CHAPTER 14

Graph Connectivity Augmentation

András Frank

Tibor Jordán

CONTENTS

14.1 INTRODUCTION

The problem of economically improving a network to meet given survivability requirements occurs in a number of areas. A straightforward problem of this type is concerned with creating more connections in a telephone or computer network so that it survives the failure of a given number of cables or terminals [1]. Similar problems arise in graph drawing [2], statics [3], and data security [4]. It is natural to model these networks by graphs or directed graphs and use graph connectivity parameters to handle the survivability requirements. This leads to the following quite general optimization problem.

Connectivity Augmentation Problem: Let $G = (V, E)$ be a (directed) graph and let $r : V \times V \to Z_+$ be a function on the pairs of vertices of G. Find a smallest (or cheapest) set F of new edges on vertex set V such that $\lambda(u, v; G') \geq r(u, v)$ (or $\kappa(u, v; G') \geq r(u, v)$) holds for all $u, v \in V$ in $G' = (V, E \cup F)$.

Here $\lambda(u, v; H)$ ($\kappa(u, v; H)$) denotes the local edge-connectivity (local vertex-connectivity, respectively) in graph H, that is, the maximum number of pairwise edge-disjoint (vertex-disjoint) paths from u to v in H. When the goal is to find a cheapest set of new edges, it is meant with respect to a given cost function on the set of possible new edges.

The connectivity augmentation problem includes a large number of different subproblems (e.g., G may be a graph or a digraph, or the requirements may involve edge- or vertex-connectivity), which are interesting on their own, and whose solutions require different methods of combinatorial optimization. Some special cases can be solved by using well-known techniques, even in the minimum cost version. For example, if there is only one vertex pair u, v for which $r(u, v)$ is positive then, after assigning zero costs to the edges in E, we can use shortest path or minimum cost flow algorithms to find an optimal solution. If $r(s, v) = k$ for some integer k for all $v \in V$, where s is a designated vertex and the graph is directed, then minimum cost s-arborescence or minimum cost matroid intersection algorithms can be used in a similar way. However, the minimum cost versions (even in the case when each edge cost is either one or ∞) are typically NP-hard. For example, a graph $H = (V, E)$ contains a Hamilton cycle if and only if the cheapest set of edges which makes the edgeless graph on V 2-edge-connected has cost $|V|$, where the cost function is defined so that the cost of each edge of H is equal to one, and all the other edge costs are ∞.

We shall focus on those efficiently solvable variants of the problem which are NP-hard in the minimum cost setting but for which polynomial-time algorithms and minimax theorems are known when the addition of any new edge has the same unit cost. That is, we shall be interested in augmenting sets of minimum size. In most cases it will be possible to add (any number of parallel copies of) a new edge ab for all $a, b \in V$, but we shall also consider tractable versions of some constrained augmentation problems, where the set of possible new edges is restricted (e.g., where we wish to augment a bipartite graph preserving bipartiteness). A brief summary of generalizations to hypergraph augmentation problems and even more abstract versions will also be given. A detailed analysis of the corresponding efficient algorithms is not in the scope of this chapter. The reader is referred to the book of Nagamochi and Ibaraki [5] for more details on the algorithmic issues. Further related survey articles and book chapters can be found in Frank [6,7], Schrijver [8], and Szigeti [9]. NP-hard versions and approximation algorithms are discussed in Khuller [10] and Gupta and Könemann [11].

14.1.1 Notation

Let $G = (V, E)$ be a graph. We shall use $d_G(X)$ to denote the *degree* of a set X of vertices. The degree of single vertex is denoted by $d_G(v)$. For two disjoint subsets $X, Y \subseteq V$ the number of edges from X to Y in G is denoted by $d_G(X, Y)$. When we deal with a directed graph D we use $\rho_D(X)$ and $\delta_D(X)$ to denote the *in-degree* and the *out-degree* of X, respectively. We omit the subscript referrring to the (directed) graph when it is clear from the context. For simplicity the directed edges (arcs) in a directed graph will also be called edges. In this case the notation reflects the orientation of the edge, that is, a directed edge uv has a *tail* u and a *head* v.

A function f defined on the subsets of V is called *submodular* if it satisfies $f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$ for all pairs $X, Y \subset V$. We say that f is *supermodular* if $-f$ is submodular. The functions d, ρ, δ defined above are all submodular.

14.2 EDGE-CONNECTIVITY AUGMENTATION OF GRAPHS

The first papers on graph connectivity augmentation appeared in 1976, when Eswaran and Tarjan [12] and independently Plesnik [13] solved the 2-edge- (and 2-vertex-)connectivity augmentation problem. Although several graph synthesis problems (i.e., augmentation problems where the starting graph has no edges) had been solved earlier (see, e.g., Gomory and

Hu [14] and Frank and Chou [15]), these papers were the first to provide minimax theorems for arbitrary starting graphs. The size of a smallest augmenting set which makes a graph 2-edge-connected can be determined as follows.

Let $G = (V, E)$ be a graph. We say that a set $U \subseteq V$ is *extreme* if $d(X) > d(U)$ for all proper nonempty subsets X of U. For example, $\{v\}$ is extreme for all $v \in V$. It is not hard to see that the extreme sets of G form a laminar family. (A set family $\mathcal{F}$ is *laminar* if for each pair of members $X, Y \in \mathcal{F}$ we have that either $X \cap Y = \emptyset$ or one of them is a subset of the other.)

An extreme set with $d(U) \leq 1$ satisfying $d(X) \geq 2$ for all proper nonempty subsets X of U is called *2-extreme*. The subgraph G' induced by a 2-extreme set U is 2-edge-connected, for if there is a nonempty subset $X \subset U$ for which $d_{G'}(X) \leq 1$, then $2+2 \leq d_G(X)+d_G(U-X) = d_{G'}(X) + d_{G'}(U - X) + d_G(U) \leq 1 + 1 + 1$, which is not possible. Furthermore, the 2-extreme sets are pairwise disjoint, since $X \cap Y \neq \emptyset$ would imply $1 + 1 \geq d_G(X) + d_G(Y) \geq d_G(X - Y) + d_G(Y - X) \geq 2 + 2$. Let $t_0(G)$ and $t_1(G)$ denote the number of 2-extreme sets of degree 0 and 1 in G, respectively.

Theorem 14.1 [12] *The minimum number γ of new edges whose addition to a graph $G = (V, E)$ results in a 2-edge-connected graph is $t_0(G) + \lceil t_1(G)/2 \rceil$.*

Proof. Shrinking a 2-extreme subset into a single vertex does not affect the values t_0 and t_1, and the minimal γ remains unchanged, as well. Therefore, we can assume that every 2-extreme set is a singleton, and hence G is a forest. In this case, t_0 is the number of isolated vertices, while t_1 is the number of leaf vertices.

In a 2-edge-connected augmentation of G, there are at least 2 new edges incident to an isolated vertex of G, and at least 1 new edge incident to a leaf vertex of G. Therefore, the number of new edges is at least $\lceil 2t_0 + t_1/2 \rceil = t_0 + \lceil t_1/2 \rceil$.

To see that the graph can be made 2-edge-connected by adding $t_0 + \lceil t_1/2 \rceil$ new edges, it suffices to show by induction that there is a new edge e such that the addition of e to G decreases the value of $t_0 + \lceil t_1/2 \rceil$. Such an edge is said to be *reducing*.

Assume first that G is disconnected. Let u and v be two vertices of degree at most one belonging to distinct components. A simple case-checking—depending on the degrees of u and v—shows that the new edge $e = uv$ is reducing.

Therefore, we can assume that G is actually a tree (and hence $t_0 = 0$) which has at least 2 vertices (and hence $t_1 \geq 2$). When $t_1 = 2$, the tree is a path, and we obtain a 2-edge-connected graph (namely, a cycle) by adding one edge connecting the end-vertices of the path. In this case, $\gamma = 1 = t_0 + \lceil t_1/2 \rceil$.

If $t_1 = 3$, then the tree consists of three paths ending at a common vertex. Let a, b, and c denote the other end-vertices of these paths. By adding the two new edges ab and ac to the tree, we obtain a 2-edge-connected graph, and hence $\gamma = 2 = t_0 + \lceil t_1/2 \rceil$.

The remaining case is when $t_1 \geq 4$. There is a path P in the tree connecting two leaf vertices such that at least two edges leave $V(P)$. (For example, a longest path including a vertex of degree at least four or a longest path including two vertices of degree three will suffice.) By adding the new edge e between the two end-vertices of P, we obtain a graph G' in which the value of t_0 continues to be 0. Furthermore, the vertex-set of $P + e$ is not 2-extreme in G', since its degree is at least 2. Thus, the addition of e to G reduces the value of t_1 by exactly 2. Consequently, e is reducing. ■

A different solution method is to (cyclically) order the 2-extreme sets as they are reached by a DFS and then connect opposite pairs by the new edges.

Next we consider the *k-edge-connectivity augmentation problem*, where the goal is to make the input graph k-edge-connected ($k \geq 2$) by adding a smallest set of new edges. Kajitani and Ueno [16] solved this problem for every $k \geq 1$ in the special case when the starting graph is a tree.

First we extend the lower bound used in Theorem 14.1 to general graphs and higher target connectivity as follows. A *subpartition* of V is a family of pairwise disjoint nonempty subsets of V. Let $G = (V, E)$ be a graph and $k \geq 2$. Let

$$\alpha(G,k) = \max\left\{\sum_{1}^{t}(k - d(X)) : \{X_1, X_2, \ldots, X_t\} \text{ is a subpartition of } V\right\} \tag{14.1}$$

and let $\gamma(G,k)$ be the size of a smallest augmenting set of G with respect to k-edge-connectivity. Every augmenting set must contain at least $k - d(X)$ edges entering X for every $X \subset V$ and every new edge can decrease this "deficiency" of at most two sets in any subpartition. Thus $\gamma(G,k) \geq \Phi(G,k)$, where $\Phi(G,k) = \lceil \alpha(G,k)/2 \rceil$. Theorem 14.1 implies that $\gamma(G,2) = \Phi(G,2)$.

Watanabe and Nakamura [17] were the first to prove that $\gamma = \Phi$ for all $k \geq 2$ by using extreme sets and constructing an increasing sequence of augmenting sets $F_1, F_2, \ldots, F_k$ such that for all $1 \leq i \leq k$ the set F_i is an optimal augmenting set of G with respect to i. Here we present a different method which was first employed by Cai and Sun [18] (and suggested already by Plesnik [13] for $k = 2$). This method is based on edge splitting. Let $H = (V+s, E)$ be a graph with a designated vertex s. By *splitting off* a pair of edges su, sv we mean the operation that replaces su, sv by a new edge uv. The resulting graph is denoted by H_{uv}. A *complete splitting* at s is a sequence of splittings which isolates s. A complete splitting exists only if $d_H(s)$ is even. The splitting off method adds a new vertex s and some new edges incident with s to the starting graph and constructs the augmenting set by splitting off all the edges from s. Frank [19] simplified and extended this method (and established the link between generalized polymatroids and augmentation problems).

Let $G = (V, E)$ be a graph. An *extension* $G' = (V + s, E')$ of G is obtained from G by adding a new vertex s and a set of new edges incident with s. An extension G' is said to be (k,s)*-edge-connected* if $\lambda(x, y; G') \geq k$ holds for every pair $x, y \in V$. G' is *minimally* (k,s)*-edge-connected* if $G' - e$ is no longer (k,s)-edge-connected for every edge e incident with s.

The following result of Lovász [20] is a key ingredient in this approach. Let $H = (V+s, E)$ be a (k,s)-edge-connected graph. We say that splitting off two edges us, sv is k*-admissible* if H_{uv} is also (k,s)-edge-connected. A *complete k-admissible splitting at s* is a sequence of k-admissible splittings which isolates s. Observe that the graph on vertex set V, obtained from H by a complete k-admissible splitting, is k-edge-connected.

Theorem 14.2 **[20]** *Let $H = (V + s, E)$ be a (k,s)-edge-connected graph for some $k \geq 2$ and suppose that $d_H(s)$ is even. Then (a) for every edge su there exists an edge sv such that the pair su, sv is k-admissible; (b) there exists a complete k-admissible splitting at s in H.* ■

Note that (b) follows by $d_H(s)/2$ repeated applications of (a).

14.2.1 Degree-Specified Augmentations

Let $G = (V, E)$ be a graph and let $m : V \to Z_+$ be a function. We say that m is a k*-augmentation vector* if there exists a graph $H = (V, F)$ for which $G+H$ is k-edge-connected and $d_H(v) = m(v)$ for every vertex v. Here $G + H$ is the graph on vertex set V with edge set $E \cup F$. For a subset $X \subseteq V$ we put $m(X) = \sum_{v \in X} m(v)$.

Theorem 14.3 **[19]** *Let $G = (V, E)$ be a graph, $k \geq 2$ an integer, and let $m : V \to Z_+$. Then m is a k-augmentation vector if an only if $m(V)$ is even and*

$$m(X) \geq k - d_G(X) \text{ for every } \emptyset \subset X \subset V. \tag{14.2}$$

Furthermore, it suffices to require (14.2) *for the extreme subsets of V.*

Proof. If m is a k-augmentation vector then there is a graph $H = (V, F)$ for which $G + H$ is k-edge-connected and $d_H(v) = m(v)$ for every vertex v. We then have $k \leq d_{G+H}(X) = d_G(X) + d_H(X) \leq d_G(X) + \sum[d_H(v) : v \in X] = d_G(X) + m(X)$, from which (14.2) follows. Since m is a degree sequence, $m(V)$ must be even.

To see sufficiency, add a new vertex s to G and $m(v)$ parallel sv-edges for every vertex v of G. It follows from (14.2) that for all $X \subset V$ we have $d_{G'}(X) = d_G(X) + m(X) \geq k$ in the extended graph G'. Thus G' is (k, s)-edge-connected. Since $m(V)$ is even, $d_{G'}(s)$ is even. Hence we can apply Theorem 14.2 to G' and conclude that there is a complete k-admissible splitting at s resulting in a k-edge-connected graph on vertex-set V. This implies that $G+H$ is k-edge-connected, where $H = (V, F)$ is the graph whose edge set F consists of the edges arising from the edge splittings. Since $d_H(v) = m(v)$ for all $v \in V$, it follows that m is a k-augmentation vector.

The last part of the theorem follows by observing that every set $X \subseteq V$ has a nonempty extreme subset U with $d(U) \leq d(X)$. ■

We define a set X ($\emptyset \subset X \subset V$) to be *tight* with respect to a function $m : V \to Z_+$ satisfying (14.2) if $m(X) = k - d_G(X)$.

Lemma 14.1 *Let $m : V \to Z_+$ be a function satisfying* (14.2) *and let T be the subset of vertices v for which $m(v) > 0$. Suppose that m is minimal in the sense that reducing $m(v)$ for any $v \in T$ destroys* (14.2)*. Then there is a subpartition $\{X_1, \ldots, X_t\}$ of V consisting of tight extreme sets which cover T.*

Proof. The minimality of m implies that each vertex $v \in T$ belongs to a tight set. Let $T(v)$ be a minimal tight set containing v. We claim that $T(v)$ is extreme. For if not, then there is a proper subset $Z \subset T(v)$ with $d_G(Z) \leq d_G(T(v))$. Then $m(Z) \geq k - d_G(Z) \geq k - d_G(T(v)) = m(T(v)) = m(Z) + m(T(v) - Z)$, from which we obtain $m(Z) = k - d_G(Z)$ and $m(T(v) - Z) = 0$. Hence Z is tight and $v \in Z$, contradicting the minimal choice of $T(v)$.

Therefore each $v \in T$ belongs to tight extreme set. Let $\{X_1, \ldots, X_t\}$ denote the maximal tight extreme sets. These sets are disjoint and cover T since the extreme subsets of V form a laminar family. ■

We are now ready to prove the following fundamental theorem, due to Watanabe and Nakamura [17]. The proof below appeared in [19].

Theorem 14.4 **[17]** *Let $G = (V, E)$ be a graph and $k \geq 2$ an integer. Then*

$$\gamma(G, k) = \Phi(G, k). \tag{14.3}$$

Proof. We have already observed that $\gamma(G, k) \geq \Phi(G, k)$. To see that equality holds choose a function $m : V \to Z_+$ for which (14.2) holds and for which $m(V)$ is as small as possible.

Claim 14.1 $m(V) \leq \alpha(G, k)$.

Proof. By the minimality of $m(V)$ we can apply Lemma 14.1 to obtain that there is a subpartition $\{X_1, \ldots, X_t\}$ of V consisting of tight extreme sets which cover every vertex v with $m(v) > 0$. Thus $m(V) = \sum_{i=1}^{t} m(X_i) = \sum_{i=1}^{t}[k - d_G(X_i)] \leq \alpha(G, k)$, as claimed. ■

If $m(V)$ is odd, increase $m(v)$ by one for some $v \in V$ to make sure that $m(V)$ is even. Now m is a k-augmentation vector by Theorem 14.3 and hence there is a graph $H = (V, F)$ for which $G + H$ is k-edge-connected and $d_H(v) = m(v)$ for all $v \in V$. Since $|F| = m(V)/2 \leq \lceil \alpha(G, k)/2 \rceil = \Phi(G, k)$, the theorem follows. ■

Note that the statement of the theorem of Watanabe and Nakamura fails to hold for $k = 1$. This can be seen by choosing G to be the edgeless graph on four vertices. It is also worth mentioning that the proof above gives rise to a polynomial algorithm since a minimally (k, s)-edge-connected extension and a complete k-admissible splitting can be computed in polynomial time by using maximum flow algorithms. For more efficient algorithms using maximum adjacency orderings, see Nagamochi and Ibaraki [5].

Naor et al. [21] came up with yet another proof (and algorithm) for Theorem 14.4. Their algorithm increases the edge-connectivity one by one and is based on extreme sets. They use the Gomory–Hu tree of G to find the extreme sets. They also show how to employ the cactus representation of minimum edge cuts [22] to find a smallest set F which increases the edge-connectivity by one. Benczúr and Karger [23] show how the so-called extreme set tree can be used to find a minimally (k, s)-edge-connected extension G' of G and a complete k-admissible splitting in G'.

14.2.2 Variations and Extensions

Frank [19] showed that, although finding an augmenting set with minimum total cost is NP-hard, the k-edge-connectivity augmentation problem with *vertex-induced* edge costs can be solved in polynomial time. In this version we are given a cost function $c' : V \to Z_+$ on the vertices of the input graph and the cost $c(uv)$ of a new edge uv is defined to be $c'(u) + c'(v)$. Several degree constrained versions of the problem are also dealt with in [19]. One of these results is the following characterization.

Theorem 14.5 **[19]** *Let $G = (V, E)$ be a graph, $k \geq 2$ an integer, and let $f \leq g$ be two nonnegative integer-valued functions on V. Then G can be made k-edge-connected by adding a set F of new edges so that $f(v) \leq d_F(v) \leq g(v)$ holds for every $v \in V$ if and only if $k - d(X) \leq g(X)$ for every $\emptyset \neq X \subset V$ and there is no partition $\mathcal{P} = \{X_0, X_1, \ldots, X_t\}$ of V, where only X_0 may be empty, for which $f(X_0) = g(X_0)$, $g(X_i) = k - d(X_i)$ for $1 \leq i \leq t$, and $g(V)$ is odd.* ■

14.3 LOCAL EDGE-CONNECTIVITY AUGMENTATION OF GRAPHS

A function $r : V \times V \to Z_+$ is called a *local requirement function* on V. We shall only consider symmetric functions, that is we shall assume that $r(u, v) = r(v, u)$ for all $u, v \in V$ when we deal with undirected graphs. Given a local requirement function r, we say that a graph H on vertex set V is *r-edge-connected* if $\lambda(x, y; H) \geq r(x, y)$ for all $x, y \in V$. In the *local edge-connectivity augmentation problem* the goal is to find a smallest set F of new edges whose addition makes the input graph $G = (V, E)$ r-edge-connected.

The local edge-connectivity augmentation problem can also be solved by using the edge splitting method. In this version we need a stronger splitting result (Theorem 14.6 below, due to Mader), and a modified lower bound counting deficiencies of subpartitions.

Let $G = (V, E)$ be a graph and let r be a fixed local requirement function on V. We define a function R on the subsets of V as follows: we put $R(\emptyset) = R(V) = 0$ and let

$$R(X) = \max\{r(x, y) : x \in X, y \in V - X\} \text{ for all } \emptyset \neq X \subset V. \tag{14.4}$$

It is not difficult to check that R is *skew supermodular*, that is, for all $X, Y \subseteq V$ we have $R(X) + R(Y) \leq R(X \cap Y) + R(X \cup Y)$ or $R(X) + R(Y) \leq R(X - Y) + R(X - Y)$ (or both).

By Menger's theorem an augmented graph G' of G is r-edge-connected if and only if $d_{G'}(X) \geq R(X)$ for every $X \subseteq V$. Let $q(X) = R(X) - d_G(X)$ for $X \subseteq V$ and let

$$\alpha(G,r) = \max\left\{\sum_{i=1}^{t} q(X_i) : \{X_1, X_2, \ldots, X_t\} \text{ is a subpartition of } V\right\}. \quad (14.5)$$

An argument analogous to that of the uniform case shows that $\gamma(G,r) \geq \Phi(G,r)$, where $\gamma(G,r)$ is the size of a smallest augmenting set and $\Phi(G,r) = \lceil \alpha(G,r)/2 \rceil$. Theorem 14.4 claims that this lower bound is achievable if $r \equiv k \geq 2$. In the local version this does not necessarily hold. For example, consider a graph with four vertices and no edges and let $r \equiv 1$. On the other hand, if

$$r(u,v) \geq 2 \quad (14.6)$$

for all $u, v \in V$ then, as we shall see, we do have the equality $\gamma = \Phi$. In the rest of this section we shall assume that (14.6) holds. In general G may contain some *marginal components* with respect to r which need to be taken care of before one may assume (14.6). This reduction is relatively easy but quite technical. Therefore we refer the reader to [19] for the details.

Let r be a local requirement function on V, let $G = (V, E)$ be a graph and let $G' = (V + s, E')$ be an extension of G. We say G' is (r,s)*-edge-connected* if $\lambda(x, y; G') \geq r(x, y)$ for every $x, y \in V$. Splitting off us, sv is r*-admissible* in G' if $\lambda(x, y; G'_{uv}) \geq r(x, y)$ for all $x, y \in V$.

Let $r_\lambda(x, y) = \lambda(x, y; G')$ be a special requirement function defined on pairs $x, y \in V$. Mader's [24] deep result, which extends Theorem 14.2 to local edge-connectivities, is as follows.

Theorem 14.6 **[24]** *Let $G' = (V + s, E')$ be a graph. Suppose that $d(s)$ is even and there is no cut-edge incident with s. Then there is a complete r_λ-admissible splitting at s.* ■

14.3.1 Degree-Specified Augmentations

The next result is the local version of Theorem 14.3.

Theorem 14.7 **[19]** *Let $G = (V, E)$ be a graph, $m : V \to Z_+$ with $m(V)$ even, and let r be a local requirement function satisfying* (14.6). *There is a graph $H = (V, F)$ for which*

$$d_H(v) = m(v) \text{ for all } v \in V \quad (14.7)$$

and

$$\lambda_{G'}(x, y) \geq r(x, y) \text{ for all } x, y \in V, \quad (14.8)$$

where $G' = G + H$, if and only if

$$m(X) \geq R(X) - d_G(X) \text{ for every } X \subseteq V. \quad (14.9)$$

Proof. If there is a graph H for which G' satisfies (14.8), then $d_G(X) + d_H(X) = d_{G'}(X) \geq R(X)$, from which $m(X) \geq d_H(X) \geq R(X) - d_G(X)$, and hence (14.9) holds.

To prove sufficiency, add a new vertex s to G and $m(v)$ parallel sv-edges for every vertex $v \in V$. In the resulting graph G', $\lambda_{G'}(x, y) \geq r(x, y)$ holds for every pair $x, y \in V$ of vertices due to (14.9). Observe that there is no cut-edge of G' incident to s by (14.6). Therefore we can apply Theorem 14.6, which asserts that there is a complete splitting at s that preserves the local edge-connectivities in V. This means that $H = (V, F)$ satisfies the requirements of the theorem where F denotes the set of edges arising from the splittings. ■

As in the global case, we can use the degree-specified version to deduce a min-max result on the size of a smallest augmenting set.

Theorem 14.8 [19] *Let $G = (V, E)$ be a graph and let r be a local requirement function on V satisfying* (14.6). *Then $\gamma(G, r) = \Phi(G, r)$.*

Proof. We have already observed that $\gamma(G, r) \geq \Phi(G, r)$. To prove that equality holds first recall that R is a skew supermodular function. Hence q is also skew supermodular, that is, for each pair of sets $X, Y \subset V$, at least one of the following two inequalities holds:

$$q(X) + q(Y) \leq q(X \cap Y) + q(X \cup Y), \tag{14.10}$$

$$q(X) + q(Y) \leq q(X - Y) + q(Y - X). \tag{14.11}$$

Let $m : V \rightarrow Z_+$ be chosen in such a way that (14.9) is satisfied and $m(V)$ is minimal in the sense that reducing any positive $m(v)$ by one destroys (14.9). ■

Claim 14.2 $m(V) \leq \alpha(G, r)$.

Proof. By the minimality of $m(V)$ every vertex v with $m(v) > 0$ belongs to a tight set where a set X is *tight* if $m(X) = q(X)$. Let $\mathcal{F} = \{X_1, \ldots, X_t\}$ be a system of tight sets which covers each vertex v with $m(v) > 0$, in which $|\mathcal{F}|$ is minimal, and with respect to this, in which $\sum[|Z| : Z \in \mathcal{F}]$ is minimal.

Suppose that $\mathcal{F}$ contains two intersecting members X and Y. If (14.10) holds, then $X \cup Y$ is tight, in which case X and Y could be replaced by $X \cup Y$, contradicting the minimality of $|\mathcal{F}|$. Therefore (14.11) must hold, which implies

$$\begin{aligned} m(X) + m(Y) &= q(X) + q(Y) \leq q(X - Y) + q(Y - X) \leq m(X - Y) + m(Y - X) \\ &= m(X) + m(Y) - 2m(X \cap Y) \end{aligned}$$

from which we can conclude that both $X - Y$ and $Y - X$ are tight and $m(X \cap Y) = 0$. That is, in $\mathcal{F}$ we could replace X and Y by $X - Y$ and $Y - X$, contradicting the minimality of $\sum[|Z| : Z \in \mathcal{F}]$.

Therefore $\mathcal{F}$ must be a subpartition. Then

$$m(V) = \sum_1^t m(X_i) = \sum_1^t q(X_i) \leq \alpha(G, r),$$

as claimed. ■

If $m(V)$ is odd, increase $m(v)$ by one for some $v \in V$ to make sure that $m(V)$ is even. Now Theorem 14.7 applies and hence there is a graph $H = (V, F)$ for which $G + H$ is r-edge-connected and $d_H(v) = m(v)$ for all $v \in V$. Since $|F| = m(V)/2 \leq \lceil \alpha(G, r)/2 \rceil = \Phi(G, r)$, the theorem follows. ■

One may also consider the fractional version of Theorem 14.8, in which edges with fractional capacities may be added and the goal is to find an augmenting set with minimum total capacity. It can be shown that the minimum total capacity is equal to $\alpha(G, r)/2$. Moreover, the fractional optimum can be chosen to be half-integral.

We say that an increasing sequence of local requirements $(r_1, r_2, \ldots, r_t)$ on V has the *successive augmentation property* if, for any starting graph $G = (V, E)$, there exists an increasing sequence $F_1 \subseteq F_2 \subseteq \ldots \subseteq F_t$ of sets of edges such that $G + F_i$ is an optimal augmentation of G with respect to r_i, for all $1 \leq i \leq t$. The proof of Theorem 14.4 by Watanabe and Nakamura [17] (and also Naor et al. [21]) implies that any increasing sequence of uniform requirements has the successive augmentation property in the edge-connectivity augmentation problem.

By using an entirely different approach, Cheng and Jordán [25] generalized this to sequences with the following property:

$$r_{i+1}(u,v) - 1 = r_i(u,v) \geq 2, \text{ for all } u,v \in V \text{ and}, 1 \leq i \leq t-1. \tag{14.12}$$

The proof is based on the fact that if $G' = (V+s, E')$ ($G'' = (V+s, E'')$) is a minimally r_{i-1}-edge-connected (minimally r_i-edge-connected, respectively) extension of G such that G' is a subgraph of G'', then any r_{i-1}-admissible splitting su, sv in G' is r_i-admissible in G''.

Theorem 14.9 **[25]** *Every increasing sequence $(r_1, r_2, \ldots, r_t)$ of local requirements satisfying* (14.12) *has the successive augmentation property in the edge-connectivity augmentation problem.* ■

A *mixed graph* $D = (V, E \cup A)$ has edges as well as directed edges. Bang-Jensen et al. [26] extended Theorem 14.6 to mixed graphs and with the splitting off method, they generalized Theorem 14.8 to the case when the edge-connectivity of a mixed graph is to be increased by adding undirected edges only. See also [9] for a list of theorems of this type.

14.3.2 Node-to-Area Augmentation Problem

Let $G = (V, E)$ be a graph and let $\mathcal{W}$ be a family of subsets of V, called *areas*. Let $r : \mathcal{W} \to Z_+$ be a requirement function assigning a nonnegative integer to each area. The *node-to-area augmentation problem* is to find a smallest set F of new edges for which $G + F$ contains at least $r(W)$ edge-disjoint paths between v and W for all $v \in V$ and $W \in \mathcal{W}$. It generalizes the k-edge-connectivity augmentation problem (make each vertex v a one-element area W_v with requirement $r(W_v) = k$).

This problem was shown to be NP-hard when $r(W) = 1$ for all $W \in \mathcal{W}$ [27], but it turned out to be polynomially solvable in the uniform case when $r(W) = r \geq 2$ for all $W \in \mathcal{W}$ [28]. The most general result proved so far is due to Ishii and Hagiwara [29] who showed that the problem can be solved even if r is not uniform but satisfies $r(W) \geq 2$ for each $W \in \mathcal{W}$.

For a proper subset X of V let

$$p(X) = \max\{r(W) : W \in \mathcal{W}, X \cap W = \emptyset \text{ or } W \subseteq X\}.$$

It is easy to see, by using Menger's theorem, that an augmented graph $G' = G + F$ is a feasible solution if and only if $d_{G'}(X) \geq p(X)$ for all proper subsets X of V. Let $\alpha(G, \mathcal{W}, r) = \max\{\sum_1^t (p(X_i) - d(X_i)) : \{X_1, \ldots, X_t\}$ is a subpartition of $V\}$ and let $\Phi(G, \mathcal{W}, r) = \lceil \alpha(G, \mathcal{W}, r)/2 \rceil$. Then we have $\gamma(G, \mathcal{W}, r) \geq \Phi(G, \mathcal{W}, r)$. This inequality may be strict but the gap can be at most one, and the instances with strict inequality (call them *exceptional configurations*) can be completely characterized [29].

Theorem 14.10 *Let $G = (V, E)$ be a graph, let $\mathcal{W}$ be a family of subsets of V, and let $r : \mathcal{W} \to Z_+$ be a requirement function satisfying $r(W) \geq 2$ for all $W \in \mathcal{W}$. Then $\gamma(G, \mathcal{W}, r) = \Phi(G, \mathcal{W}, r)$, unless $G, \mathcal{W}$, and r form an exceptional configuration, in which case $\gamma(G, \mathcal{W}, r) = \Phi(G, \mathcal{W}, r) + 1$.* ■

A shorter proof of Theorem 14.10 was given later by Grappe and Szigeti [30]. The case when the edge-connectivity requirement is given separately for each area-vertex pair remains open.

14.4 EDGE-CONNECTIVITY AUGMENTATION OF DIGRAPHS

Let $\gamma(D, k)$ denote the size of a smallest set F of new (directed) edges which makes a given directed graph D k-edge-connected. The first result on digraph augmentation is due to Eswaran and Tarjan [12], who solved the strong connectivity augmentation problem and gave the following minimax formula for $\gamma(D, 1)$.

Let $D = (V, A)$ be a digraph and let D_c be obtained from D by contracting its strong components. D_c is acyclic and it is easy to see that $\gamma(D, 1) = \gamma(D_c, 1)$ holds. Thus we may focus on acyclic input graphs. A vertex v with $\rho(v) = 0$ ($\delta(v) = 0$) is a *source* (*sink*, respectively). For a set $X \subset V$ let X^+ and X^- denote the set of sources in X and the set of sinks in X, respectively. It is clear that to make D strongly connected we need at least $|V^+|$ ($|V^-|$) new arcs.

Eswaran and Tarjan [12] proved the following min-max theorem.

Theorem 14.11 **[12]** *Let* $D_c = (V, E)$ *be an acyclic digraph. Then* $\gamma(D_c, 1) = \max\{|V^+|, |V^-|\}$.

Proof. Let $k = |V^+|$, $l = |V^-|$, and let $m = \max\{k, l\}$. We may suppose that $m = l \geq k$. We need to show that D_c can be made strongly connected by adding l new edges. We can assume that there are no isolated vertices in D_c, since an isolated vertex v can be replaced by a single edge vv', where v' is a new vertex, without changing γ, the number of sources, or the number of sinks.

The proof is by induction on m. First suppose that there is a sink-vertex t which is not reachable from some source vertex s. Then by adding the new edge ts to D we obtain a digraph that is still acyclic. Moreover, m is decreased by one. Thus the theorem follows by induction.

Next suppose that each sink vertex is reachable from each source vertex. Let $s_1, \ldots, s_k$ denote the source vertices and $t_1, \ldots, t_\ell$ the sink vertices. By adding the edges $t_1 s_1, \ldots, t_k s_k$ along with the edges $t_{k+1} s_1, t_{k+2} s_1, \ldots, t_\ell s_1$ (altogether ℓ new edges), we obtain a strongly connected augmentation of D_c. This completes the proof. ■

It follows that for arbitrary starting digraphs $\gamma(D, 1)$ equals the maximum number of pairwise disjoint sets $X_1, \ldots, X_t$ in D with $\rho(X_i) = 0$ for all $1 \leq i \leq t$ (or $\delta(X_i) = 0$ for all $1 \leq i \leq t$).

We note that there is another version of the strong connectivity augmentation problem which is nicely tractable. In this version only those edges uv are allowed to be added for which u is reachable from v in the initial digraph. The solution for this problem is based on a theorem of Lucchesi and Younger, see for example [7] and [31].

Kajitani and Ueno [16] solved the k-edge-connectivity augmentation problem for digraphs in the special case when D is a directed tree (but $k \geq 1$ may be arbitrary). The solution for arbitrary starting digraphs is due to Frank [19], who adapted the splitting off method to directed graphs and showed that, as in the undirected case, $\gamma(D, k)$ can be characterized by a subpartition-type lower bound. For a digraph $D = (V, A)$ let

$$\alpha_{in}(D, k) = \max\left\{\sum_1^t (k - \rho(X_i)) : \{X_1, \ldots, X_t\} \text{ is a subpartition of } V\right\},$$

$$\alpha_{out}(D, k) = \max\left\{\sum_1^t (k - \delta(X_i)) : \{X_1, \ldots, X_t\} \text{ is a subpartition of } V\right\}.$$

It is again easy to see that $\gamma(D, k) \geq \Phi(D, k)$, where $\Phi(D, k) = \max\{\alpha_{in}(D), \alpha_{out}(D)\}$. An *extension* $D' = (V + s, A')$ of a digraph $D = (V, A)$ is obtained from D by adding a new vertex s and a set of new edges, such that each new edge leaves or enters s. A digraph $H = (V + s, A)$ with a designated vertex s is (k, s)-*edge-connected* if $\lambda(x, y; H) \geq k$ for every $x, y \in V$. Splitting off two edges us, sv means replacing the edges us, sv by a new edge uv. Splitting off two edges us, sv is k-*admissible* in a (k, s)-edge-connected digraph H if H_{uv} is also (k, s)-edge-connected.

The next theorem, due to Mader [32], is the directed counterpart of Theorem 14.2.

Theorem 14.12 [**32**] *Let $D = (V+s, A)$ be a (k, s)-edge-connected digraph with $\rho(s) = \delta(s)$. Then (a) for every edge us there exists an edge sv such that the pair us, sv is k-admissible, (b) there is a complete k-admissible splitting at s.* ■

14.4.1 Degree-Specified Augmentations

With the help of Theorem 14.12 an optimal solution for the directed edge-connectivity augmentation problem can be obtained following the steps of the solution in the undirected case. Here we only state the key results and refer to [19] for the proofs.

Theorem 14.13 [**19**] *Let $D = (V, A)$ be a digraph and let $m_{in} : V \to Z_+$ and $m_{out} : V \to Z_+$ be in- and out-degree specifications. There is a digraph $H = (V, F)$ for which*

$$\varrho_H(v) = m_{in}(v) \text{ and } \delta_H(v) = m_{out}(v) \text{ for every } v \in V \tag{14.13}$$

and $D + H$ is k-edge-connected if and only if $m_{in}(V) = m_{out}(V)$,

$$m_{in}(X) \geq k - \varrho_D(X) \quad \text{for } \emptyset \neq X \subset V \tag{14.14}$$

and

$$m_{out}(X) \geq k - \delta_D(X) \quad \text{for } \emptyset \neq X \subset V. \tag{14.15}$$

The directed counterpart of the theorem of Watanabe and Nakamura is as follows. ■

Theorem 14.14 [**19**] *Let $D = (V, A)$ be a directed graph and let $k \geq 1$. Then $\gamma(D, k) = \Phi(D, k)$.* ■

Frank [19] also showed that the minimum cost version with vertex-induced cost functions is also solvable in polynomial time. Cheng and Jordán [25] proved that the successive augmentation property holds for any increasing sequence of uniform requirements in the directed edge-connectivity augmentation problem as well.

The local edge-connectivity augmentation problem in directed graphs is NP-hard, even if $r(u, v) \in \{0, 1\}$ for all $u, v \in V$ [19]. Bang-Jensen et al. [26] generalized Theorem 14.14 to mixed graphs and special classes of local requirements. For instance, they showed that the local version is solvable for Eulerian digraphs (i.e., for digraphs where $\rho(v) = \delta(v)$ for all $v \in V$). The proofs of these results rely on an edge splitting theorem, which is a common extension of Theorem 14.12 and a result of Frank [33] and Jackson [34] on splitting off edges in Eulerian digraphs preserving local edge-connectivities. A different version of the mixed graph augmentation problem was investigated by Gusfield [35].

14.5 CONSTRAINED EDGE-CONNECTIVITY AUGMENTATION PROBLEMS

In each of the augmentation problems considered so far it was allowed to add (an arbitrary number of parallel copies of) any edge connecting two vertices of the input graph. It is natural to consider (and in some cases the applications give rise to) variants where the set of new edges must meet certain additional constraints. In general such constraints may lead to hard problems. For example, Frederickson and Jaja [36] proved that, given a tree $T = (V, E)$ and a set J of edges on V, it is NP-hard to find a smallest set $F \subseteq J$ for which $T' = (V, E \cup F)$ is 2-edge-connected. This problem remains NP-hard even if J is the edge-set of a cycle on the leaves of T [37]. For some types of constraints, however, an optimal solution can be found in polynomial time. In this section we consider these tractable problems.

Motivated by a question in statics, Bang-Jensen et al. [3] solved the following partition-constrained problem. Let $G = (V, E)$ be a graph and let $\mathcal{P} = \{P_1, P_2, \ldots, P_r\}$, $r \geq 2$, be a partition of V. In the *partition-constrained k-edge-connectivity augmentation problem* the goal is to find a smallest set F of new edges, such that every edge in F joins two distinct members of $\mathcal{P}$ and $G' = (V, E \cup F)$ is k-edge-connected. If G is a bipartite graph with bipartition $V = A \cup B$ and $\mathcal{P} = \{A, B\}$ then the problem corresponds to optimally augmenting a bipartite graph while preserving bipartiteness. By a theorem of Bolker and Crapo [38] the solution of this bipartite version can be used to make a square grid framework highly redundantly rigid by adding a smallest set of new diagonal rods.

Let $\gamma(G, k, \mathcal{P})$ denote the size of a smallest augmenting set with respect to k and the given partition $\mathcal{P}$. Clearly, $\gamma(G, k, \mathcal{P}) \geq \gamma(G, k)$. The case $k = 1$ is easy, hence we assume $k \geq 2$. For $i = 1, 2, \ldots, r$ let

$$\beta_i = \max\left\{\sum_{Y \in \mathcal{Y}} (k - d(Y)) : \mathcal{Y} \text{ is a subpartition of } P_i\right\}. \tag{14.16}$$

Since no new edge can join vertices in the same member P_i of $\mathcal{P}$, it follows that β_i is a lower bound for $\gamma(G, k, \mathcal{P})$ for all $1 \leq i \leq r$. By combining this bound and the lower bound of the unconstrained problem we obtain $\gamma(G, k, \mathcal{P}) \geq \Phi(G, k, \mathcal{P})$, where

$$\Phi(G, k, \mathcal{P}) = \max\{\lceil \alpha(G, k)/2 \rceil, \beta_1, \ldots, \beta_r\}. \tag{14.17}$$

Simple examples show that $\gamma \geq \Phi + 1$ may hold. Consider a four-cycle C_4 and let $\mathcal{P}$ be the natural bipartition of C_4. Here we have $\Phi(C_4, 3, \mathcal{P}) = 2$ and $\gamma(C_4, 3, \mathcal{P}) = 3$. Now consider a six-cycle C_6 and let $\mathcal{P} = \{P_1, P_2, P_3\}$, where the members of $\mathcal{P}$ contain pairs of opposite vertices. For this graph and partition we have $\Phi(C_6, 3, \mathcal{P}) = 3$ and $\gamma(C_6, 3, \mathcal{P}) = 4$.

On the other hand, Bang-Jensen et al. [3] proved that we always have $\gamma \leq \Phi + 1$ and characterized all graphs (and partitions) with $\gamma = \Phi + 1$. The proof employed the splitting off method. The first step was a complete solution of the corresponding constrained edge splitting problem. Let $H = (V + s, E)$ be a (k, s)-edge-connected graph and let $\mathcal{P} = \{P_1, P_2, \ldots, P_r\}$ be a partition of V. We say a splitting su, sv is *allowed* if it is k-admissible and respects the partition constraints, that is, u and v belong to distinct members of $\mathcal{P}$. If k is even, the following extension of Theorem 14.2(b) is not hard to prove.

Theorem 14.15 [3] *Let $H = (V + s, E)$ be a (k, s)-edge-connected graph, for some even integer k, let $\mathcal{P} = \{P_1, P_2, \ldots, P_r\}$ be a partition of V, and suppose that $d(s)$ is even. There exists a complete allowed splitting at s if and only if $d(s, P_i) \leq d(s)/2$ for all $1 \leq i \leq r$.*

For k odd, however, there exist more complicated "obstacles" that prevent a complete allowed splitting at s. Let S denote the set of neighbors of s.

A partition $A_1 \cup A_2 \cup B_1 \cup B_2$ of V is called a *C_4-obstacle* if it satisfies the following properties in H for some index i, $1 \leq i \leq r$:

i. $d(A_1) = d(A_2) = d(B_1) = d(B_2) = k$;

ii. $d(A_1, A_2) = d(B_1, B_2) = 0$;

iii. $S \cap (A_1 \cup A_2) = S \cap P_i$ or $S \cap (B_1 \cup B_2) = S \cap P_i$;

iv. $d(s, P_i) = d(s)/2$. ■

C_4-obstacles exist only for k odd. It is not difficult to see that if H contains a C_4-obstacle, then there exists no complete allowed splitting at s. A more special family of obstacles, called *C_6-obstacles*, can be defined when $r \geq 3$, k is odd, and $d(s) = 6$, see [3]. These two families suffice to characterize when there is no complete allowed splitting. Note that in the bipartition constrained case only C_4-obstacles may exist.

Theorem 14.16 [3] *Let $H = (V + s, E)$ be a (k, s)-edge-connected graph with $d(s)$ even and let $\mathcal{P} = \{P_1, P_2, \ldots, P_r\}$ be a partition of V. There exists a complete allowed splitting at vertex s in G if and only if*

a. $d(s, P_i) \leq d(s)/2$ for $1 \leq i \leq r$,

b. H contains no C_4- or C_6-obstacle. ■

Bang-Jensen et al. [3] show that there exists a (k, s)-edge-connected extension $G' = (V+s, E')$ of $G = (V, E)$ with $d(s) = 2\Phi(G, k, \mathcal{P})$ for which Theorem 14.16(a) holds. If G' satisfies Theorem 14.16(b), as well, a complete allowed splitting at s yields an optimal augmenting set (of size $\Phi(G, k, \mathcal{P})$). Since $\gamma \leq \Phi + 1$, it remains to characterize the exceptions, that is, those starting graphs G (and partitions) for which any extension G' with $d(s) = 2\Phi(G, k, \mathcal{P})$ contains an obstacle (and hence $\gamma(G, k, \mathcal{P}) = \Phi(G, k, \mathcal{P}) + 1$ holds).

Let $G = (V, E)$ be a graph. A partition X_1, X_2, Y_1, Y_2 of V is a C_4-*configuration* if it satisfies the following properties in G:

i. $d(A) < k$ for $A = X_1, X_2, Y_1, Y_2$;

ii. $d(X_1, X_2) = d(Y_1, Y_2) = 0$;

iii. There exist subpartitions $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}'_1, \mathcal{F}'_2$ of X_1, X_2, Y_1, Y_2 respectively, such that for A ranging over X_1, X_2, Y_1, Y_2 and $\mathcal{F}$ the corresponding subpartition of A, $k - d(A) = \sum_{U\in\mathcal{F}}(k - d(U))$. Furthermore for some $i \leq r$, P_i contains every set of either $\mathcal{F}_1 \cup \mathcal{F}_2$ or $\mathcal{F}'_1 \cup \mathcal{F}'_2$.

iv. $(k - d(X_1)) + (k - d(X_2)) = (k - d(Y_1)) + (k - d(Y_2)) = \Phi(G, k, \mathcal{P})$.

As with C_4-obstacles, k must be odd in a C_4-configuration. A C_6-*configuration* is more specialized, since it only exists in graphs with $r \geq 3$ and $\Phi = 3$, see [3].

Theorem 14.17 [3] *Let $k \geq 2$ and let $G = (V, E)$ be a graph with a partition $\mathcal{P} = \{P_1, \ldots, P_r\}$, $r \geq 2$ of V. Then $\gamma(G, k, \mathcal{P}) = \Phi(G, k, \mathcal{P})$ unless G contains a C_4- or C_6-configuration, in which case $\gamma(G, k, \mathcal{P}) = \Phi(G, k, \mathcal{P}) + 1$.* ■

If each member of $\mathcal{P}$ is a single vertex then we are back at Theorem 14.4. The following special case solves the rigidity problem mentioned above. Let $G = (V, E)$ be a bipartite graph with bipartition $V = A \cup B$, let $\mathcal{P} = \{A, B\}$, and let

$$\beta'_1 = \sum_{v\in A} \max\{0, k - d(v)\},$$

$$\beta'_2 = \sum_{v\in B} \max\{0, k - d(v)\},$$

$$\Theta(G, k, \mathcal{P}) = \max\{\lceil \alpha(G, k, \mathcal{P})/2\rceil, \beta'_1, \beta'_2\}.$$

Theorem 14.18 [3] *Let $G = (V, E)$ be a bipartite graph with bipartition $V = A \cup B$ and let $\mathcal{P} = \{A, B\}$. Then $\gamma(G, k, \mathcal{P}) = \Theta(G, k, \mathcal{P})$ unless k is odd and G contains a C_4-configuration, in which case $\gamma(G, k, \mathcal{P}) = \Theta(G, k, \mathcal{P}) + 1$.* ■

The variant of the above problem, where the edges of the augmenting set F must lie within members of a given partition, is NP-hard [3]. The status of this variant is open if the number r of partition members is fixed, even if $r = 2$. The corresponding edge splitting problem, for $r = 2$, has been solved in [39].

A different application of Theorem 14.16 is concerned with permutation graphs. A *permutation graph* G^π of a graph G is obtained by taking two disjoint copies of G and adding a matching joining each vertex v in the first copy to $\pi(v)$ in the second copy, where π is a permutation of $V(G)$. Thus G has several permutation graphs. The edge-connectivity of any permutation graph of G is at most $\delta(G)+1$, where $\delta(G)$ is the minimum degree of G. When does G have a k-edge-connected permutation graph for $k = \delta(G) + 1$?

Creating a permutation graph of G corresponds to performing a complete bipartition constrained splitting in G', where G' is obtained from $2G$ by adding a new vertex s and precisely one edge from s to each vertex of $2G$. (For some graph H we use $2H$ to denote the graph consisting of two disjoint copies of H.) If G is simple, it can be seen that G' is (k, s)-edge-connected. Thus Theorem 14.16 leads to the following characterization, due to Goddard et al. [40]. An extension to hypergraphs was given later by Jami and Szigeti [41].

Theorem 14.19 **[40]** *Let G be a simple graph without isolated vertices and let $k = \delta(G)+1$. Then there is a k-edge-connected permutation graph of G unless $G = 2K_k$, and k is odd.* ■

The partition constrained k-edge-connectivity augmentation problem has been investigated for digraphs as well. Although the general case of this problem is still open, Gabow and Jordán presented polynomial algorithms for several special cases, including the partition constrained strong connectivity augmentation problem [42–44].

It is also natrual to consider the *planarity-preserving k-edge-connectivity augmentation problem*. In this problem we are given a planar graph $G = (V, E)$ and the goal is to find a smallest set F of new edges for which $G' = (V, E \cup F)$ is k-edge-connected and planar. The complexity of this problem is still open, even for $k = 2$. (The corresponding problem for 2-vertex-connectivity is NP-hard [2].)

A typical positive result, due to Nagamochi and Eades [45], is as follows. For $k = 2$ it was proved earlier by Kant [2]. For a simpler proof see [46].

Theorem 14.20 **[45]** *Let $G = (V, E)$ be outer-planar and let k be even or $k = 3$. Then G can be made k-edge-connected and planar by adding $\Phi(G, k)$ edges.* ■

Another natural constrained augmentation problem, which has been investigated by several authors, is the *simplicity preserving k-edge-connectivity augmentation problem*: given a simple graph $G = (V, E)$, find a smallest set F of new edges for which $G' = (V, E \cup F)$ is k-edge-connected and simple. Frank and Chou [15] solved this problem (even with local requirements) in the special case where the starting graph G has no edges. Some papers on arbitrary starting graphs G but with small target value k followed. Let us denote the size of a smallest simplicity-preserving augmenting set F by $\gamma(G, k, S)$. Clearly, we have $\gamma(G, k, S) \geq \gamma(G, k)$. Following the algorithmic proof of Theorem 14.1, it can be checked that if G is simple, so is the augmented graph G'. This proves $\gamma(G, 2, S) = \gamma(G, 2)$. Watanabe and Yamakado [47] proved that $\gamma(G, k, S) = \gamma(G, k)$ holds for $k = 3$ as well. Taoka et al. [48] pointed out that $\gamma(G, k, S) \geq \gamma(G, k) + 1$ may hold if $k \geq 4$, even if the starting graph G is $(k-1)$-edge-connected. On the other hand, they showed that for $(k-1)$-edge-connected starting graphs one has $\gamma(G, k, S) \leq \gamma(G, k) + 1$ for $k = 4, 5$. Moreover, in these special cases, we have $\gamma(G, k, S) = \gamma(G, k)$, provided $\gamma(G, k) \geq 4$. For general k, it was observed [21] that $\gamma(G, k, S) = \gamma(G, k)$ if G is $(k-1)$-edge-connected and the minimum degree of G is at least k.

Jordán [49] settled the complexity of the problem by proving that the simplicity-preserving k-edge-connectivity augmentation problem is NP-hard, even if the starting graph is $(k-1)$-edge-connected. For k fixed, however, the problem is solvable in polynomial time. This result of Bang-Jensen and Jordán [50] is based on the fact that if $\gamma(G, k)$ is large compared to k then $\gamma(G, k, S) = \gamma(G, k)$ holds.

Theorem 14.21 **[50]** *Let $G = (V, E)$ be a simple graph. If $\gamma(G, k) \geq 3k^4/2$ then $\gamma(G, k, S) = \gamma(G, k)$.* ■

The algorithmic proof of Theorem 14.21 employed the splitting off method and showed that if $\gamma(G, k)$ is large then an optimal simplicity-preserving augmentation can be found in polynomial time, even if k is part of the input. It is also proved in [50] that for any graph G we have $\gamma(G, k, S) \leq \gamma(G, k) + 2k^2$. Using this fact and some additional structural properties lead to an $O(n^4)$ algorithm for k fixed. Most of these results have been extended to the local version of the simplicity-preserving edge-connectivity augmentation problem [50].

In the *reinforcement problem*, which is the opposite of the simplicity preserving problem in some sense, we are given a connected graph $G = (V, E)$ and an integer $k \geq 2$, and the goal is to find a smallest set F of new edges for which $G' = (V, E \cup F)$ is k-edge-connected and every edge of F is parallel to some edge in E. This problem is also NP-hard [49].

We close this section by mentioning a constrained problem of a different kind. In the *simultaneous edge-connectivity augmentation problem* we are given two graphs $G_1 = (V, E)$ and $G_2 = (V, I)$, and two integers $k, l \geq 2$ and the goal is to find a smallest set F of new edges for which $G_1' = (V, E \cup F)$ is k-edge-connected and $G_2' = (V, I \cup F)$ is l-edge-connected. For this problem Jordán [46] proved that the difference between a subpartition type lower bound and the optimum is at most one. Furthermore, if k and l are both even then we have equality. The status of the simultaneous augmentation problem is still open in the case when k or l is odd. Ishii and Nagamochi [51] solved a similar simultaneous augmentation problem where the goal is to make G_1 and G_2 k-edge-connected and 2-vertex-connected, respectively.

14.6 VERTEX-CONNECTIVITY OF GRAPHS

The vertex-connected versions of the augmentation problems are substantially more difficult than their edge-connected counterparts. This will be transparent by comparing the corresponding minimax theorems, the proof methods, as well as the hardness results and open questions. The following observation indicates that the k-vertex-connectivity augmentation problem, at least in the undirected case, has a different character. Suppose the goal is to make $G = (V, E)$ k-connected, optimally, where $k = |V| - 2$. Although this case may seem very special, it is in fact equivalent to the maximum matching problem. To see this observe that F is a feasible augmenting set if and only if the complement of $G + F$ consists of independent edges. Thus finding a smallest augmenting set for G corresponds to finding a maximum matching in its complement. The case $k = |V| - 3$, which is equivalent to finding a four-cycle free 2-matching of maximum size, is still open.

As in the edge-connected case, if k is small, the k-connectivity augmentation problem can be solved by considering the tree-like structure of the k-connected components of the graph. If $k = 2$, the familiar concept of 2-connected components, or blocks, and the block-cutvertex tree helps. For simplicity, suppose that $G = (V, E)$ is connected. Let $t(G)$ denote the number of end-blocks of G and let $b(G)$ denote the maximum number of components of $G - v$ over all vertices $v \in V$. Note that the end-blocks are pairwise disjoint. Since G' is 2-connected if and only if $t(G') = b(G') - 1 = 0$, and adding a new edge can decrease $t(G)$ by at most two and $b(G)$ by at most one, it follows that at least $\Psi(G) = \max\{\lceil t(G)/2 \rceil, b(G) - 1\}$ new edges are needed to make G 2-connected. Eswaran and Tarjan [12] and independently Plesnik [13] proved that this number can be achieved. See also Hsu and Ramachandran [52]. Finding two end-blocks X, Y for which adding a new edge xy with $x \in X$ and $y \in Y$ decreases $\Psi(G)$ by one can be done, roughly speaking, by choosing the end-blocks corresponding to the end-vertices of a longest path in the block-cutvertex tree of G.

Theorem 14.22 [12,13] *Let $G = (V, E)$ be a connected graph. Then G can be made 2-connected by adding* $\max\{\lceil t(G)/2\rceil, b(G) - 1\}$ *edges.* ■

The lower bounds used in Theorem 14.22 can be extended to $k \geq 3$ and arbitrary starting graphs G as follows. Let $N_G(X)$, or simply $N(X)$ if G is clear from the context, denote the set of neighbors of vertex set X in G. A nonempty subset $X \subset V$ is a *fragment* if $V - X - N(X) \neq \emptyset$. It is easy to see that every set of new edges F which makes G k-connected must contain at least $k - |N(X)|$ edges from X to $V - X - N(X)$ for every fragment X. By summing up these "deficiencies" over pairwise disjoint fragments, we obtain a subpartition-type lower bound, similar to the one used in the corresponding edge-connectivity augmentation problem. Let

$$t(G, k) = \max\left\{\sum_{i=1}^{r}(k - |N(X_i)|) : \{X_1, \ldots, X_r\} \text{ are pairwise disjoint fragments in } V\right\}. \tag{14.18}$$

Let $\gamma(G, k)$ denote the size of a smallest augmenting set of G with respect to k. Since an edge can decrease the deficiency $k - |N(X_i)|$ of at most two sets X_i, we have $\gamma(G, k) \geq \lceil t(G, k)/2\rceil$. For $K \subset V$ let $b(K, G)$ denote the number of components in $G - K$. Let

$$b(G, k) = \max\{b(K, G) : K \subset V, |K| = k - 1\}. \tag{14.19}$$

Since the deletion of K from the augmented graph must leave a connected graph, we have that $\gamma(G, k) \geq b(G, k) - 1$. Thus we obtain $\gamma(G, k) \geq \Psi(G, k)$, where we put

$$\Psi(G, k) = \max\{\lceil t(G, k)/2\rceil, b(G, k) - 1\}. \tag{14.20}$$

Theorem 14.22 implies that $\gamma(G, 2) = \Psi(G, 2)$. Watanabe and Nakamura [53] proved that this minimax equality is valid for $k = 3$, too. Hsu and Ramachandran [54] gave an alternative proof and a linear time algorithm, based on Tutte's decomposition theory of 2-connected graphs into 3-connected components. This method was further developed by Hsu [55], who solved the problem of making a 3-connected graph 4-connected by adding a smallest set of edges. His proof relies on the decomposition of 3-conneced graphs into 4-connected components.

This approach, however, which relies on the decomposition of a graph into its k-connected components, is rather hopeless for $k \geq 5$. While k-edge-connected components have a nice structure, k-connected components are difficult to handle. Furthermore, the successive augmentation property does not hold for vertex-connectivity augmentation [25].

Although $\Psi(G, k)$ suffices to characterize $\gamma(G, k)$ for $k \leq 3$, there are examples showing that $\gamma(G, k)$ can be strictly larger than $\Psi(G, k)$. Consider for example the complete bipartite graph $K_{k-1,k-1}$ with target k. For $k \geq 4$ this graph has $\Psi = k - 1$ and $\gamma = 2k - 4$, showing that the gap can be as large as $k - 3$. Jordán [56] showed that if the starting graph G is $(k - 1)$-connected then this is the extremal case, that is, $\gamma(G, k) \leq \Psi(G, k) + k - 3$. A polynomial time algorithm to find an augmenting set with at most $k - 3$ surplus edges was also given in [56]. This gap was later reduced to $(k - 1)/2$ with the help of two additional lower bounds [57]. Cheriyan and Thurimella [58] gave a more efficient algorithm with the same approximation gap and showed how to compute $b(G, k)$ in polynomial time if G is $(k - 1)$-connected. A near optimal solution can be found efficiently even if G is not $(k - 1)$-connected. This was proved by Ishii and Nagamochi [59] and, independently, by Jackson and Jordán [60]. The approximation gap in the latter paper is slightly smaller.

Theorem 14.23 [60] *Let $G = (V, E)$ be an l-connected graph. Then* $\gamma(G, k) \leq \Psi(G, k) + (k - l)k/2 + 4$. ■

Jackson and Jordán [60] adapted the edge splitting method for vertex-connectivity. This method was subsequently employed to find an optimal augmentation in polynomial time, for k fixed.

Given an extension $G' = (V + s, E')$ of a graph $G = (V, E)$, define $\bar{d}(X) = |N_G(X)| + d'(s, X)$ for every $X \subseteq V$, where d' denotes the degree function in G'. We say that G' is (k, s)-*connected* if

$$\bar{d}(X) \geq k \text{ for every fragment } X \subset V, \tag{14.21}$$

and that it is *minimally* (k, s)-*connected* if the set of edges incident to s is inclusionwise minimal with respect to (14.21). The following result from [60] gives lower and upper bounds for $\gamma(G, k)$ in terms of $d'(s)$ in any minimally (k, s)-connected extension of G.

Theorem 14.24 [60] *Let $G' = (V + s, E')$ be a minimally (k, s)-connected extension of a graph G. Then $\lceil d'(s)/2 \rceil \leq \gamma(G, k) \leq d'(s) - 1$.* ■

Let $G' = (V + s, E')$ be a minimally (k, s)-connected extension of G. Splitting off su and sv in G' is k-*admissible* if G'_{uv} also satisfies (14.21). Notice that if G' has no edges incident to s then (14.21) is equivalent to the k-connectivity of G. Hence, as in the case of edge-connectivity, it would be desirable to know, when $d(s)$ is even, that there is a sequence of k-admissible splittings which isolates s. In this case, using the fact that $\gamma(G, k) \geq d'(s)/2$ by Theorem 14.24, the resulting graph on V would be an optimal augmentation of G with respect to k. This approach works for the k-edge-connectivity augmentation problem but does not always work in the vertex connectivity case. The reason is that complete k-admissible splittings do not necessarily exist. On the other hand, the splitting off results in [60, 61] are 'close enough' to yield an optimal solution if k is fixed.

The obstacle for the existence of a k-admissible splitting can be described, provided $d'(s)$ is large enough compared to k. The proof of the following theorem is based on a new tripartite submodular inequality for $|N(X)|$, see [60].

Theorem 14.25 [60] *Let $G' = (V + s, E')$ be a minimally (k, s)-connected extension of $G = (V, E)$ and suppose that $d'(s) \geq k^2$. Then there is no k-admissible splitting at s in G' if and only if there is a set $K \subset V$ in G such that $|K| = k - 1$ and $G - K$ has $d'(s)$ components $C_1, C_2, \ldots, C_{d'(s)}$ (and we have $d'(s, C_i) = 1$ for $1 \leq i \leq d'(s)$).* ■

Theorem 14.25 does not always hold if $d'(s)$ is small compared to k. To overcome this difficulty, Jackson and Jordán [61] introduced the following family of graphs. Let $G = (V, E)$ be a graph and k be an integer. Let X_1, X_2 be disjoint subsets of V. We say (X_1, X_2) is a k-*deficient pair* if $d(X_1, X_2) = 0$ and $|V - (X_1 \cup X_2)| \leq k - 1$. We say two deficient pairs (X_1, X_2) and (Y_1, Y_2) are *independent* if for some $i \in \{1, 2\}$ we have either $X_i \subseteq V - (Y_1 \cup Y_2)$ or $Y_i \subseteq V - (X_1 \cup X_2)$, since in this case no edge can simultaneously connect X_1 to X_2 and Y_1 to Y_2. We say G is k-*independence free* if G does not have two independent k-deficient pairs. Note that if G is $(k - 1)$-connected and (X_1, X_2) is a k-deficient pair then $V - (X_1 \cup X_2) = N(X_1) = N(X_2)$. For example (a) $(k - 1)$-connected chordal graphs and graphs with minimum degree $2k-2$ are k-independence free, (b) all graphs are 1-independence free and all connected graphs are 2-independence free, (c) a graph with no edges and at least $k + 1$ vertices is not k-independence free for any $k \geq 2$, (d) if G is k-independence free and H is obtained by adding edges to G then H is also k-independence free, and (e) a k-independence free graph is l-independence free for all $l \leq k$. In general, a main difficulty in vertex-connectivity problems is that vertex cuts can cross each other in many different ways. In the case of an independence free graph G these difficulties can be overcome.

Theorem 14.26 [61] *If G is k-independence free then $\gamma(G, k) = \Psi(G, k)$.* ■

If G is not k-independence free but $t(G,k)$ is large, then the augmentation problem can be reduced to the independence free case by adding new edges. This crucial property is formulated by the next theorem.

Theorem 14.27 [61] *Let $G = (V, E)$ be $(k-1)$-connected and suppose that $t(G,k) \geq 8k^3 + 10k^2 - 43k + 22$. Then there exists a set of edges F for G such that $t(G+F,k) = t(G,k) - 2|F|$, $G + F$ is k-independence free, and $t(G+F,k) \geq 2k-1$.* ■

These results lead to the following theorem.

Theorem 14.28 [61] *Let G be $(k-1)$-connected. If $\gamma(G,k) \geq 8k^3 + 10k^2 - 43k + 21$ then*

$$\gamma(G,k) = \Psi(G,k).$$ ■

The min-max equality in Theorem 14.28 is not valid if we remove the hypothesis that G is $(k-1)$-connected. To see this consider the graph G obtained from the complete bipartite graph $K_{m,k-2}$ by adding a new vertex x and joining x to j vertices in the m set of the $K_{m,k-2}$, where $j < k < m$. Then $b(G,k) = m$, $t(G,k) = 2m + k - 2j$ and $\gamma(G,k) = m - 1 + k - j$. However, by modifying the definition of $b(G,k)$ slightly, an analogous minimax theorem can be obtained for augmenting graphs of arbitrary connectivity. For a set $K \subset V$ with $|K| = k-1$ let $\delta(K,k) = \max\{0, \max\{k - d(x) : x \in K\}\}$ and $b^*(K,G) = b(K,G) + \delta(K,k)$. Let $b^*(G,k) = \max\{b^*(K,G) : K \subset V, |K| = k-1\}$. It is easy to see that $\gamma(G,k) \geq b^*(G,k) - 1$. Let

$$\Psi^*(G,k) = \max\{\lceil t(G,k)/2 \rceil, b^*(G,k) - 1\}.$$

Theorem 14.29 [61] *Let $G = (V,E)$ be l-connected. If $\gamma(G,k) \geq 3(k-l+2)^3(k+1)^3$ then $\gamma(G,k) = \Psi^*(G,k)$.* ■

The lower bounds, in terms of k, are certainly not best possible in Theorems 14.27 through 14.29. These bounds, however, depend only on k. This is the essential fact in the solution for k fixed. Note that by Theorem 14.24 one can efficiently decide whether $\gamma(G,k)$ (or $t(G,k)$) is large enough compared to k. The proofs of these results are algorithmic and give rise to an algorithm which solves the k-vertex-connectivity augmentation problem in polynomial time for k fixed. If $\gamma(G,k)$ is large, then the algorithm has polynomial running time even if k is not fixed. This phenomenon is similar to what we observed when we investigated the algorithm for the simplicity-preserving k-edge-connectivity augmentation problem.

Perhaps the most exciting open question of this area is the complexity of the k-vertex-connectivity augmentation problem, when k is part of the input. A recent result of Végh [62] settled the special case when the starting graph is $(k-1)$-connected. To state the result we need some new concepts.

Let $G = (V,E)$ be a $(k-1)$-connected graph. A *clump* of G is an ordered pair $C = (S, \mathcal{P})$ where $S \subset V$, $|S| = k-1$, and $\mathcal{P}$ is a partition of $V - S$ into nonempty subsets, called *pieces*, with the property that no edge of G joins two distinct pieces in C. (Note that a piece is not necessarily connected.) It can be seen that if $C = (S, \mathcal{P})$ is a clump of G then, in order to make G k-connected, we must add a set of at least $|\mathcal{P}| - 1$ edges between the pieces of C, where $|\mathcal{P}|$ is the number of pieces of $\mathcal{P}$. We shall say that a clump C *covers* a pair of vertices u, v of G if u and v belong to distinct pieces of C. A *bush* B of G is a set of clumps such that each pair of vertices of G is covered by at most two clumps in B. Thus, if B is a bush in G, then in order to make G k-connected, we must add a set of at least

$$def(B) = \left\lceil \frac{1}{2} \sum_{(S,P) \in B} (|\mathcal{P}| - 1) \right\rceil \tag{14.22}$$

edges between the pieces of the clumps in B. Two bushes B_1 and B_2 of G are *disjoint* if no pair of vertices of G is covered by clumps in both B_1 and B_2. Thus, if B_1 and B_2 are disjoint bushes, then the sets of edges which need to be added between the pieces of the clumps in B_1 and B_2 are disjoint. We can now state the theorem.

Theorem 14.30 **[62]** *Let G be a $(k-1)$-connected graph. Then the minimum number of edges which must be added to G to make it k-connected is equal to the maximum value of $\sum_{B\in D} def(B)$ taken over all sets of pairwise disjoint bushes D for G.* ■

The local vertex-connectivity augmentation problem is NP-hard even in the special case when the goal is to find a smallest augmenting set which increases the local vertex-connectivity up to k within a given subset of vertices of a $(k-1)$-connected graph. However, there exist solvable subcases and some remaining open questions. For instance, Watanabe et al. [63] gave a linear-time algorithm for optimally increasing the connectivity to 2 within a specified subset. The special case when the starting graph has no edges is an interesting open problem.

We close this section by a different generalization of the connectivity augmentation problem. In some cases it is desirable to make the starting graph $G=(V,E)$ k-edge-connected as well as l-vertex-connected at the same time, by adding a new set of edges F. In this multiple target version l is typically small while k is arbitrary. We may always assume $l \le k$. Hsu and Kao [64] solved a local version of this problem for $k=l=2$. Ishii et al. [65–67] proved a number of results for $l \le 3$ and presented near optimal polynomial-time algorithms when l as well as k can be arbitrary.

A typical result is as follows. Let $k \ge 2$ and $l=2$. By combining $\alpha(G,k)$ and $t(G,2)$ define

$$\alpha'(G,k,2) = \max\left\{\sum_{i=1}^{p}(k-d(X_i)) + \sum_{i=p+1}^{t}(2-|N(X_i)|)\right\},$$

where the maximum is taken over all subpartitions $\{X_1,\ldots,X_p,X_{p+1},\ldots,X_t\}$ of V for which X_i is a fragment for $p+1 \le i \le t$. Clearly, $\lceil \alpha'(G,k,2)/2\rceil$ is a lower bound for this multiple target problem. By applying the splitting off method (and a new operation called edge switching), a common extension of Theorems 14.4 and 14.22 can be obtained.

Theorem 14.31 **[66]** *$G=(V,E)$ can be made k-edge-connected and 2-connected by adding γ new edges if and only if* $\max\{\lceil\alpha'(G,k,2)/2\rceil, b(G,2)-1\} \le \gamma$. ■

14.7 VERTEX-CONNECTIVITY AUGMENTATION OF DIGRAPHS

From several aspects, the directed k-edge-connectivity augmentation problem is less tractable than its undirected version. This may suggest that the *directed k-vertex-connectivity augmentation problem* is harder than the (still unsolved) undirected problem. Another sign of this is the fact that after the basic result of Eswaran and Tarjan [12] on the case $k=1$ (Theorem 14.11) almost no results appeared for nearly twenty years. An exception was the following result of Masuzawa et al. [68] which solved the special case when the starting digraph $D=(V,A)$ is an arborescence (i.e., a directed tree with a root vertex r such that there is a directed path from r to every $v \in V$). Let $\gamma(D,k)$ denote the size of a smallest augmenting set with respect to the target vertex-connectivity k.

Theorem 14.32 **[68]** *Let $B=(V,A)$ be an arborescence. Then* $\gamma(B,k) = \sum_{v\in V}\max\{0, k-\delta(v)\}$. ■

In spite of the above general feeling a complete solution for the directed version has been found. For arbitrary starting digraphs $D = (V, A)$ there is a natural subpartition-type lower bound for $\gamma(D, k)$, similar to $t(G, k)$. Let $N^-(X)$ and $N^+(X)$ denote the set of in-neighbors and out-neighbors of vertex set X in D, respectively. We say $X \subset V$ is an *in-fragment* if $V - X - N^-(X) \neq \emptyset$. If $V - X - N^+(X) \neq \emptyset$ then X is called an *out-fragment*. Let

$$t_{\text{in}}(D,k) = \max\left\{\sum_{i=1}^{r}(k - |N^-(X)|) : X_1, \ldots, X_r \text{ are pairwise disjoint in-fragments in } V\right\},$$

$$t_{\text{out}}(D,k) = \max\left\{\sum_{i=1}^{r}(k - |N^+(X)|) : X_1, \ldots, X_r \text{ are pairwise disjoint out-fragments in } V\right\},$$

and let

$$\Psi(D,k) = \max\{t_{\text{in}}(D,k), t_{\text{out}}(D,k)\}.$$

It is easy to see that $\gamma(D, k) \geq \Psi(D, k)$ holds. Theorem 14.11 shows that $\gamma(D, 1) = \Psi(D, 1)$ and Theorem 14.32 implies that $\gamma(B, k) = \Psi(B, k)$ for every arborescence B and every $k \geq 1$. For $k \geq 2$, however, Jordán [69] pointed out that $\gamma(D, k) \geq \Psi(D, k) + k - 1$ may hold, even if D is $(k-1)$-connected. On the other hand, for $(k-1)$-connected starting digraphs the gap cannot be larger than $k - 1$ [69].

A stronger lower bound can be obtained by considering deficient pairs of subsets of V rather than deficient in- or out-fragments. We say that an ordered pair (X, Y), $\emptyset \neq X, Y \subset V$, $X \cap Y = \emptyset$ is a *one-way pair* in a digraph $D = (V, A)$ if there is no edge in D with tail in X and head in Y. We call X and Y the *tail* and the *head* of the pair, respectively. The *deficiency* of a one-way pair, with respect to k-connectivity, is $def_k(X, Y) = \max\{0, k - |V - (X \cup Y)|\}$. Two pairs are *independent* if their tails or their heads are disjoint. For a family $\mathcal{F}$ of pairwise independent one-way pairs we define $def_k(\mathcal{F}) = \sum_{(X,Y)\in\mathcal{F}} def_k(X, Y)$. By Menger's theorem every augmenting set F with respect to k must contain at least $def_k(X, Y)$ edges from X to Y for every one-way pair (X, Y). Moreover, these arcs are distinct for independent one-way pairs. This proves $\gamma(D, k) \geq def_k(\mathcal{F})$ for all families $\mathcal{F}$ of pairwise independent one-way pairs. Frank and Jordán [70] solved the k-vertex-connectivity augmentation problem for digraphs by showing that this lower bound can be attained.

Theorem 14.33 [70] *A digraph $D = (V, A)$ can be made k-connected by adding at most γ new edges if and only if*

$$def_k(\mathcal{F}) \leq \gamma \tag{14.23}$$

holds for all families $\mathcal{F}$ of pairwise independent one-way pairs. ■

This result was obtained as a special case of a more general theorem on coverings of bi-supermodular functions, see Theorem 14.47. We present a more direct proof in the next subsection.

The minimax formula of Theorem 14.33 was later refined by Frank and Jordán [71]. Among others, it was shown that if $def_k(\mathcal{F}) \geq 2k^2 - 1$, then the tails or the heads of the pairs in $\mathcal{F}$ are pairwise disjoint. This implies that if $\gamma(D, k) \geq 2k^2 - 1$ then the simpler lower bound $\Psi(D, k)$ suffices. With the help of this refined version, one can deduce Theorem 14.32 from Theorem 14.33 as well. A related conjecture of Frank [6] claims that $\gamma(D, k) = \Psi(D, k)$ for every acyclic starting digraph D.

If D is strongly connected and $k = 2$ then a direct proof and a simplified minimax theorem was given in [72] by applying the splitting off method. In a strongly connected digraph $D = (V, A)$ there are two types of deficient sets with respect to $k = 2$: in-fragments X with $|N^-(X)| = 1$ (called *in-tight*) and out-fragments X with $|N^+(X)| = 1$ (called *out-tight*).

Theorem 14.34 [**72**] *Let $D = (V, A)$ be strongly connected. Then $\gamma(D, 2) = \Psi(D, 2)$ holds, unless $\Psi(D, 2) = 2$ and there exist three in-tight (or three out-tight) sets B_1, B_2, B_3, such that*

$$B_1 \cap B_2 \neq \emptyset, |B_3| = 1, \quad \textit{and } V - (B_1 \cup B_2) = B_3.$$

In the latter case D can be made 2-connected by adding 3 arcs. ■

14.7.1 Augmenting *ST*-Edge-Connectivity

Theorem 14.33 can also be deduced from the solution of a directed edge-connectivity augmentation problem involving local requirements of a special kind. We present this solution in detail.

Let $D = (V, A)$ be a digraph and let S and T be two nonempty (but not necessarily disjoint) subsets of V. One may be interested in an augmentation of D in which every vertex of T is reachable from every vertex of S. This generalizes the problem of making a digraph strongly connected ($S = T = V$) which was solved by Eswaran and Tarjan (Theorem 14.11). This generalization, however, leads to an *NP*-complete problem even in the special cases when $|S| = 1$ or $S = T$. See [7] for the proof.

On the other hand, we shall show that if only ST-edges are allowed to be added, then the augmentation problem is tractable even for higher edge-connectivity. An edge with tail s and head t is an *ST-edge* if $s \in S, t \in T$. Let A^* denote the set of all ST-edges, including loops, and let $m = |A^*|$. Clearly, $m = |S||T|$. We say that a subset X of vertices is *ST-nontrivial*, or *nontrivial* for short, if $X \cap T \neq \emptyset$ and $S - X \neq \emptyset$, which is equivalent to requiring that there is an ST-edge entering X. A digraph is *k-ST-edge-connected* if the number of edges entering $X \subseteq V$ is at least k for every nontrivial X. By Menger's theorem, this is equivalent to requiring the existence of k edge-disjoint st-paths for every possible choice of $s \in S$ and $t \in T$. Note that this property is much stronger than requiring only the existence of k edge-disjoint paths from S to T.

We say that two sets X and Y are *ST-crossing* if none of the sets $X \cap Y \cap T, S - (X \cup Y), X - Y$, and $Y - X$ is empty. In the special case when $S = T = V$ this coincides with the standard notion of crossing. A family $\mathcal{L}$ is *ST-crossing* if both the intersection and the union of any two ST-crossing members of $\mathcal{L}$ belong to $\mathcal{L}$. If $\mathcal{L}$ does not include two ST-crossing members, it is said to be *ST-cross-free*. A family $\mathcal{I}$ of sets is *ST-independent* or just *independent* if, for any two members X and Y of $\mathcal{I}$, at least one of the sets $X \cap Y \cap T$ and $S - (X \cup Y)$ is empty. Note that the relation between two sets can be of three types. Either they are ST-crossing, or one includes the other, or they are ST-independent. A set F of ST-edges (or the digraph (V, F)) *covers* $\mathcal{L}$ if each member of $\mathcal{L}$ is entered by a member of F.

For an initial digraph $D = (V, A)$ that we want to make k-ST-edge-connected, define the deficiency function h on sets as follows. For a real number x we put $x^+ = \max\{x, 0\}$. Let

$$h(X) = \begin{cases} (k - \varrho_D(X))^+ & \text{if } X \text{ is } ST\text{-nontrivial} \\ 0 & \text{otherwise} \end{cases} \tag{14.24}$$

Therefore, the addition of a digraph $H = (V, F)$ of ST-edges to D results in a k-ST-edge-connected digraph if and only if F *covers* h in the sense that $\varrho_H(X) \geq h(X)$ for every $X \subseteq V$. For a set-function h and a family $\mathcal{I}$ of sets we use $h(\mathcal{I})$ to denote $\sum[h(X) : X \in \mathcal{I}]$.

Theorem 14.35 [**70**] *A digraph $D = (V, A)$ can be made k-ST-edge-connected by adding at most γ new ST-edges* (or equivalently, h can be covered by γ ST-edges) *if and only if*

$$h(\mathcal{I}) \leq \gamma \textit{ for every } ST\textit{-independent family } \mathcal{I} \textit{ of subsets of } V. \tag{14.25}$$

Equivalently, the minimum number $\tau_h = \tau_h(D)$ *of ST-edges whose addition to D results in a k-ST-edge connected digraph is equal to the maximum* $\nu_h = \nu_h(D)$ *of the sum of h-values over all families of ST-independent sets.*

Proof. We prove the second form. Since one ST-edge cannot cover two or more sets from an ST-independent family, we have $\nu_h \leq \tau_h$. For the reverse direction we proceed by induction on ν_h. If this value is 0, that is, there is if no deficient set, then D itself is k-ST-edge-connected and hence no new edge is needed. So we may assume that ν_h is positive.

First suppose that there is an edge $e \in A^*$ for which $\nu_h(D') \leq \nu_h(D) - 1$ for $D' = D + e$. Then it follows by induction that D' can be made k-ST-edge-connected by adding $\nu_h(D')$ ST-edges. But then the original D can be made k-ST-edge-connected by adding at most $\nu_h(D') + 1$ ST-edges and hence we have $\tau_h(D) \leq \tau_h(D') + 1 = \nu_h(D') + 1 \leq \nu_h(D) \leq \tau_h(D)$ from which equality follows throughout and, in particular, $\tau_h(D) = \nu_h(D)$. In this case we are done.

Thus, it remains to consider the case when for every ST-edge e, there is an ST-independent family $\mathcal{I}_e$ for which $h(\mathcal{I}_e) = \nu_h$ and e does not enter any member of $\mathcal{F}_e$. Let $\mathcal{J}'$ denote the union of all of these families $\mathcal{I}_e$ in the sense that as many copies of a set X are put into $\mathcal{J}'$ as the number of edges e for which X is in $\mathcal{I}_e$. Recall that m denotes the number of ST-edges. Then we have $h(\mathcal{J}') = m\nu_h$ and

$$\text{every } ST\text{-edge enters at most } m - 1 \text{ members.} \tag{14.26}$$

We may assume that the h-value of each member of $\mathcal{J}'$ is strictly positive since the members of zero h-values can be discarded. Now we apply the following uncrossing procedure as long as possible: if there are two ST-crossing members, replace them by their intersection and union. When a new member has zero h-value, remove it. The submodularity of the in-degree function implies that such an exchange operation preserves (14.26) and also that the h-value of the revised system is at least $h(\mathcal{J}')$.

The above uncrossing procedure terminates after a finite number of steps since the number of sets can never increase and hence it can decrease only a finite number of times. Moreover, when this number does not decrease, the sum of squares of the sizes in the family must strictly increase. Since the number of such steps is also finite, we can conclude that we arrive at an ST-cross-free family $\mathcal{J}$ after a finite number of uncrossing operations. Therefore we have $h(\mathcal{J}) \geq h(\mathcal{J}') = m\nu_h$.

We emphasize that a set $X \subseteq V$ can occur in several copies in $\mathcal{J}$. Let $s(X)$ denote the number of these copies. Evidently, the sum of the s-values over the subsets of V is exactly $|\mathcal{J}|$.

Claim 14.3 *The partial order on* $\mathcal{J}$ *defined by* $X \subseteq Y$ *admits no chain of s-weight larger than* $m - 1$.

Proof. For a contradiction suppose that there is a chain $\mathcal{C}$ of s-weight at least m. Then there are m (not necessarily distinct) members of $\mathcal{J}$ which are pairwise comparable. Since the members of a chain of ST-nontrivial sets can be covered by a single ST-edge, this contradicts property (14.26). ■

We can apply the weighted polar-Dilworth theorem asserting that the maximum weight of a chain is equal to the minimum number of antichains covering each element as many times as its weight is. It follows that $\mathcal{J}$ contains $m-1$ antichains such that $s(X)$ of them contain X for every $X \in \mathcal{J}$. Since $h(\mathcal{J}) \geq m\nu_h$, the h-sum of at least one of these antichains is larger than ν_h. However, $\mathcal{J}$ is ST-cross-free and hence this antichain is ST-independent, contradicting the definition of ν_h. This contradiction completes the proof of the theorem. ■

Note that in the special case when $S = T = V$, the members of an ST-independent family $\mathcal{I}$ of nonempty proper subsets of V are either pairwise disjoint or pairwise co-disjoint. (To see this suppose, for a contradiction, that $\mathcal{I}$ has two members which are disjoint and has two members which are co-disjoint. This implies, since any two members of $\mathcal{I}$ are disjoint or co-disjoint, that there is an $X \in \mathcal{I}$ which is disjoint from some $Y \in \mathcal{I}$ and co-disjoint from some $Z \in \mathcal{I}$. But then we must have $Y \subseteq Z$, contradicting the independence of $\mathcal{I}$.) Thus in the special case $S = T = V$ Theorem 14.35 immediately implies Theorem 14.14. It is also possible to deduce Theorem 14.33, although the proof is a bit technical, see [71].

Although it was possible to obtain a polynomial algorithm from the original proof of Theorem 14.33, it was neither combinatorial nor very efficient. Using a different (but quite complicated) approach, Benczúr and Végh [73] developed a combinatorial polynomial time algorithm. In the special of the k-vertex-connectivity augmentation problem when the initial digraph is $(k-1)$-connected, [74] describes a much simpler algorithm. Enni [75] gave an algorithmic proof of Theorem 14.35 in the special case when $k = 1$. Note that no strongly polynomial algorithm is known for the capacitated version of Theorem 14.35.

14.8 HYPERGRAPH AUGMENTATION AND COVERINGS OF SET FUNCTIONS

Connectivity augmentation is about adding new edges to a graph or digraph so that it becomes sufficiently highly connected. Applying Menger's theorem, every augmentation problem has an equivalent formulation where the goal is to add new edges so that each cut receives at least as many new edges as its deficiency with respect to the given target. Cut typically means a subset of vertices, but it may also be a pair or collection of subsets of the vertex set. The deficiency function, say $k - \rho(X)$ or $R(X) - d(X)$, is determined by the input graph and the connectivity requirements. This leads to a more abstract point of view: given a function p on subsets of a ground-set V, find a smallest cover of p, that is, a smallest set of edges F such that at least $p(X)$ edges enter every subset $X \subset V$. Deficiency functions related to connectivity problems have certain supermodular properties. This motivates the study of minimum covers of functions of this type.

This is not just for the sake of proving more general minimax theorems. In some cases (e.g., in the directed k-connectivity augmentation problem) the only known way to the solution is via an abstract result. In other cases (e.g., in the k-edge-connectivity augmentation problem) generalizations lead to simpler proofs, algorithms, and extensions (to local requirements or vertex-induced cost functions) by showing the background of the problem.

An intermediate step toward an abstract formulation is to consider hypergraphs. A *hypergraph* is a pair $\mathcal{G} = (V, E)$ where V is a finite set (the set of *vertices* of $\mathcal{G}$) and E is a finite collection of *hyperedges*. Each hyperedge e is a set $Z \subseteq V$ with $|Z| \geq 2$. The *size* of e is $|Z|$. Thus (loopless) graphs correspond to hypergraphs with edges of size two only. A hyperedge of size two is called a *graph edge*. Let $d_{\mathcal{G}}(X)$ denote the number of hyperedges intersecting both X and $V - X$. A hypergraph is *k-edge-connected* if $d_{\mathcal{G}}(X) \geq k$ for all $\emptyset \neq X \subset V$. A *component* of $\mathcal{G}$ is a maximal connected subhypergraph of $\mathcal{G}$. Let $w(\mathcal{G})$ denote the number of components of $\mathcal{G}$.

One possible way to generalize the k-edge-connectivity augmentation problem is to search for a smallest set of graph edges whose addition makes a given hypergraph k-edge-connected. Cheng [76] was the first to prove a result in this direction. He determined the minimum number of new graph edges needed to make a $(k-1)$-edge-connected hypergraph $\mathcal{G}$ k-edge-connected, by invoking deep structural results of Cunningham [77] on decompositions of submodular functions. His result was soon extended to arbitrary hypergraphs by Bang-Jensen and Jackson [78]. They employed and extended the splitting off method to hypergraphs.

A hypergraph $\mathcal{H} = (V + s, E)$ is (k, s)-*edge-connected* if s is incident to graph edges only and $d_{\mathcal{H}}(X) \geq k$ for all $\emptyset \neq X \subset V$. Splitting off two edges su, sv is k-*admissible* if $\mathcal{H}_{uv}$ is also (k, s)-edge-connected. The extension of Theorem 14.2 to hypergraphs is as follows.

Theorem 14.36 [78] *Let $\mathcal{H} = (V+s, E)$ be a (k, s)-edge-connected hypergraph with $d_{\mathcal{H}}(s) = 2m$. Then exactly one of the following statements holds.*

i. *There is a complete k-admissible splitting at s, or*

ii. *There exists a set $A \subset E$ with $|A| = k - 1$ and $w(\mathcal{H} - s - A) \geq m + 2$.*

Let $\mathcal{G} = (V, E)$ be a hypergraph. Let

$$\alpha(\mathcal{G}, k) = \max\left\{\sum_{i=1}^{t}(k - d(X_i)) : \{X_1, \ldots, X_t\} \text{ is a subpartition of } V\right\} \tag{14.27}$$

$$c(\mathcal{G}, k) = \max\{w(\mathcal{G} - A) : A \subset E, |A| = k - 1\}. \tag{14.28}$$

■

As in the case of graphs, $\lceil \alpha(\mathcal{G}, k)/2\rceil$ is a lower bound for the size of a smallest augmentation. Another lower bound is $c(\mathcal{G}, k) - 1$. Note that if $\mathcal{G}$ is a graph, $k \geq 2$, and $\mathcal{G} - A$ has $c(\mathcal{G}, k)$ components $C_1, C_2, \ldots, C_c$ for some $A \subset E$ with $|A| = k-1$ and $c \geq 2$ then $\sum_1^c (k - d(C_i)) = kc - \sum_1^c d(C_i) \geq kc - 2(k-1) = 2(c-1) + (k-2)(c-2) \geq 2(c-1)$, and hence $\alpha(\mathcal{G}, k)/2 \geq c(\mathcal{G}, k) - 1$.

Theorem 14.37 [78] *The minimum number of new graph edges which makes a hypergraph $\mathcal{G}$ k-edge-connected equals*

$$\max\{\lceil \alpha(\mathcal{G}, k)/2\rceil, c(\mathcal{G}, k) - 1\}. \tag{14.29}$$

■

Note that the successive augmentation property does not hold for hypergraphs [25]. Cosh [79] proved the bipartition constrained version of Theorem 14.37. The general partition constrained version was solved by Bernáth et al. [80]. Cosh et al. [81] proved that the local version of the hypergraph edge-connectivity augmentation problem (with graph edges) is NP-hard, even if the starting hypergraph is connected and the maximum requirement is two.

One may also want to augment a hypergraph by adding hyperedges. The minimum number of new hyperedges which make a given hypergraph $\mathcal{G} = (V, E)$ k-edge-connected is easy to determine: add l copies of the hyperedge containing all vertices of V, where $l = \max\{k - d_{\mathcal{G}}(X) : \emptyset \neq X \subset V\}$. So it is natural to either set an upper bound on the size of the new edges or to make the "cost" of a new hyperedge depend on its size. The following extension of Theorem 14.37 is due to T. Király. In a t-*uniform* hypergraph each hyperedge has size t.

Theorem 14.38 [82] *Let $\mathcal{H}_0 = (V, \mathcal{E}_0)$ be a hypergraph and $t \geq 2$, $\gamma \geq 0$ integers. There is a t-uniform hypergraph $\mathcal{H}$ on vertex-set V with at most γ hyperedges so that $\mathcal{H}_0 + \mathcal{H}$ is k-edge-connected ($k \geq 1$) if and only if*

$$\sum_{X \in \mathcal{P}} [k - d_{\mathcal{H}_0}(X)] \leq t\gamma \textit{ for every subpartition } \mathcal{P} \textit{ of } V,$$

$$k - d_{\mathcal{H}_0}(X) \geq \gamma \textit{ for every } X \subset V,$$

$$w(\mathcal{H}_0 - \mathcal{E}_0') - 1 \leq (t-1)\gamma \textit{ for every } \mathcal{E}_0' \subseteq \mathcal{E}_0,\ |\mathcal{E}_0'| = k - 1. \qquad ■$$

Szigeti [83] considered a different objective function and local requirements.

Theorem 14.39 [83] *Let $\mathcal{H}_0 = (V, \mathcal{E}_0)$ be a hypergraph and let $r(u, v)$ be a local requirement function. Then there is a hypergraph $\mathcal{H} = (V, \mathcal{E})$ so that $\lambda_{\mathcal{H}_0+\mathcal{H}}(u, v) \geq r(u, v)$ for every $u, v \in V$ and so that $\sum[|Z| : Z \in \mathcal{E}] \leq \gamma$ if and only if*

$$\sum_{X\in\mathcal{P}} [R(X) - d_{\mathcal{H}_0}(X)] \leq \gamma \text{ for every subpartition } \mathcal{P} \text{ of } V. \qquad \blacksquare$$

Benczúr and Frank [84] solved an abstract generalization of Theorem 14.37. Let V be a finite set and let $p : 2^V \to Z$ be a function with $p(\emptyset) = p(V) = 0$. p is *symmetric* if $p(X) = p(V-X)$ holds for every $X \subseteq V$. We say that p is *crossing supermodular* if it satisfies the following inequality for each pair of crossing sets $X, Y \subset V$:

$$p(X) + p(Y) \leq p(X \cup Y) + p(X \cap Y). \tag{14.30}$$

Recall that a set of edges F on V *covers* p if $d_F(X) \geq p(X)$ for all $X \subset V$.

Now suppose p is a symmetric crossing supermodular function on V and we wish to determine the minimum size $\gamma(p)$ of a cover of p consisting of graph edges. A subpartition-type lower bound is the following. Let

$$\alpha(p) = \max\left\{\sum_{i=1}^{t} p(X_i) : \{X_1, \ldots, X_t\} \text{ is a subpartition of } V\right\}.$$

Since an edge can cover at most two sets of a subpartition, we have $\gamma(p) \geq \lceil\alpha(p)/2\rceil$. This lower bound may be strictly less than $\gamma(p)$. To see this consider a ground set V with 4 elements and let $p \equiv 1$. Here $\alpha(p) = 4$ but, since every cover forms a connected graph on V, we have $\gamma(p) \geq 3$. This example leads to the following notions. We call a partition $\mathcal{Q} = \{Y_1, \ldots, Y_r\}$ of V with $r \geq 4$ *p-full* if

$$p(\cup_{Y\in\mathcal{Q}'} Y) \geq 1 \text{ for every non-empty subfamily } \mathcal{Q}' \subseteq \mathcal{Q}. \tag{14.31}$$

The maximum size of a p-full partition is called the *dimension* of p and is denoted by $\dim(p)$. If there is no p-full partition, then $\dim(p) = 0$. Since every cover induces a connected graph on the members of a p-full partition, we have $\gamma(p) \geq \dim(p) - 1$. Thus the minimum size of a cover is at least $\Phi(p) = \max\{\lceil\alpha(p)/2\rceil, \dim(p) - 1\}$.

Theorem 14.40 [84] *Let $p : 2^V \to Z$ be a symmetric crossing supermodular function. Then $\gamma(p) = \Phi(p)$.* ■

The proof of Theorem 14.40 yields a polynomial time algorithm to find a smallest cover, provided a polynomial time submodular function minimization oracle is available. The deficiency function of a (hyper)graph is symmetric and supermodular: Theorems 14.4 and 14.37 follow by taking $p(X) = k - d_{\mathcal{G}}(X)$ for all $X \subset V$ and $p(\emptyset) = p(V) = 0$, where $\mathcal{G}$ is the starting (hyper)graph, see [84]. The special case of Theorem 14.40 where $p(X) \in \{0, 1\}$ for all $X \subset V$ follows also from a result of Fleiner and Jordán [85]. Bernáth et al. [86] solved the partition-constrained version of this covering problem.

An abstract extension of Theorem 14.39 is as follows.

Theorem 14.41 [83] *Let $p : 2^V \to Z$ be a symmetric skew-supermodular function. Then*

$$\min\left\{\sum_{e\in F} |e| : F \text{ is a cover of } p\right\} = \max\left\{\sum_{1}^{t} p(X_i) : \{X_1, \ldots, X_r\} \text{ is a subpartition of } V\right\}. \tag{14.32}$$

■

Theorem 14.39 follows by taking $p(X) = q(X) = R(X) - d_{\mathcal{H}_0}(X)$. If p is an even valued skew-supermodular function, the minimum size of a cover consisting of graph edges can also be determined.

14.8.1 Detachments and Augmentations

We have seen that edge splitting results are important ingredients in solutions of connectivity augmentation problems and hence generalizations of some augmentation problems lead to extensions of edge splitting theorems. This works the other way round as well. Consider the following operation. Let $G = (V + s, E)$ be a graph with a designated vertex s. A *degree specification* for s is a sequence $\mathcal{S} = (d_1, \ldots, d_p)$ of positive integers with $\sum_{j=1}^{p} d_j = d(s)$. An $\mathcal{S}$-*detachment* of s in G is obtained by replacing s by p vertices $s_1, \ldots, s_p$ and replacing every edge su by an edge $s_i u$ for some $1 \leq i \leq p$ so that $d(s_i) = d_i$ holds in the new graph for $1 \leq i \leq p$. If $d_i = 2$ for all $1 \leq i \leq p$ then an $\mathcal{S}$-detachment corresponds to a complete splitting in a natural way. Given a local requirement function $r : V \times V \to Z_+$, an $\mathcal{S}$-detachment is called r-*admissible* if the detached graph G' satisfies $\lambda(x, y; G') \geq r(x, y)$ for every pair $x, y \in V$.

Extending an earlier theorem of Fleiner [87] on the case when $r \equiv k$ for some $k \geq 2$, Jordán and Szigeti [88] gave a necessary and sufficient condition for the existence of an r-admissible $\mathcal{S}$-detachment. We call r *proper* if $r(x, y) \leq \lambda(x, y; G)$ for every pair $x, y \in V$.

Theorem 14.42 [88] *Let r be a local requirement function for $G = (V + s, E)$ and suppose that G is 2-edge-connected and $r(u, v) \geq 2$ for each pair $u, v \in V$. Let $\mathcal{S} = (d_1, \ldots, d_p)$ be a degree specification for s with $d_i \geq 2$, $i = 1, \ldots, p$. Then there exists an r-admissible $\mathcal{S}$-detachment of s if and only if r is proper and*

$$\lambda(u, v; G - s) \geq r(u, v) - \sum_{i=1}^{p} \lfloor d_i/2 \rfloor \tag{14.33}$$

holds for every pair $u, v \in V$. ■

Theorem 14.42 implies Theorem 14.6 by letting $r \equiv r_\lambda$ and $d_i \equiv 2$. It also gives the following extension of Theorem 14.8. By *attaching* a star of degree d to a graph $G = (V, E)$ we mean the addition of new vertex s and d edges from s to vertices in V. Let $G = (V, E)$ be a graph and suppose that we are given local requirements $r(u, v)$ for each pair $u, v \in V$ as well as a set of integers $d_1, \ldots, d_p$ ($d_j \geq 2$). Can we make G r-edge-connected by attaching p stars with degrees $d_1, d_2, \ldots, d_p$? Applying Theorem 14.42 to a minimally (r, s)-edge-connected extension of G gives the following necessary and sufficient condition. Recall the definition of $q(X)$ from Section 14.2. For simplicity, we again assume that r satisfies (14.6).

Theorem 14.43 [88] *Let $G = (V, E)$ be a graph and let $r(u, v)$, $u, v \in V$ be a local requirement function satisfying* (6). *Then G can be made r-edge-connected by attaching p stars with degrees $d_1, \ldots, d_p$ ($d_j \geq 2$, $1 \leq j \leq p$) if and only if*

$$\sum_{i=1}^{t} q(X_i) \leq \sum_{j=1}^{p} d_j \tag{14.34}$$

holds for every subpartition $\{X_1, \ldots, X_t\}$ of V and $\lambda(u, v; G) \geq r(u, v) - \sum_{j=1}^{p} \lfloor d_j/2 \rfloor$ for every pair $u, v \in V$. ■

With the help of Theorem 14.43 it is easy to deduce a minimax formula for the following optimization problem: given G, r, and an integer $w \geq 2$, determine the minimum number γ for which G can be made r-edge-connected by attaching γ stars of degree w. If $w = 2$ then we are back at Theorem 14.8.

14.8.2 Directed Hypergraphs

In this subsection we consider extensions of some of the previous results on augmentations of directed graphs to directed hypergraphs and to directed covers of certain set functions.

A *directed hypergraph* (or *dypergraph*, for short) is a pair $\mathcal{D} = (V, A)$, where V is a finite set (the set of *vertices* of $\mathcal{D}$) and A is a finite collection of *hyperarcs*. Each hyperarc e is a set $Z \subseteq V$, $|Z| \geq 2$, with a specified *head vertex* $v \in Z$. We also use (Z, v) to denote a hyperarc on set Z and with head v. The *size* of e is $|Z|$. Thus a directed graph (without loops) is a dypergraph with hyperarcs of size two only. We say that a hyperarc (Z, v) *enters* a set $X \subset V$ if $v \in X$ and $Z - X \neq \emptyset$. Let $\rho(X)$ denote the number of hyperarcs entering X. A dypergraph $\mathcal{D} = (V, A)$ is *k-edge-connected* if $\rho(X) \geq k$ for every $\emptyset \neq X \subset V$. Berg et al. [89] extended Theorem 14.12 to dypergraphs and, among others, proved the following extension of Theorem 14.14.

Theorem 14.44 [89] *Let $\mathcal{D} = (V, A)$ be a dypergraph. Then $\mathcal{D}$ can be made k-edge-connected by adding γ new hyperarcs of size at most t if and only if*

$$\gamma \geq \sum_{1}^{r}(k - \rho(X_i)) \tag{14.35}$$

and

$$(t-1)\gamma \geq \sum_{1}^{r}(k - \delta(X_i)) \tag{14.36}$$

hold for every subpartition $\{X_1, X_2, \ldots, X_r\}$ of V. ■

Directed covers of set functions have also been investigated. Frank [6] proved the directed version of Theorem 14.40. We say that a set F of directed edges on ground set V *covers* a function $p : 2^V \to Z$ if $\rho_F(X) \geq p(X)$ for all $X \subset V$.

Theorem 14.45 [6] *Let $p : 2^V \to Z$ be a crossing supermodular function. Then p can be covered by γ edges if and only if*

$$\gamma \geq \sum_{1}^{t} p(X_i) \tag{14.37}$$

and

$$\gamma \geq \sum_{1}^{t} p(V - X_i) \tag{14.38}$$

hold for every subpartition $\{X_1, X_2, \ldots, X_t\}$ of V. ■

If $p(X) \in \{0, 1\}$ for all $X \subset V$, the problem corresponds to covering a "crossing family" of subsets of V by a smallest set of edges. Gabow and Jordán [43] solved the bipartition-constrained version of this special case.

Since $p(X) = k - \rho(X)$ is crossing supermodular, Theorem 14.45 implies Theorem 14.14. The following generalization of the directed k-edge-connectivity augmentation problem can also be solved by Theorem 14.45. Let $D = (V, A)$ be a directed graph with a specified root vertex $r \in V$ and let $k \geq l \geq 0$ be integers. D is called *(k, l)-edge-connected (from r)* if $\lambda(r, v; D) \geq k$ and $\lambda(v, r; D) \geq l$ for every vertex $v \in V - r$. Clearly, D is k-edge-connected if and only if D is (k, k)-edge-connected. The extension, due to Frank [90], is as follows. Let $p_{kl}(X) = \max\{k - \rho(X), 0\}$ for sets $\emptyset \neq X \subseteq V - r$ and let $p_{kl}(X) = \max\{l - \rho(X), 0\}$ for sets $X \subset V$ with $r \in X$.

Theorem 14.46 [90] *Let $D = (V, A)$ be a digraph and let $r \in V$. D can be made (k, l)-edge-connected from r by adding γ new arcs if and only if*

$$\gamma \geq \sum_{1}^{t} p_{kl}(X_i) \tag{14.39}$$

and

$$\gamma \geq \sum_{1}^{t} p_{kl}(V - X_i) \tag{14.40}$$

hold for every partition $\{X_1, X_2, \ldots, X_t\}$ of V. ■

Finally we state a result on directed covers of pairs of sets, due to Frank and Jordán [70], which led to the solution of the k-vertex-connectivity augmentation problem for directed graphs.

Let V be a ground set and let $p(X, Y)$ be an integer-valued function defined on ordered pairs of disjoint subsets $X, Y \subset V$. We call p *crossing bi-supermodular* if

$$p(X, Y) + p(X', Y') \leq p(X \cap X', Y \cup Y') + p(X \cup X', Y \cap Y')$$

holds whenever $X \cap X', Y \cap Y' \neq \emptyset$. A set F of directed edges *covers* p if there are at least $p(X, Y)$ arcs in F with tail in X and head in Y for every pair $X, Y \subset V$, $X \cap Y = \emptyset$. Two pairs $(X, Y), (X', Y')$ are *independent* if $X \cap X'$ or $Y \cap Y'$ is empty.

Theorem 14.47 [70] *Let p be an integer-valued crossing bi-supermodular function on V. Then p can be covered by γ arcs if and only if $\sum_{(X,Y)\in\mathcal{F}} p(X, Y) \leq \gamma$ holds for every family $\mathcal{F}$ of pairwise independent pairs.* ■

Let $D = (V, A)$ be a digraph. By taking $p(X, Y) = k - |(V - (X \cup Y)|$ for one-way pairs (X, Y) we can deduce Theorem 14.33. Furthermore, Theorem 14.47 implies Theorems 14.12, 14.14 and 14.35 as well as Edmonds' matroid partition theorem, a theorem of Győri on covering a rectilinear polygon with rectangles, and a theorem of Frank on $K_{t,t}$-free t-mathcings in bipartite graphs, see [70,91] for more details. A recent application to the jump number of two-directional orthogonal ray graphs can be found in [92].

The idea of abstract formulations may also lead to graph augmentation problems with somewhat different but still connectivity related objectives. A recent result of Frank and Király [93] solves the problem of optimally augmenting a graph G by adding a set F of edges so that $G + F$ is (k, l)-partition-connected. A graph $G = (V, E)$ is called *(k, l)-partition-connected* if the number of cross edges is at least $k(|\mathcal{P}| - 1) + l$ for all partitions $\mathcal{P}$ of V. With this definition (k, k)-partition-connectivity is equivalent to k-edge-connectivity while $(k, 0)$-partition-connectivity is equivalent, by a theorem of Tutte, to the existence of k-edge-disjoint spanning trees.

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