in this context. In order to do so, it is useful to introduce the notion of linear extension of a set-function.

Let b be a set-function on a groundset S for which $b(\emptyset) = 0$. For any vector $c \in \mathbf{R}^{|S|}$, arrange the elements of S in such a way that $c(s_1) \geq \dots \geq c(s_n)$. Let $S_i := \{s_1, \dots, s_i\}$ and define $\hat{b}(c)$ by $\hat{b}(c) := c(s_n)b(S_n) + \sum_{i=1}^{n-1} [c(s_i) - c(s_{i+1})] b(S_i)$. The function $\hat{b} : \mathbf{R}^S \to \mathbf{R}$ defined this way is called the **linear extension** of b. It was introduced also by Lovász in 1983 [43] and therefore often the term *Lovász extension* is used. It should be noted that the correctness of the matroid greedy algorithm is equivalent to

stating that the maximum c-weight of bases of a matroid with rank function r equals $\hat{r}(c)$, or more generally, Edmonds' polymatroid greedy algorithm is equivalent to the assertion that, given a fully submodular function b, $\max\{cx : x \in B(b)\} = \hat{b}(c)$, where $B(b) := \{x \in \mathbf{R}^S : x(Z) \leq b(Z)$ for every $Z \subset S$ and $x(S) = b(S)\}$ is the so called base-polyhedron.

The solution of Lovász in [39] to Problem 11 contains implicitly the following.

Lemma 4.1. *Let b be a fully submodular function on a ground-set S and $\hat{b}$ its linear extension. Then, for any collection $\{X_1, X_2, \dots, X_m\}$ of subsets of S,*

$$\sum_i^m b(X_i) \geq \hat{b}\Big(\sum_i^m \chi_{X_i}\Big), \tag{13}$$

where χ_X denotes the characteristic function of X.

Proof. Apply the uncrossing procedure to the family $\{X_1, \dots, X_m\}$, that is, as long as there are two uncomparable sets in the current family, replace them by their intersection and union. Due to the submodularity of b, the sum of the b-values of the members never increases, while the sum of the characteristic vectors of the members stay unchanged.

Since the number of uncomparable sets in the family during an uncrossing step strictly decreases, the uncrossing procedure terminates in a finite number of steps. The final family is a chain $\{Z_1 \subsetneq Z_2 \subsetneq \dots \subsetneq Z_m\}$ of subsets for which $\sum_i \chi_{X_i} = \sum_i \chi_{Z_i}$, and hence $\sum_i b(X_i) \geq \sum_i b(Z_i) = \hat{b}(\sum_i \chi_{Z_i}) = \hat{b}(\sum_i \chi_{X_i})$. ■

The inequality in (13) may be called generalized submodular inequality. (We note that the even more general inequality $\sum \hat{b}(c_i) \geq \hat{b}(\sum c_i)$ also holds true for arbitrary vectors $c_1, \dots, c_m \in \mathbf{R}^S$.) To see the usefulness of (13), we make a little detour and derive in a few lines the following elegant result on matroids from the partition theorem.

Theorem 4.2 (Greene és Magnanti). *Let B_1 and B_2 be bases of a matroid M and $\{Z_1, Z_2, \dots, Z_m\}$ a partition of B_1. Then there is a partition $\{Y_1, \dots, Y_m\}$ of B_2 for which $B_1 - Z_i \cup Y_i$ is a basis for each subscript $i = 1, \dots, m$.*

Proof. We may assume that B_1 and B_2 are disjoint for otherwise their intersecion can be contracted and the theorem for the contracted matroid implies that for M. Let k denote the rank of M. For each i, consider the matroid $M_i = (B_2, r_i)$ arising from M by contracting first $B_1 - Z_i$ and restricting then the resulting matroid to B_2. For any subset $X \subseteq B_2$, let $X_i := B_1 - Z_i \cup X$. Then $\sum_i \chi_{X_i} = (m-1)\chi_{(B_1 \cup X)} + \chi_X$, and by (13) we have $\sum_i r(X_i) \geq \hat{r}\big(\sum_i \chi_{X_i}\big) = \hat{r}\big[(m-1)\chi_{(B_1\cup X)} + \chi_X\big] = (m-1)r(B_1 \cup X) + r(X) = (m-1)k + |X|$. From $r_i(X) = r(X_i) - r(B_1 - Z_i) = r(X_i) - |B_1 - Z_i|$, we obtain $\sum_i r_i(X) = \sum_i \big[r(X_i) - |B_1 - Z_i|\big] = \sum_i r(X_i) - (km-k) \geq (m-1)k + |X| - (km-k) = |X|$.

By the matroid partition theorem of Edmonds and Fulkerson [11], B_2 can be partitioned into sets $Y_1, Y_2, \ldots, Y_m$ so that Y_i is independent in M_i. By the definition of M_i, $|Y_i| \leq |Z_i|$ for each i, and hence $\sum |Z_i| = \sum |Y_i|$. Therefore $|Y_i| = |Z_i|$, and then $B_1 - Z_i \cup Y_i$ is a basis of M. ■

4.1. Orientations and augmentations through submodular flows

A general and flexible framework concerning sub- or supermodular functions is the notion of submodular flow. In [23] a rather exhaustive survey was given to show how basic results on submodular flows can be applied to orientation problems. By an **orientation of a mixed graph** $M = (V, A + E)$, with directed and undirected edge-sets A and E respectively, we mean a directed graph $(V, A + \vec{E})$ arising from M by orienting each undirected edge and leaving alone the directed ones.

Before exhibiting a characterization of mixed graphs having k-edge-connected orientations, let us consider the special case $k = 1$.

4.1.1. Strongly connected orientation of mixed graphs. A straightforward generalization of Robbins' theorem, with a fairly easy proof, is due to F. Boesch and R. Tindell [7].

Theorem 4.3. *A mixed graph $M = (V, A + E)$ has a strongly connected orientation if and only if M has no cut-edge and no subset $\emptyset \subset X \subset V$ of nodes so that neither directed nor undirected edges leave X.*

Proof. We show that the undirected edges can be oriented greedily one by one, taking care only to avoiding the creation of a directed cut. There is nothing to prove if E is empty. Let $e = uv \in E$ be an undirected edge. If orienting e toward v (toward u, respectively) creates a directed cut, then there is a $u\bar{v}$-set X (a $v\bar{u}$-set Y) so that no directed edge leaves X (Y) and e is the only undirected edge leaving X (Y). Then neither $X \cap Y$ nor $V - (X \cup Y)$ admits a leaving edge and hence they must be empty. Therefore X and Y are complementary sets and e is the only edge connecting X and Y, contradicting the assumption on the non-existence of cut-edges. ■

The simplicity of this result may suggest that Nash-Williams' Theorem 2.8 on k-edge-connected orientability of $2k$-edge-connected undirected graphs can also be extended to mixed graphs in a straightforward way. But this is not the case even for $k = 2$.

4.1.2. An example for $k = 2$. It turns out that in this case the natural cut-type or partition-type necessary conditions are not sufficient anymore. To see this, define a mixed graph $M = (V_4, A + E)$ as follows. Let $V_4 = \{v_1, v_2, v_3, v_4\}$, let E consist of two edges $e_1 = v_1v_2$, $e_2 = v_3v_4$, and let A consist of the following nine edges: v_1v_3, v_1v_3, v_3v_1, v_2v_3, v_2v_3, v_3v_2, v_2v_4, v_2v_4, v_4v_2.

The digraph $D = (V_4, A)$ is strongly connected, that is, every in-deficient set (with respect to 2-edge-connectivity) is of indegree one, and there are exactly three such sets:

$$X_1 := \{v_1\}, \quad X_2 := \{v_1, v_2, v_3\}, \quad X_3 := \{v_2, v_4\}.$$

Let $\mathcal{A}_3 := \{X_1, X_2, X_3\}$. In order to have a 2-edge-connected orientation of M, one has two orient the two edges of $G = (V_4, E)$ so that each member of $\mathcal{A}_3$ admits at least one newly oriented entering edge. An easy case checking shows that no such orientation may exist. Note, however, that for every two members of $\mathcal{A}_3$, there is an orientation of G in which the indegree of these two members is at least 1. This implies that any certificate of the nonexistence of a 2-edge-conneced orientation of M which consists of in-deficient sets must include all the three members of $\mathcal{A}_3$.

Note that $\mathcal{A}_3$ is neither a partition nor a co-partition of any subset of V. The example therefore indicates why one needs more general families of sets in the characterization of k-edge-connected orientable mixed graphs. The result will also show that the use of submodular functions is unavoidable in the solution of this purely graph-theoretic problem. The approach easily extends to (k, l)-edge-connected orientability.

4.1.3. Tree-compositions. For a proper non-empty subset S of V we introduce the notion of a tree-composition of S. Let $\{S_1, \dots, S_\alpha\}$ be a partition of S and $\{Z_1, \dots, Z_\beta\}$ a partition of $V - S$ $(\alpha, \beta \geq 1)$. Let $T = (U, F)$ be a directed tree such that $U := \{s_1, \dots, s_\alpha, z_1, \dots, z_\beta\}$ and each directed edge goes from a z_j to an s_i. For each edge f of the tree, let T_f denote the set of nodes of that component of $T - f$ which is entered by f. The family $\mathcal{A} := \{\varphi^{-1}(T_f) : f \in F\}$ is called a **tree-composition of** S where $\varphi(v) = s_i$ if $v \in S_i$ and $\varphi(v) = z_j$ if $v \in Z_j$. We will also say that a partition or a co-partition of V is a **tree-composition of** V. Note that a tree-composition $\mathcal{A}$ of S is cross-free and every element of S belongs to the same number t of members of $\mathcal{A}$ and every element of $V - S$ belongs to $t - 1$ members. (If $\alpha = \beta = 1$, then $\mathcal{A}$ consists of the single set S. If $\beta = 1 < \alpha$, then $\mathcal{A}$ is a partition of S. If $\alpha = 1 < \beta$, then $\mathcal{A}$ is a co-partition of S.)

Let us consider the subset $S := \{v_1, v_2\}$ in the example above. We claim that the family $\mathcal{A}_3$ forms a tree-composition of S. This can be seen by defining $S_1 := \{v_1\}$, $S_2 := \{v_2\}$, $Z_1 := \{v_3\}$, $Z_4 := \{v_4\}$ and by letting T be a directed tree on node set $\{s_1, s_2, z_1, z_2\}$ having three edges: $f_1 = v_3v_1$, $f_2 = v_3v_2$, $f_3 = v_4v_2$. Now $T_{f_1} = s_1$, $T_{f_2} = \{z_2, s_2\}$ and $T_{f_3} = \{s_1, s_2, z_1\}$. Let $\varphi(v_1) = s_1$, $\varphi(v_2) = s_2$, $\varphi(v_3) = z_1$, $\varphi(v_4) = z_2$. Then $\mathcal{A}_3$ indeed arises in the form described in the definition of tree-composition.

Suppose now that $G = (V, E)$ is an arbitrary undirected graph. Let $\mathcal{A}$ be a tree-composition of a subset $S \subseteq V$ and $j = uv$ an edge of G. Let $e_{u\bar{v}}(\mathcal{A})$ denote the number of $u\bar{v}$-sets in $\mathcal{A}$. That is, $e_{u\bar{v}}(\mathcal{A})$ is the number of sets in $\mathcal{A}$ entered by the directed edge with tail v and head u. Let $e_j(\mathcal{A}) := \max\{e_{u\bar{v}}(\mathcal{A}), e_{\bar{u}v}(\mathcal{A})\}$ and

(14) $$e_G(\mathcal{A}) := \sum_{j \in E} e_j(\mathcal{A}).$$

Note that $|e_{u\bar{v}}(\mathcal{A}) - e_{v\bar{u}}(\mathcal{A})| \leq 1$ with equality if and only if $|S \cap \{u, v\}| = 1$. The quantity $e_j(\mathcal{A})$ indicates the (maximal) possible contribution of an edge $j = uv$ to the sum $\sum[\varrho_{\vec{G}}(X) : X \in \mathcal{A}]$ for any orientation $\vec{G}$ of G. Hence $e_G(\mathcal{A})$ measures the total of these contributions and we have

(15) $$\sum_{X \in \mathcal{A}} \varrho_{\vec{G}}(X) \leq e_G(\mathcal{A})$$

for any orientation $\vec{G}$ of G. Let $D = (V, A)$ be a digraph and $M = (V, A+E)$ a mixed graph. Let s be a root-node of M. For integers $0 \leq l \leq k$ define $p_{kl}(X) := (k - \varrho_D(X))^+$ if $\emptyset \subset X \subset V - s$ and $p_{kl}(X) := (l - \varrho_D(X))^+$ if $s \in X \subset V$.

Theorem 4.4 [23]. *A mixed graph M has a (k, l)-edge-connected orientation (with respect to root-node s) if and only if*

(16) $$\sum[p_{kl}(X) : X \in \mathcal{A}] \leq \sum[e_G(\mathcal{A}) : e \in E]$$

holds for every tree-composition $\mathcal{A}$.

In the example above, where $k = l = 2$, $\mathcal{A}_3$ violates (16) since $p_{kl}(X) = 1$ for each $X \in \mathcal{A}_3$ while $e_G(\mathcal{A}_3) = 2$ since each of the two edges of G can contribute to the indegree of the sets in $\mathcal{A}_3$ by one.

4.1.4. Special cases. While tree-compositions are inevitable in general, in some important special cases they are not, as we have already seen in Theorems 2.8 and 2.9. We now exhibit a common generalization of these last two results when partition type conditions turn out to be sufficient. We investigate the orientation problem when l-edge-connectivity and rooted k-edge-connectivity are simultaneously required (that is, we want a (k, l)-edge-connected orientation).

Theorem 4.5 [18]. *Let $0 \leq l \leq k$ be integers. An undirected graph $G = (V, E)$ has a (k, l)-edge-connected orientation if and only if G is (k, l)-partition-connected.*

Another special case of the mixed graph (k, l)-edge-connected orientation problem when only partition type conditions are required is the case of $l \leq 1$. The case $l = 0$, which is a generalization of Theorem 2.9, appeared in [16].

Theorem 4.6. *A mixed graph $D + G = (V, A + E)$ with a root-node s has a $(k, 0)$-edge-connected (that is, s-rooted k-edge-connected) orientation if and only if the number of cross-edges of G is at least*

(17) $$\sum_{i=1}^{t} [k - \varrho_D(V_i)]$$

for every partition $\{V_0, V_1, \dots, V_t\}$ of V into non-empty parts with $s \in V_0$.

The case $l = 1$ appeared in [23].

Theorem 4.7. *A mixed graph $D + G = (V, A + E)$ with a root-node s has a $(k, 1)$-edge-connected orientation (that is, strongly connected and s-rooted k-edge-connected) if and only if the number of cross-edges of G is at least $\sum_{i=1}^{t}[k - \varrho_D(V_i)] + 1$ for every partition $\{V_0, V_1, \dots, V_t\}$ of V into non-empty parts with $s \in V_0$.*

4.1.5. An augmentation result. The rooted edge-connectivity augmentation problem (in digraphs) behaves nicely in the sense that even the minimum cost version is tractable. Suppose that we are given a digraph with a special root-node s and we want to augment the digraph by adding a minimum cost of new edges so as to have a rooted k-edge-connected digraph. At the beginning of section 2, we mentioned that the minimum cost subgraph problem is equivalent to the minimum cost augmentation problem, and in this case the subgraph problem (:find in a digraph a minimum cost rooted k-edge-connected subgraph) can be solved with the help of submodular flows, see [17] and [54]. Here we mention only one consequence of this:

Theorem 4.8. *Let $D = (V, E)$ and $H = (V, A)$ be two digraphs so that their union $D + H = (V, E \cup A)$ is k-edge-connected from a root-node s. The minimum number of edges of H whose addition to D results in a s-rooted k-edge-connected digraph is equal to the maximum of $\sum [k - \varrho_D(X) : X \in \mathcal{F}]$, where the maximum is taken over all laminar families $\mathcal{F}$ of non-empty subsets of $V - s$ for which no edge of H enters more than one member of $\mathcal{F}$.*

4.2. Connectivity orientation and augmentation combined

Now comes an account on some new developments making possible to combine certain orientation and augmentation problems. In subsection 2.2 we have already mentioned this type of results: Theorem 2.10 characterized undirected graphs which can be augmented by adding at most γ edges so as to have a $(k, 0)$-edge-connected orientation. We also remarked that even the minimum cost augmentation was tractable by using matroid techniques. Here we consider the same problem for mixed graphs (where those matroid techniques do not work.) Let us consider Theorem 4.6 and suppose that the required orientation does not exists, that is, the necessary and sufficient condition in (17) fails to hold. How many new undirected edges should be added to M so as to have a $(k, 0)$-edge-connected orientation. Or more generally, what is the minimum cost of required new edges? By considering the existing undirected edges having zero cost, this latter problem is equivalent to the following.

Given a mixed graph with a root node s endowed with a non-negative cost function on the set of undirected edges, delete a maximum cost of edges so that the resulting mixed graph has a $(k, 0)$-edge-connected orientation.

S. Khanna, J. Naor and F. B. Shepherd [35] solved this problem in an even more general form when the directed edges may also have costs and the two possible directions $e' = uv$ and $e'' = vu$ of an undirected edge uv may have different costs.

To be more specific, let $M = (V, A + E)$ be a mixed graph consisting of a digraph $D = (V, A)$ and an undirected graph (V, E). Let s be a root-node of M and let $A_1 := A \cup \{e', e'' : e \in E\}$. Furthermore we are given a nonnegative cost function $c : A_1 \to \mathbf{R}_+$. We say that a subset $F \subseteq A_1$ of directed edges (or the subdigraph $D' := (V, F)$) is **orientation-constrained** if F may contain at most one of the two possible directions e' and e'' of any undirected edge $e \in E$.

The $(k, 0)$**-orientable subgraph problem** consists of finding a minimum cost $(k, 0)$-edge-connected orientation-constrained subdigraph $D' = (V, F)$ of $D_1 := (V, A_1)$.

Khanna, Naor and Shepherd considered the following linear program:

$$\min \sum [c(f)x(f) : f \in A_1] \tag{18}$$

subject to

$$0 \le x(f) \le 1 \quad \text{for every directed edge} \quad f \in A_1 \tag{19}$$

$$x(e') + x(e'') \le 1 \quad \text{for every edge} \quad e \in E \tag{20}$$

(21)
$$\sum [x(f) : f \in A_1,\ f \text{ enters } Z] \ge k \quad \text{for every subset} \quad \emptyset \subset Z \subseteq V - s.$$

Let P denote the polytope described by the three constraints. Clearly, an integer vector in P is actually $0-1$-valued and the $0-1$ vectors of P are precisely the characteristic vectors of orientation constrained $(k, 0)$-edge-connected subdigraphs of D_1.

The main result of [35] is as follows:

Theorem 4.9 (Khanna, Naor, and Shepherd). *The vertices of polytope P are $0-1$ vectors, or equivalently, P is the convex hull of (characteristic vectors) of orientation-constrained $(k, 0)$-edge-connected subdigraphs of D_1.*

By relying on linear programming duality, this theorem provides a min-max formula for the minimum cost of a solution. We avoid formulating this since the result can be even further improved [29]. We emphasize, however, that the improvement uses only known ideas, and the main point here is the recognition of Khanna, Naor, and Shepherd that even this general framework is tractable by standard techniques.

Theorem 4.10. *The linear inequality system of* (19), (20), *and* (21) *is totally dual integral (implying the integrality of P). Moreover, P is a submodular flow polyhedron.*

This theorem enables us to solve the problem algorithmically by invoking a submodular flow algorithm. Furthermore, one has a better structured duality theorem. For the sake of simplicity we formulate it only for $0-1$-valued cost functions.

Theorem 4.11. *Let $M = (V, A+E)$ be a mixed graph with a root-node s endowed with a $0-1$ valued cost function $c : A \cup E \to \{0,1\}$. The minimum cost of a mixed subgraph of M which has a $(k,0)$-edge-connected orientation is equal to the maximum of*

$$tk - e_G(\mathcal{F}) - \sum \left[\varrho_D(X) : X \in \mathcal{F}\right] + q(\mathcal{F}),$$

where the maximum is taken over all laminar families $\mathcal{F}$ of t ($t \geq 0$) subsets of $V - s$. Here $G = (V,E)$ is the undirected part of M, $e_G(\mathcal{F})$ is defined in (14), *and $q(\mathcal{F})$ denotes the number of (directed or undirected) edges of cost 1 which enter at least one member of $\mathcal{F}$.*

This is a common generalization of Theorems 4.6 and 4.8. When c is zero on all directed edges, we are back at our starting problem of finding a smallest set of new undirected edges to be added to a mixed graph to have a $(k,0)$-edge-connected orientation.

So, we can solve quite reassuringly the combined orientation/augmentation problem in mixed graphs when the target is $(k,0)$-edge-connectivity. Wouldn't it be natural to lift our horizon to (k,l)-edge-connectivity? The directed (k,l)-edge-connectivity augmentation problem is solved by Theorem 3.16. The (k,l)-edge-connectivity orientation problem is solved for undirected graphs by Theorem 4.5 (and even for mixed graphs by Theorem 4.4). We show now how to solve the problem of augmenting an undirected graph by adding undirected edges so that the resulting graph has a (k,l)-edge-connected orientation. Due to the relatively complicated nature of tree-compositions in Theorem 4.4, so far we have not taken courage to try to attack the corresponding augmentation problem for mixed graphs. And even for undirected graphs the minimum cost version is out of question

because the NP-complete problem of finding a Hamiltonian circuit problem is a special case. We consider the degree-specified and the minimum augmentation problems as well. The following results are taken from [28].

Theorem 4.12. *Let $G = (V,E)$ be an undirected graph, $k \geq l \geq 0$ integers, and $m := V \to \mathbf{Z}_+$ a degree-specification for which $m(V)$ is even. There exists a graph $H = (V,A)$ so that $d_H(v) = m(v)$ for every $v \in V$ and so that $G+H$ is (k,l)-tree-connected (= (k,l)-partition-connected = (k,l)-edge-connected orientable) if and only if*

$$m(V)/2 \geq (t-1)k + l - e_G(\mathcal{F}) \tag{22}$$

and

$$\min_{X \in \mathcal{F}} m(V-X) \geq (t-1)k + l - e_G(\mathcal{F}) \tag{23}$$

hold for every partition $\mathcal{F}$ of V into $t \geq 2$ non-empty parts.

Let us indicate briefly the proof of necessity. If $G+H$ has a (k,l)-edge-connected orientation, then it is (k,l)-partition-connected, that is, $e_{G+H}(\mathcal{F}) \geq k(t-1)+l$ and hence $e_H(\mathcal{F}) \geq k(t-1)+l-e_G(\mathcal{F})$. If H satisfies the degree-specification, then $m(V)/2 = |A| \geq e_H(\mathcal{F})$ and $m(V-X) \geq e_H(\mathcal{F})$ for every $X \in \mathcal{F}$ from which both (22) and (23) follow.

This result might be interesting even in the special case of $l = 0$:

Corollary 4.13. *Let $G = (V,E)$ be an undirected graph, $k \geq 1$ an integer, and $m := V \to \mathbf{Z}_+$ a degree-specification for which $m(V)$ is even. There exists a graph $H = (V,A)$ so that $d_H(v) = m(v)$ for every $v \in V$ and so that $G+H$ is k-tree-connected if and only if*

$$m(V)/2 \geq (t-1)k - e_G(\mathcal{F}) \tag{24}$$

and

$$\min_{X \in \mathcal{F}} m(V-X) \geq (t-1)k - e_G(\mathcal{F}) \tag{25}$$

hold for every partition $\mathcal{F}$ of V into $t \geq 2$ non-empty parts.

The following theorem is a bit out of the main line of the paper since the target of the augmentation is not a connectivity property. As a counterpart to tree-packing in corollary 4.13, here our target is tree-covering:

Theorem 4.14 [28]. *Let $G = (V,E)$ be an undirected graph, $k \geq 1$ an integer, and $m := V \to \mathbf{Z}_+$ a degree-specification for which $m(V)$ is even. There exists a graph $H = (V,A)$ so that $d_H(v) = m(v)$ for every $v \in V$ and so that $G + H$ is the union of k forests if and only if*

$$m(X) - m(V)/2 \leq k(|X|-1) - i_G(X) \tag{26}$$

for every $\emptyset \subset X \subseteq V$, where $i_G(X)$ denotes the number of edges of G induced by X.

Again it is useful to prove the necessity. If H is a graph for which $G+H$ is the union of k forests, then $e_{G+H} \leq k(|X|-1)$ holds for every subset $X \subseteq V$, that is, $i_H(X) \leq k(|X|-1) - i_G(X)$. If H satisfies the degree-specification, then $|A| = m(V)/2$ and at most $m(V-X)$ edges may be incident with an element of $V - X$. So at least $m(V)/2 - m(V-X)$ edges are induced by X in H and hence $m(X) - m(V)/2 = m(V)/2 - m(V-X) \leq i_H(X) \leq k(|X|-1) - i_G(X)$.

To conclude this subsection, we cite a result from [28] on the minimization form of (k,l)-tree-connectivity augmentation.

Theorem 4.15. *Let $G = (V,E)$ be an undirected graph. It is possible to add at most γ new edges to G so that the resulting graph G^+ is (k,l)-tree-connected (that is, G^+ has a (k,l)-edge-connected orientation) if and only if*

$$\gamma \geq k(t-1) + l - e_G(\mathcal{F}) \tag{27}$$

holds for every partition $\mathcal{F}$ of V with t members, and

$$2\gamma \geq t_1 k + t_2 l - e_G(\mathcal{F}) \tag{28}$$

holds whenever $\mathcal{F}$ is the union a partition $\mathcal{F}_1$ of a subset $Z \subsetneq V$ and a co-partition $\mathcal{F}_2$ of Z so that $|\mathcal{F}_i| = t_i$ $(i = 1,2)$ and so that $\mathcal{F}_1$ is a finer partition of Z than partition $\{X : V - X \in \mathcal{F}_2\}$.

4.3. Directed edge-connectivity augmentation

In [25] we proved a general min-max formula concerning minimum coverings of a so-called bi-supermodular function by directed graphs. This result implies Theorem 3.16 (which has had an independent and simpler proof) and implies the following, as well.

Theorem 4.16. *Let $D = (V,A)$ be a directed graph and S, T two (not necessarily disjoint) non-empty subsets. It is possible to add at most γ ST-edges so that the resulting digraph is k-edge-connected from S to T if and only if*

$$\sum [k - \varrho_D(X) : X \in \mathcal{F}] \leq \gamma \tag{29}$$

holds for every family $\mathcal{F}$ of pairwise ST-independent sets, where two sets X, Y are ST-independent if $X \cap Y \cap T = \emptyset$ or $S - (X \cup T) = \emptyset$.

In sharp contrast with the existence of a good characterization in Theorem 3.12 concerning local edge-connectivity augmentations of undirected graphs, the directed counterpart of this problem is NP-complete [19] even in the special case when the requirement is one between the nodes of a specified subset T of nodes and zero otherwise. (That is, given a digraph, add a minimum number of new edges so that there is a path from every element of T to every other element of T.) Recently, however, I found the following characterization for $|T| = 2$ [24]. (This result seems to be independent of the rather general main theorem of [25].)

Theorem 4.17. *Let $D = (V,E)$ be a digraph with two specified nodes s, t and let k, l be two non-negative integers. Let S, T be non-empty subsets of V so that every $s\bar{t}$-set X with $\varrho_D(X) < k$ and every $t\bar{s}$-set X with $\varrho_D(X) < l$ is entered by an ST-edge. D can be augmented by adding at most γ (possibly parallel) ST-edges so that in the resulting digraph there are k edge-disjoint paths from s to t and there are l edge-disjoint paths from t to s if and only if $\gamma \geq k - \varrho_D(X)$ whenever $t \in X \subseteq V - s$, $\gamma \geq l - \varrho_D(X)$ whenever $s \in X \subseteq V - t$, and $\gamma \geq (l - \varrho_D(X)) + (k - \varrho_D(Y))$ holds whenever $s \in X, t \in Y$ and $X \cap Y \cap T = \emptyset$ or $X \cup Y \supseteq S$.*

5. Constructive characterizations

We have already seen constructive characterizations of k-edge-connected graphs and digraphs (Theorems 2.13, 3.2, 2.14), of $(k,0)$-edge-connected digraphs (2.15) and k-tree-connected graphs (2.16). For integers $0 \le l < k$ we offer the following:

Conjecture 5.1. *A directed graph D is (k,l)-edge-connected if and only if it can be built from a node by the following two operations:* (j) *add a new edge,* (jj) *pinch i $(l \le i < k)$ existing edges with a new node z, and add $k-i$ new edges entering z and leaving existing nodes. An undirected graph is (k,l)-tree-connected ($=(k,l)$-partition-connected) if and only if it can be built from a node by the following two operations:* (j) *add a new edge,* (jj) *pinch i $(l \le i < k)$ existing edges with a new node z, and add $k-i$ new edges connecting z with existing nodes.*

Note that by Theorem 4.5 the undirected version of the conjecture follows from the directed one. As mentioned above, the case $l=0$ is settled by Theorem 2.15. Jointly with Zoltán Király [27], we characterized $(k,k-1)$-edge-connected digraphs (and hence $(k,k-1)$-partition-connected graphs, as well). At the other end of the range of l, recently in [31] we proved the case $l=1$. All other cases of the conjecture are open (for example, when $k=4$, $l=2$).

The theorem in [27] concerning the case $l=k-1$, in turn, can be used to derive the following orientation result. Let $G=(V,E)$ be an undirected graph. A subset T of nodes is called **G-even** if $|T|+|E|$ is even. We call an orientation of G **T-odd** if the indegree of a node v is odd precisely when v belongs to T. The following is taken from [27].

Theorem 5.2. *An undirected graph G has a k-edge-connected and T-odd orientation for every G-even subset T if and only if G is $(k+1,k)$-partition-connected.*

Corollary 5.3. *A $(2k+2)$-edge-connected graph always admits a k-edge-connected orientation in which the indegree of all nodes but possibly one are odd.*

As mentioned above, the proof is based on the constructive characterization of $(k+1,k)$-partition-connected graphs. It would be interesting to have a simple direct proof of the corollary, even for the special case $k=1$ when it asserts that a 4-edge-connected graph has a strongly connected orientation in which every node but possibly one is of odd indegree.

The motivation behind such a theorem is the natural attempt to have a better understanding of problems where both parity and connectivity are involved. In Theorem 5.2 we charaterized graphs having a certain orientation for *every* G-even subset T. It would be interesting to know the necessary and sufficient condition of the existence of a k-edge-connected T-odd orientation of a graph G for *one* specified G-even subset T. This is open. However, the analogous question concerning k-tree-connectivity has been settled in [26].

Theorem 5.4. *Let $G=(V,E)$ be a graph with a root-node s. Let T be a G-even subset of $V-s$. G has a $(k,0)$-edge-connected ($=s$-rooted k-edge-connected) T-odd orientation if and only if the number of cross edges of every partition $\mathcal{P} := \{V_1,\dots,V_t\}$ of V into at least two non-empty parts is at least*

$$k(t-1)+o(\mathcal{P}),$$

where $o(\mathcal{P})$ (which depends also on G, k, and T) denotes the number of those parts X of $\mathcal{P}$ for which $|X \cap T| - i_G(X) - k$ is odd.

As a possible counterpart to Corollary 5.3, we can derive:

Corollary 5.5. *Let $G=(V,E)$ be an undirected graph with $|E|+|V|$ even. If G is $(k+1)$-tree-connected, then G has a $(k,0)$-edge-connected V-odd orientation.*

But this is straightforward anyway since we can take $k+1$ edge-disjoint trees, orient the edges of k of these away from a root node s, orient the remaining edges not in the last tree F_{k+1} arbitrarily, and finally, orient the edges of F_{k+1} so as to meet the parity prescription.

A problem related to the constructive characterization of k-edge-connected digraphs is to find a characterization of (acyclic) digraphs whose all directed cuts admit at least k edges. Such an approach could perhaps be used to prove D. Woodall's long-standing conjecture:

Conjecture 5.6. *If every directed cut of a digraph D has at least k edges, then the edge-set of D can be partitioned into k parts so that each part has at least one edge from every directed cut.*

Woodall's conjecture can easily be seen to be true for $k = 2$ but no answer is known even for $k = 3$ and for planar digraphs. (In which case, after planar dualization, the conjecture reads as follows: *in a simple planar digraph, the edge-set can be coloured by three colours so that every directed triangle contains each colour.*) A straightforward generalization of Woodall's conjecture concerning a crossing family of directed cuts was disproved by A. Schrijver [53] even for $k = 2$.

We call a graph $G = (V, E)$ **nearly k-tree-connected** if $G + e$ is the union of k edge-disjoint spanning trees for every possible new edge $e = uv$ ($u, v \in V$). It follows that such a graph has exactly $k(|V|-1)-1$ edges and that every subset $X \subseteq V$ with $|X| \geq 2$ induces at most $k(|X|-1)-1$ edges. A theorem of Nash-Williams [49] implies that these properties actually characterize nearly k-tree-connected graphs.

This notion for $k = 2$ (under different name) has been introduced in the theory of graph rigidity. By combining theorems of L. Henneberg [33] and of G. Laman [37], one obtains the following constructive characterization of nearly 2-tree-connected graphs.

Theorem 5.7 (Henneberg and Laman). *A graph G is nearly 2-tree-connected if and only if G can be constructed from one (non-loop) edge by the following two operations:* (i) *add a new node z and connect z to two distinct existing nodes,* (ii) *subdivide an existing edge uv by a node z and connect z to an existing node distinct from u and v.*

Jointly with László Szegő [31], we were able to extend this result for general k.

Theorem 5.8. *A graph G is nearly k-tree-connected if and only if G can be constructed from an initial graph, consisting of two nodes and $k-1$ parallel edges, by the following operation: choose a subset F of j existing edges ($0 \leq j \leq k-1$), pinch the elements of F with a new node z, and add $k-j$ new edges connecting z with other nodes so that there are no k parallel edges among these new edges.*

$(k, 1)$-tree-connectivity has meant that the graph has k disjoint spanning trees even after *deleting* any edge. What can be said about graphs which can be covered by k forests even after *adding* any new edge? We call such a graph **k-sparse.** By a theorem of Nash-Williams, we know that a graph $G = (V, E)$ is k-sparse if and only if every subset X of nodes with at least two elements induces at most $k(|X|-1)-1$ edges. Note that k-sparse graphs with $k(|V|-1)-1$ edges are exactly the nearly k-tree-connected graphs.

Theorem 5.9 [31]. *An undirected graph $G = (V, E)$ is k-sparse if and only if G can be built from a single node by applying the following operations.* (i) *add a new node z and at most k new edges ending at z so that no k parallel edges can arise,* (ii) *choose a subset F of i existing edges ($1 \leq i \leq k-1$), pinch the elements of F with a new node z, and add $k-i$ new edges connecting z with other existing nodes so that there are no k parallel edges in the resulting graph.*

6. Hypergraphs

So far our interest has been fully occupied by graphs and digraphs. In this last section we let hypergraphs take over the center stage. A hypergraph $H = (V, \mathcal{F})$ consists of a ground-set V and a family $\mathcal{F}$ of (not necessarily distinct) subsets of V, called hyperedges. The cardinality $|Z|$ of a hyperedge Z is called its **size.** We are naturally back at undirected graphs when each hyperedge is of size two. Such a hyperedge will be referred as a graph-edge. The maximum size of a hyperedge is called the **rank** of H. Throughout we will assume that the size of every hyperedge is at least two.

It is often useful to associate a bipartite graph $B = B_H = (V, U_{\mathcal{F}}; E)$ with hypergraph H as follows. The elements of $U_{\mathcal{F}}$ correspond to the hyperedges of H and a node $v \in V$ is connected to a node $u_X \in U_{\mathcal{F}}$ precisely if $u \in X$. In this correspondence the size of a hyperedge Z will be the degree of its corresponding node u_Z in B.

For a subset $X \subseteq V$ let $d_H(V)$ denote the number of hyperedges of H intersecting both X and $V - X$. For a specified subset $R \subseteq V$, a hypergraph H is called **k-edge-connected in** R if $d_H(X) \geq k$ for every subset $X \subset V$ separating R. (X is said to **separate** R if $X \cap R \neq \emptyset, R - X \neq \emptyset$.) If $R = V$, the hypergraph itself is called **k-edge-connected.** When $k = 1$ we simply say that H is connected.

From the definitions it follows that H is k-edge-connected in R if and only if the elements of R belong to one component of the graph arising from the associated bipartite graph $(V, U_{\mathcal{F}}; E)$ by deleting at most $k-1$ elements of $U_{\mathcal{F}}$. By a version of Menger's theorem, it follows that B has this property if and only if there are k paths between any pair of nodes u, v of R so that each node of $U_{\mathcal{F}}$ belongs to at most one of these paths (but the paths may share freely elements of V).

This implies that a hypergraph H is k-edge-connected in R if and only if there are k hyperedge-disjoint hyperpaths between every pair of nodes $u, v \in R$. Here a hyperpath means a sequence $\{u_1 := u, F_1, u_2, F_2, \ldots, u_t, F_t, u_{t+1} := v\}$ so that $u_i, u_{i+1} \in F_i \in \mathcal{F}$ for $i = 1, \ldots, t$.

Theorem 2.4 has been extended by J. Bang-Jensen and B. Jackson to hypergraphs [2].

Theorem 6.1 (Bang-Jensen and Jackson). *A hypergraph $H = (V, A)$ can be made k-edge-connected by adding at most γ new graph-edges if and only if $\sum(k - d_H(X) : X \in \mathcal{P}) \leq 2\gamma$ holds for every sub-partition $\mathcal{P}$ of V and $c(H') - 1 \leq \gamma$ for every hypergraph $H' = (V, A')$ arising from H by leaving out $k-1$ hyperedges where $c(H')$ denotes the number of components of H'.*

In [4] we extended this to the case when the target is k-edge-connectivity in a specified subset $R \subseteq V$.

For $q \geq 3$, T. Király [36] recently to characterized hypergraphs which can be made k-edge-connected by adding at most γ hyperedges of size at most q. The special case, when H is already $(k-1)$-edge-connected, was solved by T. Fleiner and T. Jordán [14].

Let r be again a requirement function on the set of unordered pairs of nodes, We say that H is **r**-edge-connected if there are at least $r(u, v)$ edge-disjoint hyperpaths between every pair of nodes u, v. Again by Menger's theorem, this is equivalent to requiring $d_H(X) \geq R_r(X)$ for every non-empty subset $X \subset V$.

Since local edge-connectivity augmentation is nicely tractable for undirected graphs, one may want to extend this to hypergraphs and determine the minimum number of new graph edges whose addition to H results in an **r**-edge-connected hypergraph. However, B. Cosh, B. Jackson and Z. Király [8] pointed out that this problem is NP-complete even if r is $(1-2)$-valued. For 3-uniform hypergraphs, however, the local edge-connectivity augmentation problem is tractable in the case when the newly added hyperedges are of size three or size two and for both types the number of new hyperedges are specified. This follows from Theorem 3.22 of Jordán and Szigeti and is based on the observation that intuitively says that the contribution of a hyperedge $\{a, b, c\}$ of size three to the edge-connectivity is the same as that of a star graph with three edges, that is, a graph with node set $\{z, a, b, c\}$ and edge set $\{za, zb, zc\}$.

Another interesting version of the local edge-connectivity augmentation of hypergraphs was solved nicely by Z. Szigeti [55].

Theorem 6.2 (Szigeti). *Given a requirement function r, a hypergraph H can be made **r**-edge-connected by adding hyperedges with total size at most γ if and only if $\sum_i \left(R_r(X_i) - d_H(X_i)\right) \leq \gamma$ holds for every subpartition $X_1, \ldots, X_t$ of V.*

The material below is taken from [30]. A hypergraph $H = (V, \mathcal{E})$ is called **connected** if there is a hyperedge intersecting both X and $V - X$ for every non-empty, proper subset X of V. The hypergraph is **partition-connected** if there are at least $t-1$ hyperedges intersecting at least two parts for every t-partition of V. For graphs these two notions coincide but for hypergraphs they do not (consider the hypergraph on three elements a, b, c having a single hyperedge $\{a, b, c\}$).

The connectivity of a hypergraph is equivalent to the connectivity of the bipartite graph associated with H. Therefore deciding whether a hypergraph is connected is an easy task. Testing a hypergraph for partition-connectivity is not so straightforward. To this end we call a hypergraph $H = (V, \mathcal{F})$ **wooded** if it is possible to select two elements from each hyperdege of H so that the selected pairs, as graph edges, form a forest.

Theorem 6.3 (Lovász). *A hypergraph $H = (V, \mathcal{F})$ is wooded if and only if H satisfies the strong form of the Hall condition, that is, the union of any j hyperedges $(j \geq 1)$ has at least $j + 1$ nodes.*

Proof. (outline) The necessity is staightforward. To see the sufficiency, consider the bipartite graph $B = (V, U; E)$ associated with H. Since the Hall condition is satisfied, there is a matching M of B covering the elements of U. Let S denote the set of nodes not covered by M. Orient the elements of M toward V while all other edges toward U. It follows from the strong form of the Hall condition that each node of B is reachable from S. Hence there is a spanning branching of B rooted at S and this determines the required forest. ■

Theorem 6.4 (Lorea, [38]). *Given a hypergraph $H = (V, \mathcal{E})$, the wooded subhypergraphs of H form a family of independent sets of a matroid on ground-set $\mathcal{E}$.*

Theorem 6.5 [30]. *A hypergraph $H = (V, \mathcal{E})$ is partition-connected if and only if H contains a wooded subhypergraph $(V, \mathcal{F})$ with $|V| - 1$ hyperedges.*

A hypergraph is k-**partition-connected** if there are at least $k(t-1)$ hyperedges intersecting at least two parts for every t-partition of V.

Tutte's Theorem 1.2 characterizes those graphs that can be decomposed into k edge-disjoint connected (or equivalently, partition-connected) spanning subgraphs, asserting that exactly the k-partition-connected graphs have this property. The problems of decomposing a hypergraph into k connected or into k partition-connected spanning subhypergraphs are not equivalent anymore. The first one can be shown to be NP-complete, while the second one is tractable.

Theorem 6.6 [30]. *A hypergraph $H = (V, \mathcal{F})$ can be decomposed into k partition-connected subhypergraphs if and only if H is k-partition-connected.*

The following corollary is well-known for graphs (case $q = 2$).

Corollary 6.7. *If a hypergraph H of rank at most q is (kq)-edge-connected, then H can be decomposed into k partition-connected (and thus connected) spanning subhypergraphs.*

Proof. By Theorem 6.6 it suffices to show that H is k-partition-connected. Let $\mathcal{P} = \{V_1, \dots, V_t\}$ be a partition of V. There are at least kq hyperedges intersecting both V_i and its complement for each i. Since every hyperedge is of cardinality at most q, the total number of hyperedges intersecting at least two members of $\mathcal{P}$ is at least $kqt/q = kt \geq k(t-1)$. Therefore H is indeed k-partition-connected and Theorem 6.6 applies. ■

6.1. Directed hypergraphs

There may be several choices to define directed hypergraphs, we work with the following definition. A **directed hyperedge** (Z, z) is a pair of a subset Z of the ground-set V and an element z of Z. The element z is called the **head** of Z. By a **directed hypergraph** we mean a collection of directed hyperedges. This obviously generalizes the notion of directed graphs. A disadvantage of this definition is that the symmetry between the head and the tail of a directed graph edge is lost. On the positive side of this definition is that several results concerning edge-connectivity of directed graphs can be carried over nicely to directed hypergraphs.

We say that a directed hyperedge (Z, z) **enters** a subset $X \subseteq V$ if the head z is in X but $Z - X \neq \emptyset$. A directed hypergraph is called k-**edge-connected** if there are at least k hyperedges entering each non-empty proper subset of V. More generally, for integers $0 \leq l \leq k$, a directed hypergraph is called (k, l)-**edge-connected** if there is a node $s \in V$ so that each non-empty subset $X \subseteq V - s$ is entered by at least k hyperedges and each subset $X \subset V$ containing s is entered by at least l hyperedges.

By orienting an (undirected) hypergraph we mean the operation that consists of assigning a head to every hyperedge.

Theorem 6.8 [29]. *A hypergraph has a (k, l)-edge-connected orientation if and only if there are at least $kt - k + l$ hyperedges intersecting more than one part of every t-partite partition of V.*

Finally we mention that Edmonds' Theorem 1.3 can also be carried over to hypergraphs. To this end we say that a directed hypergraph H is a spanning hyper-arborescence of root s if H has $|V| - 1$ hyperedges whose heads are distinct elements of $V - s$ and H is $(1, 0)$-edge-connected.

Theorem 6.9 [29]. *A directed hypergraph contains k disjoint spanning hyper-arborescences of root s if and only if H is $(k, 0)$-edge-connected (with respect to s).*

Note that the special case $l = 0$ of Theorem 6.8 combined with Theorem 6.9 immediately implies Theorem 6.6 (without using matroids).

The paper [5] of Berg, Jackson and Jordán contains extensions of Mader's directed splitting lemma and of the directed augmentation Theorem 2.6 to directed hypergraphs.

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