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An undirected (directed) graph is called *Eulerian* if $d(v)$ is even ($\varrho(v) = \delta(v)$) for every node v.

Where S is a finite set, $X \subseteq S$ and $h : S \to R$ is a function we use the notation $h(X) := \sum_{x \in X} h(x)$.

In an undirected graph $G = (V, E)$ *splitting off* two edges uv and vz means the operation of replacing uv and vz by a new edge uz. Similarly, in a directed graph splitting off two arcs uv and vz is an operation that replaces uv and vz by a new arc uz. If $u = z$, we leave out the resulting loop uz.

The following two equalities will prove extremely useful. The first concerns directed graphs while the second is for undirected graphs. For $A, B \subseteq V$

(1.1) $$\varrho_G(A) + \varrho_G(B) = \varrho_G(A \cap B) + \varrho_G(A, B) + d_G(A, B)$$

(1.2) $$d_G(A) + d_G(B) = d_G(A \cap B) + d_G(A \cup B) + 2d_G(A, B)$$

The proof consists of showing that the contribution of any of the edges to the two sides of the equality is the same.

An obvious consequence of (1.1) and (1.2) is the submodular property of ϱ and d:

(1.1′) $$\varrho_G(A) + \varrho_G(B) \geq \varrho_G(A \cap B) + \varrho_G(A \cup B)$$

(1.2′) $$d_G(A) + d_G(B) \geq d_G(A \cap B) + d_G(A \cup B)$$

Sometimes more complicated relations are needed. Suppose that the node set V is partitioned into 5 sets; A, M, N, X, Y. Then

(1.3) $$d(X \cup M) + d(Y \cup M) + 2d(A, N) = d(X \cup N) + d(Y \cup N) + 2d(A, M).$$

The proof is an easy exercise.

The starting point of the whole theory is Menger's (1927) theorem. In what follows s and t are two specified nodes of the graph or digraph $G = (V, E)$ in question.

Theorem 1.1. *a, In a digraph (graph) there are k arc-disjoint (edge-disjoint) st-paths if and only if every $t\bar{s}$-set has at least k entering edges.*
b, In a digraph (graph) if there is no arc (edge) from s to t, there are k openly disjoint st-paths if and only if the paths from s to t cannot be covered by less than k nodes distinct from s and t.

(A set of paths is called openly disjoint if they are disjoint except for their end nodes).

Actually here we have four theorems according to whether directed or undirected and edge-(arc-)disjoint or openly disjoint st-paths are considered. Menger originally proved the undirected, openly disjoint version.

Although this theorem is included in almost every book concerning graphs, here we exhibit a proof since its basic idea, splitting off a pair of adjacent edges and the use of submodularity, is extensively used throughout the whole paper.

Proof. Let us first consider the arc-disjoint case. Let the minimum in question be $k > 0$. Call a $t\bar{s}$-set T *tight* if $\varrho(T) = k$.

Lemma. *If A and B are tight, then both $A \cap B$ and $A \cup B$ are tight, furthermore $d(A, B) = 0$.*

Proof. By (1.1) we have $k + k = \varrho(A) + \varrho(B) = \varrho(A \cap B) + \varrho(A \cup B) + d(A, B) \geq k + k + d(A, B)$ from which the lemma follows. □

We use induction on the number of edges. Let $e = uv$ be an edge with $v \neq t$. (If there is no such an edge, the theorem is trivial.) We can assume that e enters a tight set, for otherwise, by deleting e, we are done by induction. By the lemma there is a unique minimal tight set T that is entered by e. Now there is an edge vz with $z \in T$, for otherwise $\varrho(T - v) < k$. There is no tight set Z containing z but not containing u and v since then $Z \cap T$ is tight by the lemma contradicting the minimal choice of T. There is no tight set Z containing u and z but not v since then $d(Z, T) > 0$ contradicting the lemma again.

Therefore if we split off uv and vz, no $t\bar{s}$-set can arise with indegree less than k. By induction the resulting graph includes k edge-disjoint paths from s to t. Replacing back the new edge uz by uv and vz we obtain k edge-disjoint paths in G.

From the directed edge-version the other three cases of the Menger theorem follow by elementary construction. Namely, in case a, replace each edge by a pair of oppositely directed arcs and observe that if there is a set of k arc-disjoint paths in the resulting digraph, then there is one that does not use both arcs assigned to an original edge. The same construction yields the undirected openly-disjoint version from the directed one.

To see the directed openly-disjoint version construct a new digraph D' from D as follows. Replace each node v of D ($\neq s, t$) by a pair of new nodes v' and v''. Let $v'v''$ be an arc of D' and for an arc uv of D let $u''v$ be an arc of D'. Arc-disjoint st-paths in D' correspond to openly disjoint paths in D. Moreover, if there are k arcs in D' covering all st-paths, then these arcs can be assumed to be of type $v'v''$ and this set of arcs corresponds to a set of k nodes of D covering all st-paths. □

There exist other versions of Menger's theorem. For example, given a graph and two disjoint subsets S, T of its node set, there are k disjoint paths between S and T if and only if there are no $k-1$ nodes covering all such paths. By elementary construction this result easily follows from the original Menger theorem.

Since this paper is about paths and circuits let us close this introductory first section by mentioning a recent application of the Menger theorem.

Theorem 1.2 (Egawa, Kaneko and Matsumoto 1988). *In an undirected graph there are k edge-disjoint circuits passing through two specified nodes s and t if and only if every cut separating s and t contains at least $2k$ edges and after deleting any node distinct from s and t every cut separating s and t contains at least k edges.*

2. Disjoint Paths Problem

In this section we address the following problem, called the *disjoint paths problem*. Let us given a connected graph $G = (V, E)$ or a digraph and k pairs of nodes $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$. Find k pairwise disjoint paths connecting the corresponding pairs (s_i, t_i). If we are interested in finding edge-disjoint paths we speak about the *edge-disjoint paths problem.*

First, let us concentrate on undirected edge-disjoint paths. Sometimes it is convenient to mark the terminal pairs to be connected by an edge. The graph $H = (U, F)$ formed by the marking edges is called a *demand graph* while the original graph $G = (V, E)$ is the *supply graph.* (Of course, H may not be connected). In this terminology the edge-disjoint paths problem is equivalent to seeking for $|F|$ edge-disjoint circuits in $G+H$ each of which contains exactly one edge of F.

A natural necessary condition is the cut criterion:

CUT-CRITERION $d_G(X) \geq d_H(X)$ for every $X \subseteq V$.

Since any cut of G can be partitioned into bonds cut criterion holds if we require the inequality above only for subsets X for which both X and $V - X$ induce a connected subgraph.

We call $d_H(X)$ the *congestion* and the difference $s(X) := d_G(X) - d_H(X)$ the *surplus* of cut $\nabla(X)$ The cut criterion is equivalent to saying that the surplus of every cut is non-negative. A cut $\nabla(X)$ is called *tight* or *saturated* if $s(X) = 0$.

The cut criterion is sufficient if the demand graph consists of a set of parallel edges (in which case we are back at the undirected edge-version of Menger's theorem), or if H is a star (that is, the demand edges share a common endpoint). (This immediately follows from Menger).

The cut criterion is not sufficient, in general, as the following simple example shows (Figure 2.1).

REDUCTION PRINCIPLE. Let us introduce a simple device by which the edge disjoint paths problem can be reduced to a case when every degree in $G + H$ is at most 4. First replace each demand edge by a path of three edges such that the middle edge is a demand edge, the two other edges are supply edges. As a result, no demand edge is incident to a node of degree bigger than 2. Next let v be a node with degree at least 5. Replace this node and the incident edges as is shown in the picture 2.2.

It is easy to see that the edge-disjoint paths problem is solvable in the original graph if and only if it is solvable in the new graph. Applying this reduction at one node v as long as the degree of v is bigger than 4, we see that v is replaced by a subgraph displayed in Figure 2.2a.

The problem we obtain by eleminating all nodes of degree at least five is not only equivalent to the original problem but its size is a polynomial of the original size. Indeed, every node has been replaced by $O(d(v)^2)$ new nodes of degree four. For applications of the reduction principle, see Sections 3 and 4.

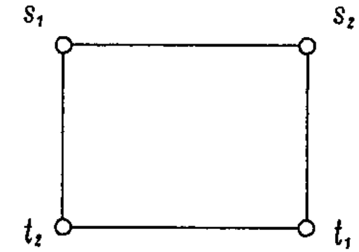


Fig. 2.1

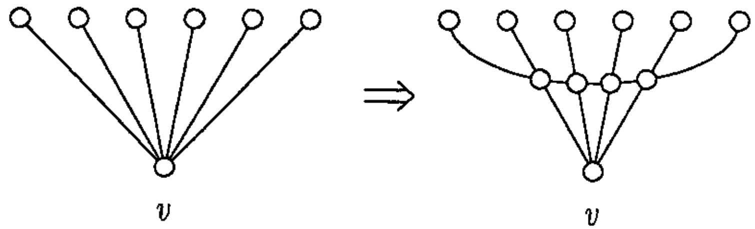


Fig. 2.2

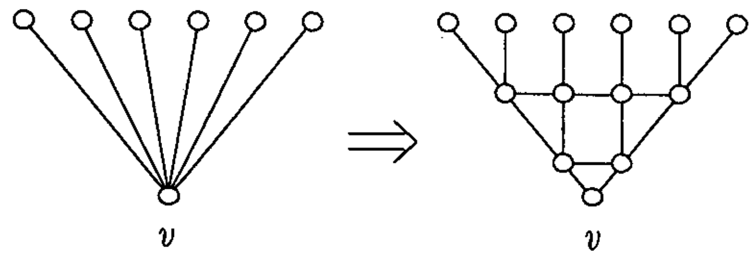


Fig. 2.2a

There is a natural relaxation of the edge-disjoint paths problem called *multicommodity flow*, or for short, the *multiflow* problem. Let G be undirected. The problem is to assign non-negative variables to paths connecting the prescribed terminal pairs $s_1t_1, s_2t_2, \ldots, s_kt_k$ so that for each terminal pair s_it_i the sum of variables assigned to paths connecting s_i and t_i is at least one and the sum of variables assigned to paths passing through any edge of G is at most one. (In the general multiflow problem one may have capacities on the edges).

Obviously a solution to the edge-disjoint paths problem is a 0-1 solution to the multiflow problem and vice versa. This is why we say that the edge-disjoint path problem *has a fractional solution* when its multiflow relaxation has a solution. Notice that the problem in Figure 2.1 has a fractional solution (assign 1/2 to the 4 paths of length 2).

One way to formulate the multiflow problem as a linear program is the following. Let A be a 0,1 matrix the rows of which correspond to the edges of G the colums correspond to the good circuits. An entry (i,j) is 1 if the edge corresponging to i is in the circuit corresponding to j and 0 otherwise. Similarly let B be a 0,-1 matrix the rows of which correspond to the edges of H the columns correspond to the good circuits. An entry (i,j) is -1 if the edge corresponding to i is in the circuit corresponding to j and 0 otherwise. (The structure of B is simple: every column has exactly one non-zero entry). The multiflow problem is equivalent to the following linear inequality system. $Ax \leq \underline{1}$, $Bx = -\underline{1}$, $x \geq 0$, where $\underline{1}$ and $-\underline{1}$ is appropriately sized vectors of 1's and -1's, respectively.

By Farkas' lemma this system has no solution if and only if there is a vector w in R^E_+ and a vector z in R^F such that $\Sigma(w(e) : e \in E) - \Sigma(z(f) : f \in F) < 0$ and such that $\Sigma(w(e) : e \in C - f) - z(f) \geq 0$ holds for every demand edge f and every circuit C for which $C \cap F = \{f\}$. Obviously, if there is such a w and z, then z can be chosen so as to satisfy $z(f) = dist_w(u,v)$ where $f = uv$ and $dist_w(u,v)$ is the minimum w-weight of a path in G connecting the end nodes of demand edge f.

Theorem 2.0 *The multiflow problem has a solution if and only if*

$$\text{DISTANCE CRITERION } \Sigma(dist_w(u,v) : uv \in F) \leq \Sigma(w(e) : e \in E)$$

holds for every vector $w \in \mathbf{R}^E_+$.

By chosing d to be 1 on the edges of a cut and 0 otherwise we see that the distance criterion implies the cut criterion. But not the other way round! In the next figure one can check by inspection that the cut criterion holds true but the distance criterion does not: choose w to be 1 everywhere.

This example also shows that the cut criterion is not sufficient in general even if $G + H$ is Eulerian. The next example, due to Éva Tardos, shows that even the stronger distance criterion is not sufficient (Figure 2.4).

Actually this is not surprising in the view of the following.

Theorem 2.1 (R. Karp 1972). *The undirected (edge-) disjoint paths problem (when k can vary) is NP-complete.*

The disjoint paths problem remains NP-complete for planar G and even for gridgraphs (a gridgraph is an induced subgraph of a rectilinear grid) (Richards), (Kramer and Leeuwen).

Even, Itai and Shamir (1976) proved that the problem is NP-complete in the special case when the demand graph consists of two sets of parallel edges. Recently, Middendorf and Pfeiffer (1989) proved that both the edge-disjoint

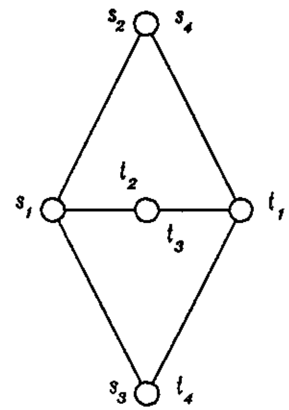


Fig. 2.3

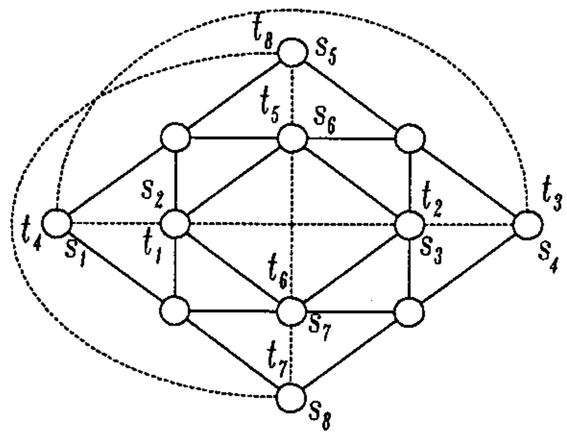


Fig. 2.4

and the node-disjoint paths problem is NP-complete if $G + H$ is planar even if every degree in $G + H$ is restricted to be at most 3. They also showed that the half-integer multicommodity flow problem is NP-complete. This implies that the edge-disjoint paths problem is NP-complete even if $G + H$ is Eulerian.

To consider the arc-disjoint paths problem in directed graphs let $D = (V, A)$ be a digraph and let (s_i, t_i) $(i = 1, 2, \ldots, k)$ be ordered pairs of terminals. The problem is to find arc-disjoint paths from s_i to t_i.

Let $H = (U, F)$ denote the demand digraph, where $F = \{(t_i s_i : i = 1, 2, \ldots, k\}$. Then the problem can be reformulated as follows: Find k arc-disjoint circuits in $G + H$ each of which contains exactly one demand edge.

Again a natural necessary condition is available:

DIRECTED CUT CRITERION $\varrho_G(X) \geq \delta_H(X)$ for every $X \subseteq V$.

If $s_1 = \ldots = s_k$ and $t_1 = \ldots = t_k$, then the directed cut criterion is sufficient as well (directed arc version of Menger's theorem). It remains true, via an elementary construction, if we require only $s_1 = \ldots = s_k$.

For general digraphs one has the following negative result.

Theorem 2.2 (Fortune, Hopcroft and Wyllie 1980). *The (arc-) disjoint paths problem is NP-complete for $k = 2$.*

To close this section we formulate a necessary condition for the disjoint path problem (in an undirected graph).

NODE-CUT CRITERION. The counterpart of the cut condition requires that a subset S of nodes must not separate more than $|S|$ terminal pairs.

This condition is sufficient if the terminal pairs share a common node (a node-version of the Menger theorem) but not in general. Another special case when the node-cut criterion is sufficient is the following.

Theorem 2.3 (N. Robertson and P. Seymour 1986). *Suppose that G is planar and the terminals are on the outer face. This disjoint paths problem has a solution if and only if the node-cut condition holds and there are no two "crossing" terminal pairs (that is, any two pairs (s_1, t_1) and (s_2, t_2) are in this order on the outer face: s_1, t_1, s_2, t_2).*

(The proof of this result is easy).

3. $G + H$ is Eulerian

In Section 2 we saw how submodularity can be used for proving Menger's theorem. Let us start this section by claiming a simple lemma that makes possible some more sophisticated uses of submodularity.

Let $G = (V, E)$ and $H = (V, F)$ be two graphs for which the cut criterion holds, that is $d_G(X) \geq d_H(X)$ for every $X \subseteq V$. Call a subset X of nodes tight if $d_G(X) = d_H(X)$.

Lemma 3.1. *a, If A and B are tight and $d_H(A, B) = 0$, then both $A \cap B$ and $A \cup B$ are tight and $d_G(A, B) = 0$. b, If A and B are tight and $d_H(A, \overline{B}) = 0$, then both $A - B$ and $B - A$ are tight and $d_G(A, \overline{B}) = 0$.*

Proof. By applying (1.2) to G and H we have
$d_H(A) + d_H(B) = d_G(A) + d_G(B) = d_G(A \cap B) + d_G(A \cup B) + 2d_G(A, B) \geq d_H(A \cap B) + d_H(A \cup B) + 2d_G(A, B) = d_H(A) + d_H(B) + 2(d_G(A, B) - d_H(A, B))$
from which part a, follows. We obtain part (b) if (a) is applied to A and $V - B$. □

In this section we outline the edge-disjoint paths problem when $G + H$ is Eulerian. It was already mentioned that the edge-disjoint paths problem can be formulated in terms of packing of circuits. When $G + H$ is Eulerian, the problem is equivalent to finding a partition of the edge set of $G + H$ into circuits each of which contains at most one edge from H. Such a partition will be called *good*. Figure 2.3 shows that the cut criterion is not sufficient in general even if G is planar.

However, there are important special cases when the cut criterion is sufficient. In one class of examples the supply graph G is planar and there are additional restrictions on H. In another class G is arbitrary but the demand graph H is rather restricted.

First let us survey the results concerning planar G.

Theorem 3.2 (Okamura and Seymour 1981). *Suppose that G is planar, $G + H$ is Eulerian, and each terminal is on one face of G. Then the cut criterion is necessary and sufficient for the solvability of the edge-disjoint paths problem.*

Proof (Okamura and Seymour 1981). By induction on the number of edges of G. Let G be embedded in the plane. We can assume that G is 2-connected. Then every face is bounded by a circuit. Let C denote the circuit bounding the infinite face and let the subscripts of the nodes $v_1, \ldots, v_k$ of C reflect the cyclic order. Assume that the terminals are on C.

Choose an edge e of C which is in a tight cut (if there is no tight cut any $e \in E(C)$ will do) and renumber the nodes of C such that $e = v_h v_1$. Let A be a minimal tight set containing v_1 but not v_h. Choose a demand edge $f = v_i v_j$ $(i < j)$ such that $v_i \in A$, $v_j \notin A$ and j is as big as possible. (If there is no tight set at all, any demand edge will do).

Delete e from G and replace f by $v_1 v_i$ and $v_j v_h$. We are going to show that the cut criterion holds with respect to the resulting $\overline{G}$ and $\overline{H}$. This will imply the theorem since $\overline{G}$ has one less edge than G and the other hypotheses of the theorem hold for $\overline{G}$ and $\overline{H}$. So by induction we have the edge-disjoint paths in $\overline{G}$. This provides the required edge-disjoint paths in G if we observe that glueing together the path between v_1 and v_i and the path between v_j and v_h and the edge e we obtain a path between v_i and v_j.

If the cut criterion, indirectly, fails to hold for $\overline{G}$ and $\overline{H}$, then there is a set B which is tight with respect to G and H and, among the four nodes v_1, v_i, v_j, v_h, B contains exactly (i) v_1, (ii) v_h, (iii) v_1 and v_h.

By Lemma 3.1 if A and B are tight and $d_H(A, B) = 0$, then both $A \cap B$ and $A \cup B$ are tight and $d_G(A, B) = 0$.

By the choice of f in each case we have $d_H(A, B) = 0$ so Lemma 3.1 applies. Thus $A \cap B$ is tight which in Cases (i) and (iii), contradicts the minimal choice of A. Lemma 3.1 also implies that ~~saved~~$d_G(A, B) = 0$ showing that Case (ii) cannot occur either (in Case (ii) $d_G(A, B) > 0$ because of edge e). □

Remark. Around the same time when Okamura and Seymour proved their theorem S. Lins (1981) showed that the maximum number of edge-disjoint non-separating circuits in an Eulerian graph embedded into the projective plane is equal to the minimum cardinality of a non-separating cut. This theorem in the present context is nothing but the theorem of Okamura and Seymour's theorem in the special case when all the terminals are distinct and they are positioned around the specific face in the cyclic order $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$. However, it can be shown by a simple trick that this special case implies the Okamura-Seymour theorem. Indeed, if there are two terminals s_1, s_2 sitting at the same node u, then add two new nodes v_1, v_2 and two new edges uv_1, uv_2 to the graph and move terminal s_1 to v_1 and s_2 to v_2. Applying this operation we can ensure that the terminals are distinct. The requirement on the cyclic order of the terminals is equivalent to saying that any two terminal pairs s_it_i and s_jt_j crosses each other, that is, their cyclic order is s_i, s_j, t_i, t_j If this is not the case, then there are two non-crossing terminal pairs s_it_i and s_jt_j such that one of s_i and t_i, say s_i, and one s_j and t_j, say s_j, are consecutive in the cyclic order (this is an easy exercise). Now modify the graph and the position of s_i and s_j as is depicted in Figure 3.1.

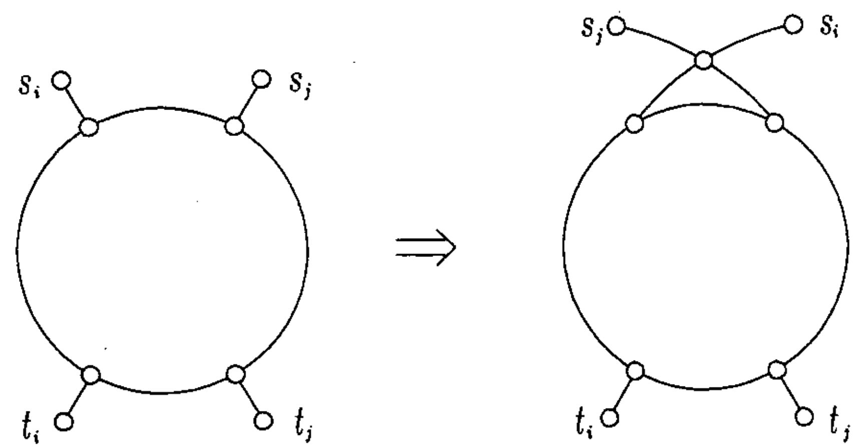


Fig. 3.1

It is easily seen that the cut criterion is satisfied for the new problem if it is satisfied for the old and if the required paths exist in the new problem, then so do they in the old. Furthermore the number of crossing terminal pairs is one bigger in the new problem. Applying this technique as long as there are non-crossing terminal pairs finally we obtain a problem which is equivalent to the original one and the terminals satisfy Lins' requirement.

H. Okamura generalized the theorem of herself and Seymour in two directions. The first one is:

Theorem 3.3 (Okamura 1983). *Suppose that G is planar, $G+H$ is Eulerian, and there are two faces C_1, C_2 such that each demand edge connects two nodes of either C_1 or C_2. Then the cut criterion is necessary and sufficient for the solvability of the edge-disjoint paths problem.*

The proof below (due to Gábor Tardos (Tardos 1984)) is a slightly simplified version of Okamura's original proof.

Proof. Again, we can assume that G is 2-connected. We say that a set K of nodes crosses a face if K contains a node of the face but not all. If there is a tight set crossing only one of the two specified faces C_1 and C_2, then the reduction step used in the proof of Okamura and Seymour's theorem can be applied. (Notice that the crucial equality $d_H(A, B) = 0$ in that proof cannot spoil down since every terminal pair is either on C_1 or on C_2.)

So assume that every tight set crosses both C_1 and C_2. Assume that a terminal pair st is in C_1 and that C_1 is the outer face of G. (It will cause no confusion that we use the same term C_i to denote the graph-circuit of G bounding the face C_i.) The nodes s and t divide C_1 into two paths P and Q connecting s and t.

First, delete the edges of P from G and remove the demand edge st from H. For the resulting G_1 and H_1 the hypotheses of the theorem hold and then we are done if the cut criterion is satisfied. So assume this is not the case. Then there is a set K which is tight with respect to G and H such that $s, t \notin K$ and K intersects P.

Second, delete the edges of Q from G and remove the demand edge st from H. Analogously to the first case, we are in trouble only if there is a set L tight with respect to G and H such that $s, t \notin L$ and L intersects Q.

Let $Z := V - (K \cup L)$. Since both K and L cross C_2, in the subgraph induced by Z there is no path connecting s and t. Therefore there is a partition of Z into two sets A and N with $s \in N$, $t \in A$ such that $d_G(A, N) = 0$ Let us introduce the following notation: $M := K \cap L$, $X := K - L$, $Y := L - K$. If M is non-empty, then at least one of A and N, say A, is disjoint from C_2. Theorefore $d_H(A, M) = 0$ and this is also true if $M = \emptyset$.

We will apply formula (1.3) from Section 1:

$$d(X \cup M) + d(Y \cup M) + 2d(A, N) = d(X \cup N) + d(Y \cup N) + 2d(A, M)$$

Now $X \cup M$ and $Y \cup M$ are tight and $d_H(A, M) = 0 = d_G(A, N)$, thus we have

$$0+0 = s(X \cup M) + s(Y \cup M) = s(X \cup N) + s(Y \cup N) + 2[d_G(A, M) + d_H(A, N)] \geq 0.$$

Therefore each term is 0, in particular, $d_H(A, N) = 0$. But this is impossible since the demand edge st leads between A and N. □

Okamura's other generalization of Okamura and Seymour's theorem is as follows.

Theorem 3.4 (Okamura 1983). *Let G be planar, $G+H$ Eulerian, C a specified face of G and s is a node of C. Suppose that each terminal pair has either both members on C or one member at s. Then the cut criterion is necessary and sufficient for the solvability of the edge-disjoint paths problem.*

There is a recent result by A. Schrijver of similar vein concerning path-packing problems in a planar graph.

Theorem 3.5 (Schrijver 1988b). *Let G be planar, $G+H$ Eulerian and let C_1 and C_2 be two specified inner faces of G. Assume that the demand edges $s_1t_1, \ldots, s_kt_2$ are such that each s_i is on C_1 and each t_i is on C_2 and their cyclic order is the same. Then the cut criterion is necessary and sufficient for the solvability of the edge-disjoint paths problem. (Notice that if C_1 is chosen to be the outer face of G then the cyclic orders should be opposite.)*

In Theorems 3.3 and 3.5 G is planar $G+H$ is Eulerian and the terminals are on two specified faces. Figure 2.3 shows that if we do not impose some extra conditions on the terminals, then the cut condition is not sufficient, in general. In the example in Figure 2.3 even no fractional solution exists. Thus one may suspect that under the circumstances above the existence of a fractional solution already implies solvability. However this is not the case as is shown in Figure 2.4.

Here is yet another fundamental result concerning planar graphs.

Theorem 3.6 (Seymour 1981). *Suppose that $G+H$ is planar and Eulerian. Then the cut criterion is necessary and sufficient for the solvability of the edge-disjoint paths problem.*

Proof (Z. Zubor 1989). We can assume that every edge $e \in E$ is in a tight cut since otherwise e can be moved from E into F without destroying the cut criterion. By the reduction principle we can assume that in $G+H$ every degree is 2 or 4. Suppose that $G+H$ is a counter–example with a minimum number of nodes of degree 4. Define

$$w : E \cup F \to \{+1, -1\} \text{ by}$$

$$w(e) := \begin{cases} +1 & \text{if } e \in E \\ -1 & \text{if } e \in F. \end{cases}$$

The cut criterion is equivalent to: $d_w(X) \geq 0$ for every $X \subseteq V$. We need the following observation of A. Sebő (1987b).

Claim. *Let $A \subseteq V$ be tight, i.e. $d_w(A) = 0$, and define*

$$w'(e) = \begin{cases} w(e) & \text{if } e \notin \nabla(A) \\ -w(e) & \text{if } e \in \nabla(A). \end{cases}$$

Then $d_{w'}(X) \geq 0$ for every $X \subseteq V$.

Proof. We have $d_{w'}(X) = d_w(A \oplus X) - d_w(A) = d_w(A \oplus X) \geq 0$. ($A \oplus X$ denotes $(A - X) \cup (X - A)$.) □

By *interchanging along* a cut C we mean an operation that replaces F by $F \oplus C$ and E by $E \oplus C$. By the Claim the theorem holds for $G+H$ if and only if it holds after interchanging along a tight cut.

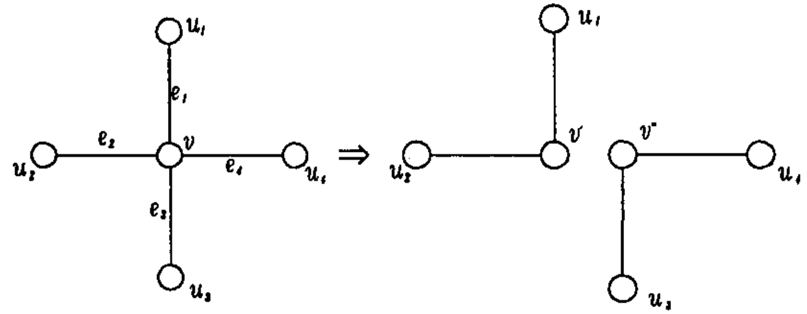


Fig. 3.2

Let vu_1 be a demand edge. Assume that the four edges $e_i = vu_i$ $(i = 1,2,3,4)$ incident to v are indexed in cyclic order so that $e_1 \in F, e_2 \in E$. Modify slightly the "splitting off" operation as follows. Replace v by v' and v'' so that v' is connected to u_1 and u_2 and v'' is connected to u_3 and u_4 (Figure 3.2).

Let $G' = (V', E')$ and $H' = (V', F')$ denote the resulting graphs. If there were a solution to the edge–disjoint paths problem in $G' + H'$, there would be one in $G+H$. Thereby there is a bond $\nabla'(A)$ for which $d_{G'}(A) < d_{H'}(A)$. We can assume that $v' \in A$. Since the cut criterion holds for $G+H$ we have

(*) $v'' \notin A$ and an edge e_i $(i = 1,2,3,4)$ belongs to $\nabla'(A)$ precisely if $e_i \in F$.

These are two cases.

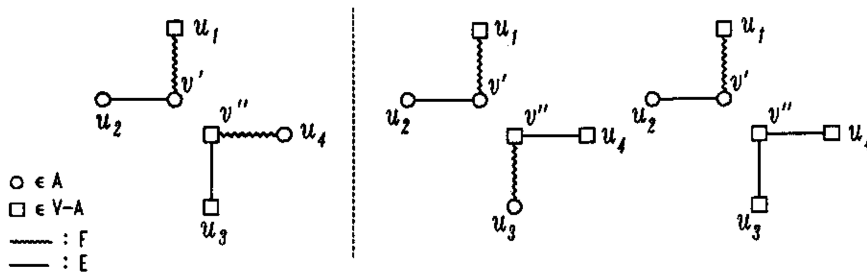


Fig. 3.3

Case 1. $e_4 \in F$. By (*) $u_2, u_4 \in A$ and $u_1, u_3 \notin A$. Both A and $V' - A$ induce a connected subgraph of $G' + H'$ contradicting the planarity of $G' + H'$.

Case 2. $e_4 \in E$ By (*) $u_2 \in A$ and $u_1, u_4 \notin A$. Now $A - v'$ is tight in $G+H$. By interchanging along $\nabla(A - v')$ (and re–indexing the e_i's) we are at Case 1. □

It is a challenging open problem to find a unified theorem that implies all the "planar" results above.

Actually Seymour proved a result more general than Theorem 3.6. If we take planar dual, then the role of circuits and cuts is interchanged, in particular Eulerian turns into bipartite. It turns out that planarity can be left out from the hypotheses:

Theorem 3.6′ (Seymour 1981a). *Suppose that $G+H$ is bipartite. There are $|F|$ edge-disjoint cuts in $G+H$, each containing exactly one element of F if and only if every circuit of $G+H$ contains as many edges from G as from H.*

In Section 8 (Theorem 8.1) we will prove this result along with some generalizations. Also a strongly polynomial-time algorithm will be provided for the more general weighted case. P. Seymour found another generalization of Theorem 3.6.

Theorem 3.7 (Seymour 1981b). *Suppose that $G+H$ is Eulerian and no subgraph of it can be contracted to K_5 (complete graph on 5 nodes). Then the cut criterion is necessary and sufficient for the solvability of the edge-disjoint paths problems.*

Let us now turn to another class of graphs when, supposing $G+H$ Eulerian, the cut criterion is sufficient.

For a given demand graph $H=(V,F)$, H' will denote the graph arisen from H by replacing each (maximal) set of parallel edges by one edge. Let us call a graph a *double star* if there are at most two nodes that cover all the edges. In what follows K_n denotes the complete graph on n nodes and C_5 denotes a circuit on 5 nodes. Let K_2+K_3 denote a graph on 5 nodes with components K_2 and K_3. Similarly $3K_2$ denotes a graph consisting of three disjoint edges.

Theorem 3.8. *Suppose that $G+H$ is Eulerian and H' is either a double-star or K_4 or C_5. Then the cut criterion is necessary and sufficient for the solvability of the edge-disjoint paths problem.*

The case when H' is $2K_2$ was proved by Rothschild and Whinston (1966b), sharpening earlier results of T. C. Hu (1963). From this the theorem easily follows for double-stars (see below). The K_4 case was proved by P. Seymour (1980a) and M. Lomonosov (1979), independently. The C_5 case is due to Lomonosov (1979).

The theorem is sharp in the sense that if H' is different from each of the three graphs in the theorem, then there is a G and H such that $G+H$ is Eulerian, the cut criterion holds but there is no solution to the edge-disjoint paths problem. To see this, observe that the example in Figure 2.3 shows that H' must not contain K_2+K_3 as a subgraph. The example in Figure 3.4 shows that H' must not contain $3K_2$. This is due to Papernov (1976).

It is an easy exercise to show that if a graph contains neither of these two forbidden graphs, then it is either a double star or K_4 or C_5.

Proof of Theorem 3.8. Suppose the $G+H$ is counterexample with a minimum number of edges. Obviously $G+H$ is connected. We need some preparation that is useful for each case.

Claim 1. There are no edges $e \in E$ and $f \in F$ that are parallel.

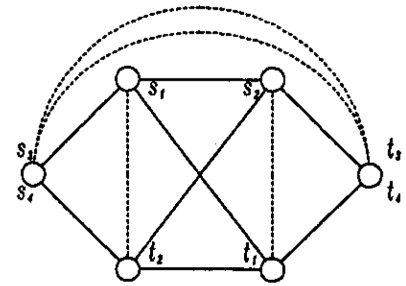


Fig. 3.4

Proof. Deleting e and f does not destroy the cut criterion and then a good circuit-partition of the smaller graph along with circuit $\{e,f\}$ would form a good circuit-partition of $G+H$. □

Lemma 3.1 implies:

Claim 2. Let vz be an edge of G where v is not a terminal node. Let A and B be two (distinct) maximal tight $z\bar{v}$–sets. Then $d_H(A,\overline{B})>0$ and $d_H(A,B)>0$. □

Let $A_1,A_2,\dots,A_k$ be the maximal tight $z\bar{v}$–sets. Suppose that there is an edge uv which does not occur in any tight cut. Then splitting off uv and vz does not destroy the cut criterion. So the resulting graph has a good circuit-partition. But this provides a good circuit-partition of $G+H$ which is impossible. Therefore every edge vu enters some A_i.
Claim 3. $k \geq 3$.

Proof. If $k=1$, then every neighbour of v is in $\nabla(A_1)$ but then $\nabla(A_1+v)$ would violate the cut condition. If $k=2$ and $d_G(v,A_1)\ge d_G(v,A_2)$, say, then $d_G(A_1+v) < d_G(A_1)$ (because of edge vz) and therefore $\nabla(A_1+v)$ violates the cut criterion. □

Turning to the different cases of the theorem let us first assume that $H'=2K_2$ and the demand edges are between s_i and t_i ($i=1,2$).

We claim that there is a node v which is not a terminal. Indeed, if no such a node exists, then, by Claim 1, G must be a four-circuit with possible parallel edges. But this cannot be a counterexample as is seen by inspection (or by the theorem of Okamura and Seymour).

Therefore there is an edge vz of G where $z=s_1$ and v is not a terminal node. By Claim 3 there are at least three maximal tight $z\bar{v}$-sets A_1,A_2,A_3. By Claim 2 $d_H(A_i,A_j)>0$ $(1\le i<j\le 3)$ but this is impossible since $s_1\in A_1\cap A_2\cap A_3$.

Next we show how the double star case reduces to the case $H'=2K_2$. Let s_1 and s_2 be the two nodes covering the edges of a double-star H. First subdivide each demand edge s_1t_i by a node t_i' such that s_1t' belongs to the demand graph and $t_i't_i$ to the supply graph. Then contract the nodes t_i' into one node. Finally do the same with the demand edges incident to s_2. This way we obtain a new problem which is equivalent to the original one and the demand graph consists of two sets of parallel edges.

Suppose that $H' = K_4$ and the four terminal nodes are $s_1, \ldots, s_4$. By Claim 1 there is no edge in G connecting two terminal nodes.

Let us denote s_1 by z. If there is no tight set containing s_1, s_2 and not containing s_3, s_4, then let vz be any edge in G with vs_i. If there is one, then the intersection Z of such sets is tight by Lemma 3.1. We claim that there is a $v \in Z - \{s_1, s_2\}$ such that vz is an edge of G. For otherwise, $d_G(Z) = d_G(z) + d_G(Z - z) \geq d_H(z) + d_H(Z - z) = d_H(Z) + 2d_H(z, Z - z) = d_G(Z) + 2d_H(z, Z - z) > d_G(Z)$, a contradiction.

By Claim 3 there are at least three maximal tight $z\bar{v}$–sets A_1, A_2, A_3. By Claim 2 $d_H(A_i, A_j) > 0$ $(1 \leq i < j \leq 3)$. But this is possible only if each of A_1, A_2 and A_3 contains a terminal node which is not in the union of the two others. Assume that A_2 contains s_2. Then A_2 is a tight set containing s_1, s_2 and not s_3, s_4 and v. This contradicts the choice of v and the definition of Z and thus the case of K_4 is settled.

Finally let us assume that $H' = C_5$. If $|V| = 5$, then, by Claim 1 G is a subgraph of a 5-circuit with possible parallel edges. But then the Okamura-Seymour theorem shows that $G + H$ cannot be a counterexample. So let vz be an edge of G where v is not a terminal.

By Claim 3 there are at least three maximal tight $z\bar{v}$-sets A_1, A_2, A_3. By Claim 2 (*) $d_H(A_i, A_j) > 0$ and $d_H(A_i, \overline{A}_j) > 0$ $(1 \leq i < j \leq 3)$. Then each A_i contains 2 or 3 terminals. The complement of A_i is also tight so we can assume that there are three tight sets B_1, B_2, B_3 for which (*) holds and each of them contains exactly two terminals. Now if $B_1 \cap B_2 \cap B_3$ contains a terminal node, then each of B_1, B_2, B_3 contains a terminal node which is not in the union of the two others. But then these three terminals must form a triangle in H' which is impossible.

Suppose now that $B_1 \cap B_2 \cap B_3$ contains no terminal node. Since $B_i \cap B_j$ contains a terminal node $(1 \leq i < j \leq 3)$ the other two terminal nodes must be outside $\cap B_i$ and then we must have again a triangle in H', a contradiction. □

Each of Theorem 3.2 through 3.8 has a fractional version as an easy consequence. For example:

Theorem 3.8′ (Papernov 1976). *Let G be arbitrary and H as in Theorem 3.8. Then the cut criterion is necessary and sufficient for the solvability of the multiflow problem.*

There is a very useful device by which the reverse implication can also be proved. The idea, noticed by van Hoesel and Schrijver (1990), is as follows. (For more details, see (Schrijver 1988a)).

Proof of Theorem 3.8 from Theorem 3.8′. Let x be a solution to the multiflow problem and P a path for which $x(P) > 0$. Let v be any inner node of P and uv and vz the two edges of P incident to v. We claim that uv and vz can be split off without violating the cut criterion. Indeed, if the cut criterion does not hold after the splitting, there is a tight cut of G that contains both uv and vz. But this is impossible since a simple argument shows that any tight cut and any path Q with $x(Q) > 0$ have at most one edge in common. □

One can similarly proceed to derive Theorems 3.2–3.6 from their corresponding fractional version. However, in order to maintain planarity, certain care is required while chosing the pair of edges to be split off:

Proof of Theorems 3.2–3.6 from the corresponding fractional versions. First, by the reduction principle described in Section 2 we assume that in $G + H$ every node has degree four. Let x be a solution to the corresponding multiflow problem (in either of Theorems 3.2–3.6). If there is a path P and an inner node v of P such that $x(P) > 0$ and the two edges uv and vz of P are in the same face of G, then splitting off these edges preserves not only the cut criterion but also the planarity. If no such a path exists (that is, for every inner node v of any path P with $x(P) > 0$ goes "across" v), then for every terminal pair (s, t) there can be only one path P with $x(P) > 0$ connecting s and t. Consequently, x is 0-1 valued, that is, x itself is a solution to the corresponding edge-disjoint paths problem. □

Of course, the reduction method above can be considered useful only if there is a direct way to prove the "fractional" theorems. In Section 8 we indicate such a method.

By applying the splitting off technique to directed graphs a directed counterpart of the theorem of Rothschild and Whinston can be proved. By a *(directed) star* we mean a directed graph in which either all the edges enter the same node or all the edges leave the same node.

Theorem 3.9 (Frank 1985). *Suppose that $G + H$ is an Eulerian digraph and H is the union of two stars. Then the directed cut criterion is necessary and sufficient for the solvability of the undirected edge-disjoint paths problem.*

The following figure shows some small H which are not the union of two stars, the directed cut condition holds but there is no solution.

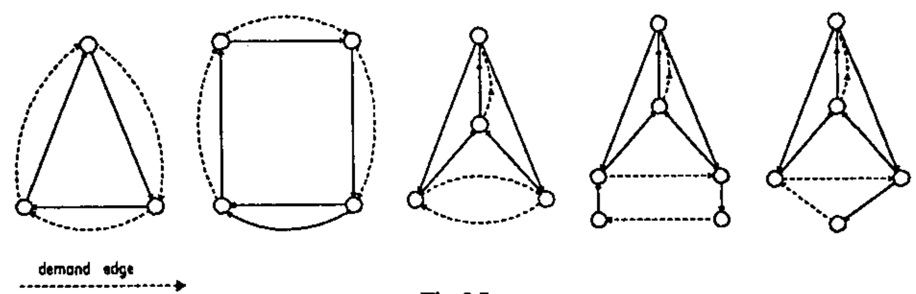


Fig. 3.5

Let us conclude this section by citing two recent results of Karzanov concerning undirected G and H.

Theorem 3.10 (Karzanov 1987). *Suppose that $G+H$ is Eulerian and the demand edges form a K_5. Then the distance criterion is necessary and sufficient for the solvability of the edge-disjoint paths problem. (In other words, if there is a fractional solution, there is an integral one.)*

Theorem 3.11 (Karzanov 1989a). *Suppose that $G+H$ is Eulerian, G is planar and the demand edges are on three faces of G. Then the distance criterion is necessary and sufficient for the solvability of the edge-disjoint paths problem.*

4. Further Necessary Conditions

The purpose of this section is to introduce some further necessary conditions concerning the (edge-) disjoint path problem. They belong to two classes. The first one is a kind of topological obstruction while the second is based on parity arguments.

Let G and H be undirected. We know that the cut-criterion is sufficient when H is a star. Suppose now that H consists of two disjoint edges. The following characterization appears in three different papers: E.A. Dinits and A.V. Karzanov (1979), P. Seymour (1980) and C. Thomassen (1980).

Theorem 4.1. *Let G be a graph such that no cut edge separates both of the two terminal pairs (s_1,t_1) and (s_2,t_2). There is no two edge-disjoint paths between the corresponing terminal pairs if and only if some edges of G can be contracted so that the resulting graph G' is planar, the four terminals have degree two while the other nodes are of degree 3 and the terminals are positioned on the outer face in this order: s_1,s_2,t_1,t_2.*

Figure 4.1 shows a typical example where the two edge-disjoint paths do not exist.

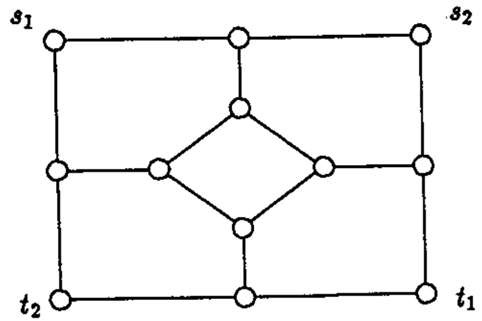


Fig. 4.1

Recall that if one wants k_i paths between s_i and t_i $(i=1,2)$, then the problem becomes NP-complete. The necessity of the condition in the theorem depends on three observations. Namely, edge-contraction does not destroy solvability, a node of degree three can be used by at most one path, and two curves in the plane connecting antipodal pairs of points of a circle must intersect each other.

Actually this theorem immediately follows from the following node-version by considering the line graph.

Theorem 4.2 (Thomassen 1980a, Seymour 1980). *Let G be a graph such that no node separates s_1 from t_1 and s_2 from t_2. There are no disjoint paths between s_1 and t_1 and between s_2 and t_2 if and only if G arises from a planar graph G', where the four terminals are one the outer face in this order s_1,s_2,t_1,t_2, by placing an arbitrary graph into some faces of G' bounded by two or three edges.*

For directed graphs the two disjoint paths problem is NP-complete. However, Thomassen (1985) found a complete description of acyclic digraphs have no solution to the 2-disjoint paths problem. The core of his result os as follows.

Theorem 4.3. *Let us be given an acyclic digraph $D=(V,A)$ (with no cut-node and parallel edges) and terminal pairs (s_1,t_1), (t_2,t_2) such that $|V|\geq 5$, $\varrho(v),\delta(v)\geq 2$ for each non-terminal node v and $\varrho(s_1)=\varrho(s_2)=\delta(t_1)=\delta(t_1)=\delta(t_2)=0$. If there are no disjoint paths from s_1 to t_1 and from s_2 to t_2, then D is planar and has a plane representation in such a way that s_1,t_2,t_1,s_2 are on the outer face occuring in that cyclic order.*

Notice that in these theorems the hypotheses are purely graphical and topological arguments come only in the characterization. But one can be interested in disjoint paths in a graph embedded in a plane with certain holes such that the paths must satisfy a certain homotopy requirement. (That is, the topological way how the paths have to go around the holes is spedified.) This general problem is precisely the central topic of A. Schrijver's article in this volume.

Let us turn to the other class of necessary conditions and consider the edge-disjoint paths problem in an undirected graph. In the preceding section we have considered special classes when $G+H$ is Eulerian, that is, when $d_{G+H}(X)$ is even for every subset of V. Let us now call a set X *odd* (or the cut ∇_G odd) if $d_{G+H}(X)$ is odd. It is useful to observe that the number of odd nodes is always even and that a set X is odd if and only if X contains an odd number of odd nodes.

The crucial observation concerning odd cuts is that, given an odd set X and any solution to the edge-disjoint paths problem, an odd number of edges of $\nabla_G(X)$, in particular at least one edge, can not be used by the paths in the solution. (Actually, we have already relied on an special case of this idea when we argued after Theorem 4.1 that no two edge-disjoint paths can go through a node of degree three.)

Thus this parity argument provides a kind of force that intuitively prevents a solution to use too many edges. On the other hand, in a tight cut all of the edges are necessarily used. Or more generally, for a set X with surplus $s(X)$ ($= d_G(X) - d_H(X)$) there may be at most $s(X)$ edges in $\nabla_G(X)$ which are not used by a solution. Thus this surplus argument provides a kind of force that intuitively prevents a solution to use too few edges.

These two forces of opposite directions are the basis of each of the following necessary conditions. For example, it is necessary that

(4.1) $\nabla_G(D)$ cannot be covered by two tight cuts for any odd set D.

Observe that (4.1) is not satisfied by the graph in Figure 2.1. We mention three cases when (4.1) is sufficient, as well.

Suppose first that H consists of two sets of parallel edges, that is, there are two terminal pairs. We call a set X of nodes (and the cut $\nabla(X)$)) *separating* if $\nabla(X)$ separates both terminal pairs. Two separating sets X and Y are called *parallel* if either $X - Y$ or $X \cap Y$ is separating. Otherwise they are *non-parallel.* We say that a set X *crosses* C if $X \cap C$ and $C - X$ are non-empty.

Theorem 4.4 (Seymour 1981a). *Suppose that $G+H$ is planar and H consists of two sets or parallel edges. The edge-disjoint paths problem has a solution if and only if the cut criterion holds and (**) $d_{G+H}(S \cap T)$ is even for any two tight non-parallel separating sets S, T.*

Proof. We can assume that G is 2-connected. Assume that there are k_i demand edges connecting s_i and t_i, the terminal pair (s_i, t_i) is on face C_i ($i = 1, 2$) and that C_i is the outer face. We can assume that both $k_1 \geq k_2 > 0$, since otherwise Menger's theorem applies. Let us recall that if the cut criterion does not hold, then there is bond $\nabla(K)$ violating it. Because of planarity K divides any facial circuit of G into at most two paths.

The nodes s_1 and t_1 divide C_1 into two paths P and Q connecting s_1 and t_1. First, delete the edges of P from G and remove one demand edge connecting s_1 and t_1 from H.

Assume first that the resulting G_1 and H_1 satisfy the cut criterion. One can observe that if X and Y violate (**) for G_1 and H_1, then X and Y violate (**) with respect to G and H. Then, by induction, there is a solution with respect to G_1 and H_1 and this solution along with path P yields a solution with respect to G and H. So we can assume that there is a set K violating the cut criterion with respect to G_1 and H_1. Then $s_1, t_1 \notin K$ intersects P and the surplus $s(K) \leq 1$. Then $\nabla(K)$ necessarily separates s_2 and t_2 and K crosses C_2.

Similarly, delete the edges of Q from G and remove one demand edge connecting s_1t_1 from H. Analogously to the first case, we are in trouble only if there is a set L with surplus $s(L) \leq 1$ such that $s_1, t_1 \notin L$ and L intersects Q.

Let $Z := V - (K \cup L)$. Since both K and L cross C_2, in the subgraph of G induced by Z there is no path connecting s_1 and t_1. Therefore there is a partition of Z into two sets A and N with $s_1 \in N$, $t_1 \in A$ such that $d_G(A, N) = 0$. Let us introduce the following notation: $M := K \cap L$, $X := K - L$, $Y := L - K$. If M is non-empty, then at least one of A and N, say A, is disjoint from C_2. Therefore $d_H(A, M) = 0$ and this is also true if $M = \emptyset$.

We will apply formula (1.3) to both G and H. Exploiting that $d_H(A, M) = 0 = d_G(A, N)$, we have

$$1 + 1 \geq s(X \cup M) + s(Y \cup M) = s(X \cup N) + s(Y \cup N) +$$
$$2[d_G(A, M) + d_H(A, N)] \geq 0 + 0 + 2[0 + 1] = 2. \tag{4.2}$$

Therefore equality holds everywhere and, in particular, $s(X \cup M) = s(Y \cup M) = 1$, $s(X \cup N) = s(Y \cup N) = 0$, $d_G(A, M) = 0$, $d_G(A, N) = 1$. The last equality shows that $k_1 = 1$. Since $k_1 \geq k_2 > 0$, we have $k_2 = 1$. (This means that the two edges leaving K are common edges of C_1 and C_2).

Since $s(K) = s(L) = 1$ it follows that $d_G(K) = d_G(L) = 2$ and that $d_{G+H}(K) = 3$. Now $M = K \cap L$ must be empty for if a node v is in $K \cap L$, then there is a path in K from v to P. But such a path leaves L along an edge that is not in C_1. So we would have $d_G(L) \geq 3$. See Figure 4.2. We see that $K = X$ and $L = Y$.

Since M is empty $d_H(N, M) = 0$. (4.2) can be applied with interchanging A and N. We obtain that $Y \cup A$ is tight. Let $S =: V - (Y \cup A)$ ($= K \cup N$) and $T := V - (Y \cup N) = (K \cup A)$. Now S and T violate (**) since S and T are tight, $K = S \cap T$ and $d_{G+H}(K)$ is 3, an odd number. □

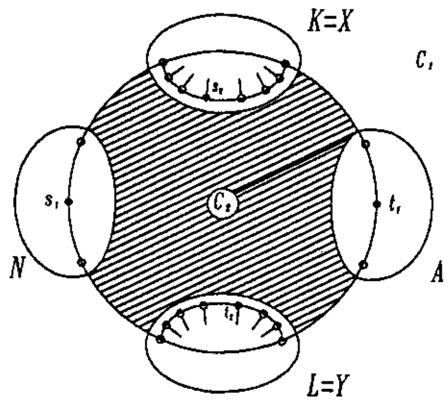


Fig. 4.2

Remark. The original proof of Seymour relies on the concept of T-cuts. The proof outlined above has the advantage that it can be extended to obtain the following generalization of Seymour's theorem.

Theorem 4.4a (Frank 1988). *Suppose that $G+H$ is planar and the edges of H are in two faces of G. The edge-disjoint paths problem has a solution if and only if the cut criterion holds and $d_{G+H}(S \cap T)$ is even for any two tight sets S, T.*

A direct conseqeuence of Theorem 4.4 is that the problem has a solution if the cut criterion holds with strict inequality on any separating cut. In Theorem 4.6 we shall see that in an extension of Okamura and Seymour's theorem, when no parity restriction is imposed on the nodes of the outer face, a similar type of results holds. The statement is not true if H has three disjoint edges as is shown by the following example of E. Korach where there is no tight cut at all. (Compare Theorem 4.4a to this example: here there are only three demand edges but they are on three faces.) M. Middendorf and F. Pfeiffer recently showed that the edge-disjoint paths problem is NP-complete if $G + H$ is planar.

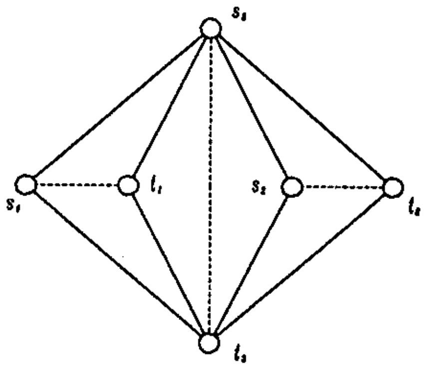


Fig. 4.3

The following theorem deals with a very special graph but it will find a nice application in the next section.

Theorem 4.5 (Frank and Tardos 1984). *Suppose that G is a circuit with parallel edges. The edge-disjoint paths problem has a solution if and only if the cut-criterion and (4.1) hold.*

Proof. If $G+H$ is Eulerian, then the cut criterion itself is sufficient by the Okamura-Seymour theorem (Theorem 3.2). So assume that the set $T = \{a_1, a_2, \ldots, a_{2h}\}$ of odd nodes is non-empty. The idea behind the proof is that we want to reduce the problem to Theorem 3.2 be eliminating the odd nodes. In order to do so first add the following h new demand edges to H: $a_1a_2, a_3a_4, \ldots, a_{2h-1}a_{2h}$. Let H_1 denote the extended demand graph. Obviously, $G + H_1$ is Eulerian, so we are done by Theorem 3.2 if the cut criterion holds in $G + H_1$. If this is not the case, then there is a bond $\nabla_G(X_1)$ which is tight in $G+H$ where $X_1 = \{a_i, a_{i+1}, \ldots, a_j\}$ $(1 < i < j)$, i is even and j is odd.

Second, add $a_2a_3, a_4a_5, \ldots, a_{2h}a_1$ as new demand edges to H obtaining this way H_2. If the cut criterion does not hold in $G + H_2$, then there is a bond $\nabla(X_2)$ which is tight in $G+H$ where $X_2 = \{a_k, a_{k+1}, \ldots, a_l\}$ $(1 < k < l)$, k is odd and l is even. But now the component of $G - (\nabla_G(X_1) \cup \nabla_G(X_2))$ containing a_1 contains and odd number of odd nodes and therefore it violates (4.1). □

This idea of pairing off the odd nodes can be used to prove the following consequence of the Okamura-Seymour theorem when no parity restriction is imposed at the nodes of the outer face.

Theorem 4.6. *Suppose that G is planar, the terminals are on the outer face and the degree of every node not on the outer face is even. If the cut criterion holds in a strong form, that is, $d_G(X) > d_H(X)$ for every $\emptyset \neq X \subset V$, then the edge-disjoint paths problem always has a solution.*

Proof. If there are no odd nodes, we are done by Theorem 3.2. Otherwise let $T = \{a_1, a_2, \ldots, a_{2h}\}$ be the set of odd nodes. Extend the demand graph by the following new edges: $a_1a_2, \ldots, a_{2h-1}a_{2h}$. Observe that the cut criterion continuous to hold and apply Theorem 3.2. □

There is a pretty consequence of Theorem 4.6. Suppose that given a big triangle R in a triangular grid which is bounded by lattice lines. R defines a graph G_R in the natural way.

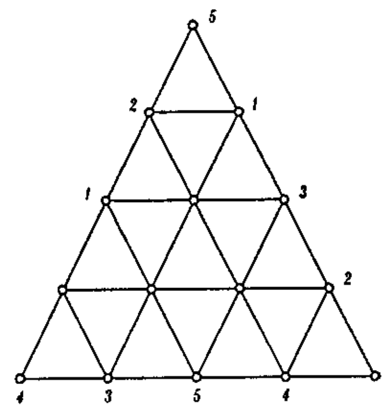


Fig. 4.4

Corollary. *If the terminals are on the boundary of R and are distinct, then the edge-disjoint paths problem has a solution.*

Actually, we can have a complete characterization for the case considered in Theorem 4.6:

Theorem 4.7 (Frank 1985). *Suppose that G is planar, the terminals are on the outer face and the degree of every node not on the outer face is even. The edge disjoint paths problem has a solution if and only if $\Sigma(s(C_i) \geq 1/2q$ for every family $(C_1, C_2, \ldots, C_l)$ of cuts $(l \leq |V|)$ where q denotes the number of components in $G - C_1 - C_2 \ldots - C_l$ which are odd (in $G + H$) and $s(C)$ is the surplus of C.*

Note that this theorem provides a characterization for the edge-disjoint paths problem when the supply graph G is outerplanar.

Proof. (Outline) To show the necessity of the condition suppose that there is a solution and let $Q_1, Q_2, \ldots, Q_q$ be the odd components in question. For each Q_i

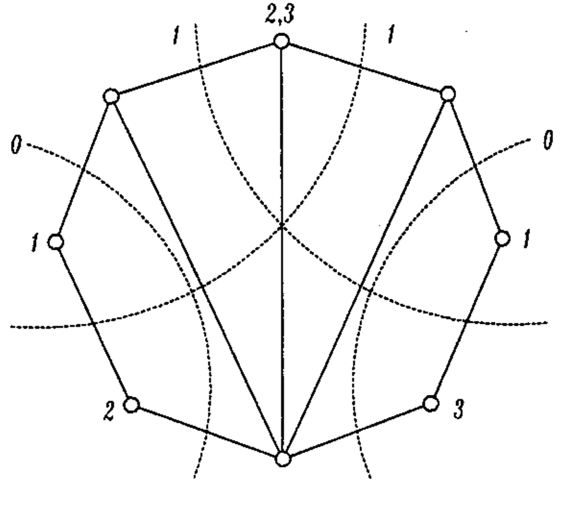


Fig. 4.5

at least one edge in $\nabla_G(Q_i)$ is not used by the solution. Because any edge of G may belong to (at most) two $\nabla_G(Q_i)'s$, we see that at least $1/2q$ edges must not be used. On the other hand all of these edges are in $\cup C_i$ therefore $\Sigma s(C_i) \geq 1/2q$.

(For example there is no solution to the edge-disjoint paths problem in the graph depicted in the following figure. The four cuts violate the necessary condition of the theorem since their sum of surpluses is 2 while the removal of them gives rise to $q = 8$ odd components and $2 \not\leq 8/2$.) The sufficiency of the condition can be proved with the same idea we have used for proving Theorem 4.6. The only difference is that this time finding the appropriate pairing of the odd nodes needs a little more care.

Namely we proceed as follows. If there is no tight cut, then we are back at Theorem 4.6. For simplicity suppose that G is 2-connected and let C denote the outer circuit of G. Call a tight set X *minimal* if $V(C) \cap X$ is minimal for inclusion. The basic step of the pairing algorithm is that we find a tight set X which is minimal and find the odd nodes $a_1, a_2, \ldots, a_j$ (in this order along C) in $V(C) \cap X$. (It can be shown that j is even). Now extend the demand graph H by the following new terminal pairs: $a_1a_2, \ldots, a_{j-1}a_j$. The crucial observation is that the original problem has a solution if and only if the new one has. Therefore we can keep going on this pairing operation. If in the course of the procedure a cut arises which violates the cut condition with respect to the current (enlarged) demand graph, then this cut and the minimal tight cut used by the procedure in the previous steps violate the condition of Theorem 4.7. Let us consider a possible run of the pairing procedure on following example:

The odd nodes are indicated by solid points. X_1, X_2, X_3 are the current minimal tight sets. In the fourth step $\nabla_G(X_4)$ violates the cut condition with respect to

the enlarged demand graph. The four cuts $C_i : \nabla_G(X_i)$ $(i = 1, 2, 3, 4)$ violate the condition in the theorem since the sum of their original surplus is 2 while the number of odd components in $G - (UC_i)$ is 8. See Figure 4.7. □

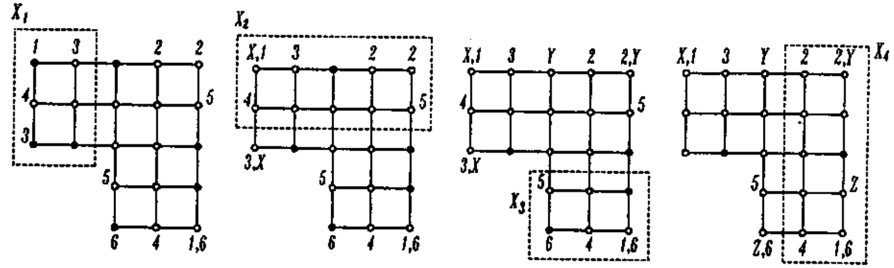


Fig. 4.6

In Theorem 4.7 the parity restriction on the inner nodes cannot be dropped as is shown by the example in Figure 4.1.

Our last result to demonstrate the use of parity conditions is due to P. Seymour. Let G be again arbitrary. The cut criterion is sufficient if H is a star. The next two simplest demand graphs are $2K_2$ and K_3. As another application of the "parity-versus-surplus" principle we exhibit a characterization when H is a K_3 with parallel edges.

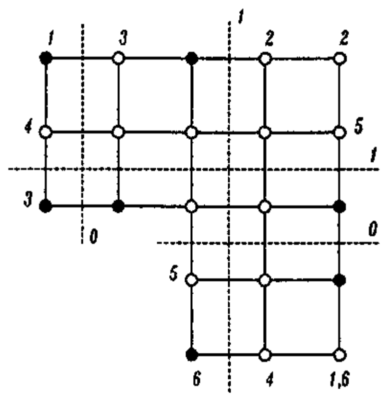


Fig. 4.7

Theorem 4.8 (Seymour 1980b). *If the demand graph H consists of three sets of parallel edges between nodes s_1, s_2 and s_3, the edge-disjoint paths problem has a solution if and only if the cut-criterion holds and*

$$q(V_1 \cup V_2 \cup V_3) \leq s(V_1) + s(V_2) + s(V_3) \tag{4.3}$$

for every choice of disjoint sets V_i with $s_i \in V_i$ $(i = 1,2,3)$ where $s(X)$ denotes the surplus and $q(X)$ denotes the number of components C in $G - X$ for which $d_G(C) + d_H(C)$ is odd.

Proof. Let $X = V_1 \cup V_2 \cup V_3$. If there is a solution, then a least $q(X)$ edges of $\nabla_G(X)$ are not used. On the other hand at most $s(V_1) + s(V_2) + s(V_3)$ edges are avoided by the paths in the solution. Therefore (4.3) is necessary.

To see the sufficiency we invoke Mader's edge-disjoint A-paths theorem (see Theorem 7.3). Extend G by adding three new nodes a_1, a_2, a_3 and $k_{i,i+1} + k_{i,i-1}$ parallel edges between a_i and s_i (where $i = 1,2,3$) and the subscript are meant modulo 3). Let $A = \{a_1, a_2, a_3\}$. If there are $h := k_{1,2} + k_{2,3} + k_{2,3}$ edge-disjoint A-paths, then their restriction to G provides the desired edge-disjoint paths in G. So suppose there are no h edge-disjoint A-paths. By Mader's theorem there are three disjoint sets U_i with $a_i \in U_i$ $(i = 1,2,3)$ for which

$$h > \text{value}(U_1, U_2, U_3) \tag{4.4}$$

where $\text{value}(U_1, U_2, U_3) := 1/2(\Sigma d_G(U_i) - q(\cup U_i))$.

We can assume that (i) each U_i induces a connected subgraph. Indeed, if the subgraph induced by U_1, say, has a component C not containing a_1, then $\text{value}(U_1 - C, U_2, U_3) \leq \text{value}(U_1, U_2, U_3)$.

We can assume that (ii) each component C of $V - \cup U_i$ is connected to every U_i $(i = 1,2,3)$. Indeed, if a component C is not connected to U_3, say, but it is connected to U_1, then $\text{value}(U_1 \cup C, U_2, U_3) \leq \text{value}(U_1, U_2, U_3)$. (Notice that if U_1 is connected, then so is $U_1 \cup C$.) If $s_i \in U_i$ for each $i = 1,2,3$, then the sets $V_i := U_i - a_i$ $(i = 1,2,3)$ violate (4.4). If $s_1 \notin U_1$, say, then $U_1 = \{a_1\}$ because of (i) and then $q(\cup U_i) \leq 1$ because of (ii).

But now we have $h >$ value $(U_1, U_2, U_3) \geq 1/2(d(U_i) + 1) \geq h + 1/2$, a contradiction. (Notice that the cut-criterion was used in the last inequality.) □

5. Problems on Grids

We devote this section to disjoint paths problems when the supply graph G is a subgraph of a rectilinear grid.

We are given a closed rectangle T (bounded by lattice lines). T defines a finite subgraph G of the plane grid in the natural way which has $m * n$ nodes when m horizontal and n vertical grid lines intersect T. Assume that the terminals are on the perimeter of T, that is on the outer face of G. Then the edge-disjoint paths problem is a special case of the problem solved in Theorem 4.7. However, exploiting the simpler structure of G we can obtain simpler theorems. By a *column (row)* of G we mean a cut consisting only of horizontal (vertical) edges. Obviously there are $n-1$ columns and $m-1$ rows.

Let us first restrict ourselves to the case when each terminal is on the upper or on the lower horizontal lines bounding T. We call this case *two-sided.* See Figure 5.1. Figure 2.1 shows that the cut criterion is not sufficient even for the two-sided problem. However a tiny restriction makes the cut criterion sufficient:

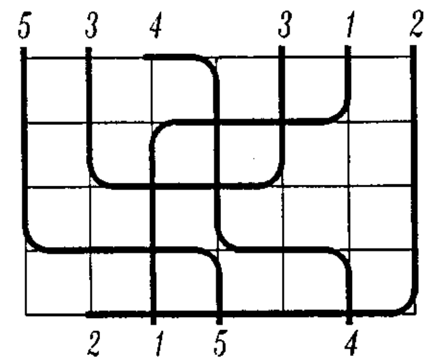


Fig. 5.1

Theorem 5.1. *In the two-sided case if the terminals are distinct and at least one of the assumptions (i) and (ii) holds, then the edge-disjoint paths problem has a solution if and only if the cut criterion holds for every column where*

(i) at least one corner point of T is not a terminal,
(ii) at least one terminal pair is such that both members of it are on the same horizontal line.

Part (i) was proved in (Frank 1982) with the help of a polynomial time algorithm. A slight modification of the same idea led to (ii) (Abos, Frank, Tardos 1982).

Notice that if there is an inner node v on the upper boundary of G such that neither v nor its opposite node u on the lower boundary is a terminal, then the vertical line connecting u and v can be left out. Similarly we can leave out horizontal lines as long as the cut criterion holds for columns. This way we can preassume that $m \leq k$ and $n \leq 2k$. In applications it is desirable that the paths do not have too many bends since each bend corresponds to a via in a layout realization.

Theorem 5.2 (Preparata and Lipski 1984). *Assume that one member of each terminal pair is on the upper boundary line of T, the other member is on the lower boundary line and that there is no terminal on the $l = d/2$ right-most vertical lines where d denotes the maximum congestion of a column. If $m \geq d$ (that is, the cut criterion holds for the columns) then there is a solution to the edge-disjoint paths problem such that each path has one of the following shape (Figure 5.2).*

Let us turn to the case when the terminals are still distinct (with the possible exception that there may be two terminals sitting at one corner) but they are arbitrarily positioned on the boundary of T. Let c be an arbitrary column and let $\{r_1, r_2, \ldots, r_t\}$ be the set of tight rows $(t \geq 0)$. Let $T_1, T_2, \ldots, T_{t+1}$ be the components of $G - (c \cup r_1 \cup \ldots \cup r_t)$ that are on the left-hand side of c. The *parity*

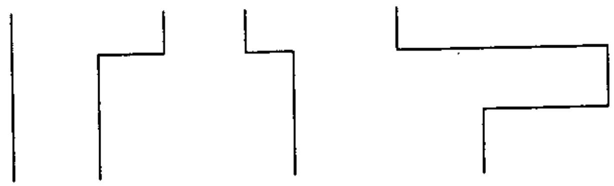


Fig. 5.2

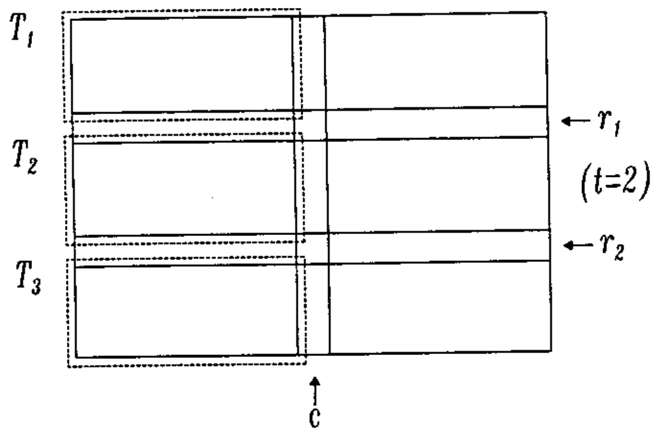


Fig. 5.3

congestion of c is the number of odd sets T_i. The parity congestion of a row is defined analogously. By the parity-versus-surplus principle we see that the

REVISED CUT CRITERION: the parity congestion of a row or column cannot exceed the surplus is necessary for the solvability.

We have

Theorem 5.3 (Frank 1982). *We are given a rectangle in a rectilinear grid and k pairs of distinct terminals on its boundary. The edge-disjoint paths problem has a solution if and only if the revised cut criterion holds for every row and column.*

In Figure 5.4 two examples are shown differing only in the position of terminal "1". The first example has a solution but the second does not since the column c indicated in the picture violates the revised cut criterion.

A further advantage of Theorem 4.7 is that it makes possible to handle certain capacitated cases. For example, suppose that each vertical and horizontal line has a positive integer capacity (not necessarily the same). Let the capacity of an edge e of the grid-graph be the capacity of the line containing e. Instead of seeking for edge-disjoint paths we require that no edge is contained in more paths than its capacity. Obviously Theorem 4.7 can be applied since the sum of capacities of the edges incident to an inner node is even.

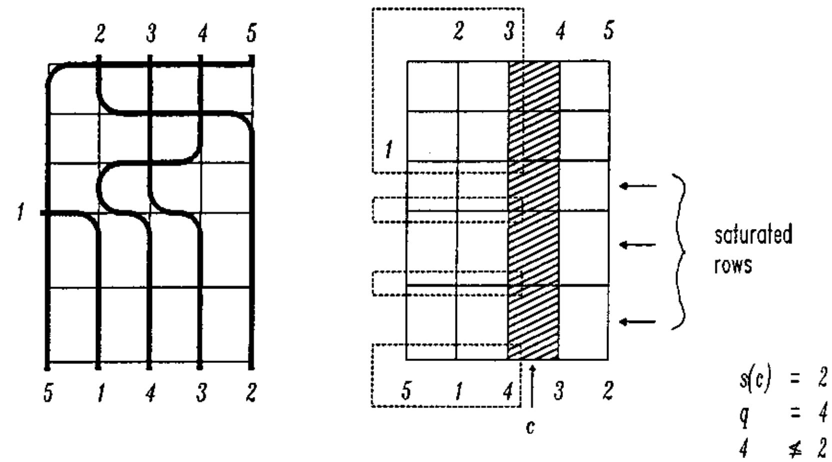


Fig. 5.4

In applications sometimes one needs regions of the grid more general than rectangles. As long as the terminals are on the outer face the problem is still a special case of the problem answered by Theorem 4.7. But the revised cut criterion is not sufficient in general as was shown by (Lai and Sprague 1987). See the problem in Figure 4.6.

With the help of the pairing method described in the proof of Theorem 4.7 we can reduce the problem to the Eulerian case. When the boundary region is x-convex, that is, any horizontal line intersects it in a segment, then there is an extremely simple algorithm due to M. Kaufmann.

We close the section by an application of Theorem 4.5. (The material is taken from (Frank and Tardos 1984). Let O and I be two closed rectangles bounded by lattice lines such that I is in the inside of O. The graph we consider is the subgraph of the rectilinear grid between O and I. The k terminal pairs to be connected are on the perimeter of I. The problem is to find edge-disjoint paths connecting the corresponding terminal pairs which, in addition, do not touch. That is, if a path bends at a certain node v, then v must not be used by other paths. This constraint is imposed in order to model two-layer routing problems where one layer is used for horizontal segments, the other for vertical ones and a bend corresponds to a via hole between the two layers.

Figure 5.5 shows an instance of the problem along with a solution.

A version of this problem was solved by LaPaugh (1980) when only the inner rectangle I is given and the problem is to find a surrounding rectangle O of minimum area such that the paths exist between O and I.

We need the following well known result. Let $\mathcal{F}$ be a family of closed intervals of a segment S. The *density* of $\mathcal{F}$ is the maximum number of intervals covering a point of S.

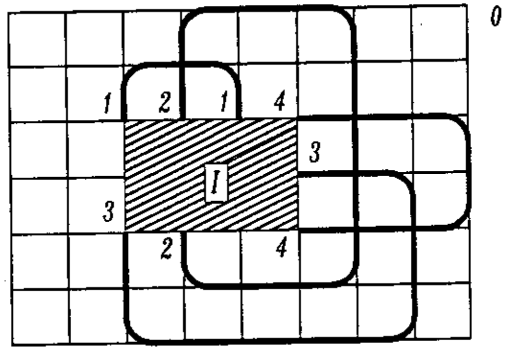


Fig. 5.5

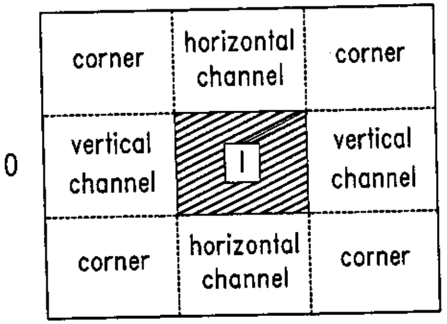


Fig. 5.6

Lemma (Gallai 1962). *$\mathcal{F}$ can be partitioned into d classes consisting of pairwise disjoint segments if and only if the density of $\mathcal{F}$ is at most d. (Furthermore the partition can be found in $O(|\mathcal{F}|\log|\mathcal{F}|)$ time.)*

The four straight lines of the boundary segments of I divide $O - I$ into four channels and four corners.

The *width* of the channels above and below I (resp. left and right to I) is the number of their horizontal (resp. vertical) lines.

Observe that each path has two different homotopies. Suppose for a moment that the homotopies have already been specified. Then they define four interval systems as is shown in Figure 5.7a and b.

If (*) the density of each of these interval systems is at most the width of the corresponding channel, then by Gallai's lemma the intervals can be placed on the available lines, and the resulting segments belonging to the same homotopy can be connected in the corners so as to form the desired paths (Figure 5.7 c).

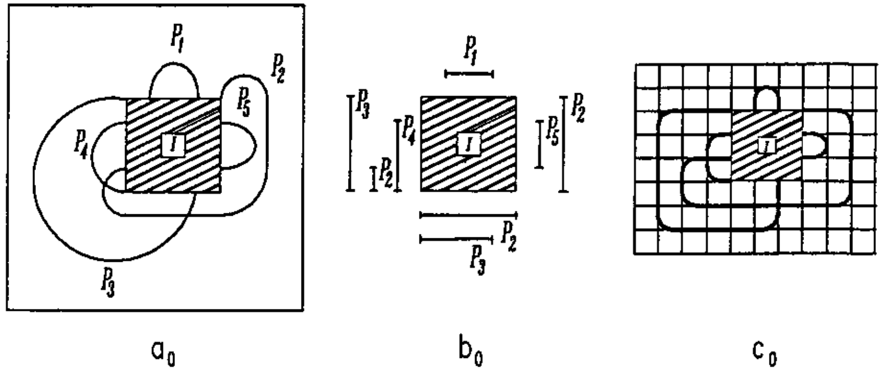


Fig. 5.7

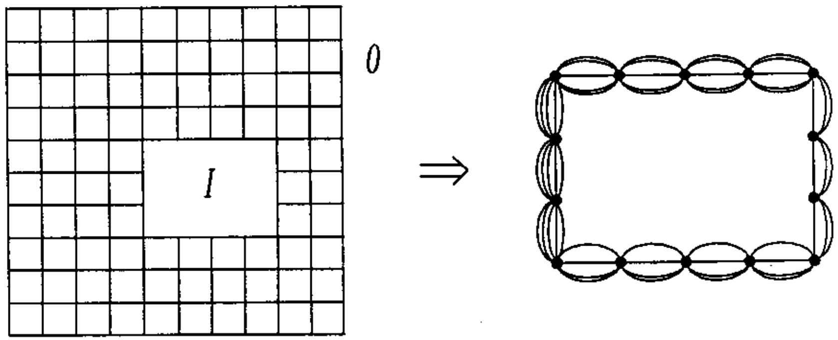


Fig. 5.8

Therefore the only problem we are encountered is to find the homotopies that satisfy (*). But this problem can immediately be solved if we apply Theorem 4.5 to the graph obtained by contracting all the edges in the four corners, the vertical edges in the two horizontal channels and the horizontal edges in the two vertical channels.

6. When the Disjoint Paths Problem is Tractable

In this section we survey restrictions of the (edge-) disjoint paths problem when either a polynomial time algorithm is available or a sufficient condition (or both).

Theorem 6.1 (Robertson-Seymour 1986b). *For fixed k the undirected (edge-) disjoint paths problem can be solved in polynomial time.*

Actually this theorem is the central topic of Robertson and Seymour's paper in this volume. As they remark the algorithm is completely out of the range of practical usability when $k > 2$. For acyclic digraphs an analogous result holds.

Theorem 6.2 (Fortune, Hopcroft and Wyllie 1980). *In acyclic digraphs the (arc-) disjoint paths problem can be solved in polynomial time if k is fixed.*

Unlike the undirected case, the algorithm of Fortune, Hopcroft and Wyllie is quite reasonable for small k. The idea behind it is reformulated in (Thomassen 1985) as follows. Let the set $S = \{s_1, \dots, s_k\}$ of source nodes and the set $T = \{t_1, \dots, t_k\}$ of target nodes be disjoint. We introduce an auxiliary digraph G^* the nodes of which correspond to the k–tuples of distinct nodes of G. There is an arc in G^* from $X = \{x_1, \dots, x_k\}$ to $Y = \{y_1, \dots, y_k\}$ if and only if there is a $j \in \{1, \dots, k\}$ such that $x_i y_i$ for $i \neq j$ and G contains an arc $x_j y_j$ and contains no directed path from x_i to x_j $(i \neq j)$. It can be shown that there are k pairwise disjoint paths in G from s_i to t_i $(i = 1, \dots, k)$ if and only if G^* has a path from S to T.

A by-product of Schrijver's disjoint homotopic paths theory (see his paper in this volume) is the following.

Theorem 6.3. *The disjoint paths problem is solvable in polynomial time when G is planar and the terminals are on a bounded number of faces of G.*

Note that in this case no restriction is put on the size of the demand graph. The status of the corresponding edge-disjoint paths problem is not known.

Theorem 6.4 (Sebő 1988). *The integer multicommodity flow problem is solvable in polynomial time if $G+H$ is planar and there is a bounded number of demand edges (with arbitrary big demand values).*

The same question remains open if only the number of faces of G covering the terminal nodes is bounded.

Next we list results where connectivity assumptions prove to be sufficient for the (edge-) disjoint paths problem.

Let us call a graph *k-linked on the edges (or weakly k-linked)* if for any choice of k pairs of terminals there are k edge-disjoint paths connecting the corresponding terminal pairs. Let $g(k)$ denote the minimal number m such that every m-edge-connected graph is k-linked on the edges.

C. Thomassen has a nice conjecture asserting that $g(2k+1) = g(2k) = 2k+1$.

Theorem 6.5. $g(3) = 3$, $g(4) = 5$, $g(5) \leq 6$, $g(6) \leq 8$, $g(7) \leq 9, g(3k) \leq 4k$, *and* $g(3k+1) \leq g(3k+2) \leq 4k+2$ $(k \geq 2)$.

Here $g(3) = 3$ is due to (Okamura 1984), $g(4) = 5$ to (T. Hirata-K. Kubota-O. Saito 1984) and to (Mader 1985), the other results are due to (Okamura 1988).

Surprisingly for directed graphs the analogous situation is much simpler. The following was observed by Shiloach (1979). Let us call a digraph $D = (V, A)$ *k-linked on the arcs* if for any choice of k pairs $\{(s_1, t_1), \dots, (s_k, t_k)\}$ of (not necessarily distinct) terminals there are arc-disjoint paths p_i from s_i to t_i $(i = 1, \dots, k)$. Obviously such a digraph is strongly k-arc connected (that is every non-empty proper subset of nodes has k entering arcs.)

Theorem 6.6. *A strongly k-arc connected digraph is k-linked on the arcs.*

Proof. Add a new node r to D and new arcs rs_i $(i = 1, 2, \dots, k)$ and apply Edmonds' disjoint arborescence theorem (Edmonds 1973). □

We call a graph *k-linked* if for any choice of k pairs of terminals there are k openly disjoint paths connecting the corresponding terminal pairs.

Theorem 6.7 (Jung 1970), (Larman and Mani 1970). *A 2^{3k} connected graph is k-linked.*

It is not known if 2^{3k} can be replaced by a linear bound. The natural $2k+2$ is not enough as can be seen from a K_{3k-1} with edges $x_1 y_1, \dots, x_k y_k$ removed (an example due to (Strange and Toft 1983)).

In certain cases the cut condition is not strong enough to ensure the existence of all required paths but the demands can almost be met:

Theorem 6.8 (Korach and Penn 1985). *Suppose that $G+H$ is planar and that the demand edges are on k faces of G. If the cut criterion holds, there are edge-disjoint paths connecting all but $k-1$ terminal pairs so that for one face F, specified in advance, all the terminal pairs on F are connected while for each other face F' the terminal pairs on F' with one possible exception are connected.*

Actually, Korach and Penn proved a more general result. There is an important corollary to Theorem 6.8.

Corollary 6.9. *Suppose that $G+H$ is planar and H consists of k demand edges (s_i, t_i) endowed with integer demands d_i. The supply edges e have integer capacities $c(e)$ so that the cut criterion holds. Then there are d_1 paths connecting s_1 and t_1 and $d_i - 1$ paths connecting s_i and t_i $(i = 2, 3, \dots, k)$ so that each supply edge e is used by no more than $c(e)$ among these $\sum d_i - k + 1$ paths.*

Another result of similar flavour is the following.

Theorem 6.10 (Itai and Zehavi 1984). *In a graph s_i, t_i are terminal pairs $(i = 1, 2)$ such that there are k edge-disjoint paths connecting s_i and t_i $(i = 1, 2)$. Then for each m, $0 \leq m < k$ there are k edge-disjoint paths $P, S_1, S_2, \dots, S_m, Q_1, Q_2, \dots, Q_{k-m-1}$ such that each S_i connects s_1 and t_1, each Q_j connects s_2 and t_2 and P connects either s_1 and t_1 or s_2 and t_2.*

7. Maximization

In combinatorial optimization sometimes we are interested in the existence of a certain configuration (e.g., is there a perfect matching in a graph) other times we need the biggest (or smallest) configuration (e.g. find the biggest matching). Not surprisingly, the corresponding feasibility and maximization problem often correlate and typically (though not always) the maximization problem is more difficult.

For example, in the matching case Tutte's theorem on the existence of a perfect matching is a direct consequence of the so-called Berge-Tutte formula on the maximum cardinality of a matching. Conversely, the Berge-Tutte formula can be derived from Tutte's theorem via an elementary construction.

There are however other cases when a good answer to the feasibility problem exists but the corresponding maximization problem is NP-complete. For example, suppose that G has a perfect matching. Then the problem of finding a stable set of $|V|/2$ elements is tractable, but to find a maximum cardinality stable set is NP-complete.

As far as (edge-) disjoint paths problems are concerned we have studied so for problems of feasibility type. In the *maximization problem* we want to find the maximum total number M of (edge-) disjoint paths connecting the corresponding terminal pairs $s_1t_1, s_2t_2, \ldots, s_kt_k$ (allowing many paths between one terminal pair).

In what follows we discuss, among others, some feasibility problems where the corresponding maximization problem is solvable but the derivation needs some work.

Let $V_1, V_2, \ldots, V_t$ be a family $\mathcal{P}$ of disjoint subsets of V such that each demand edge connects different V_i's. By a *multicut* defined by $\mathcal{P}$ we mean the set of edges uv of G such that $u \in V_i$, $v \notin V_i$ for some i. The capacity m of a multicut is defined to be $1/2\Sigma d(V_i)$. Let m_1 denote the minimum cardinality of a cut separating each terminal pair. Obviously, $m_1 \geq m \geq M$. If $k = 1$, then $m_1 = M$ by Menger's theorem.

First, we will present two theorems for $k = 2$. In the first one, due to B. Rothschild and A. Whinston (1966a), we assume that the degree of every non-terminal node is even, in the second one, due to M. Lomonosov (1983), we assume that G together with the two edges s_1t_1, s_2t_2 is planar.

For both cases let c_i denote the cardinality of a minimum cut separating s_i and t_i but not separating $s_{3-i}t_{3-i}$. Let c_{12} denote the minimum cardinality of a cut separating both terminal pairs. (Then $c_{12} = m_1$). Obviously $c_1 + c_2 \geq c_{12}$.

Theorem 7.1 (Rothschild-Whinston 1966a). *If $k = 2$ and $d_G(v)$ is even for each non-terminal node, then $m_1 = M$.*

Proof. Assume first that c_{12} is even.

Case 1. $d(s_1)$ and $d(t_1)$ have the same parity (equivalently $d(s_2)$ and $d(t_2)$ have the same parity). Define a demand graph H to consist of c_1 parallel edges between s_1 and t_1 and $c_{12} - c_1$ parallel edges between s_2 and t_2. By Theorem 3.8 we are done since $G + H$ is Eulerian and the cut criterion holds.

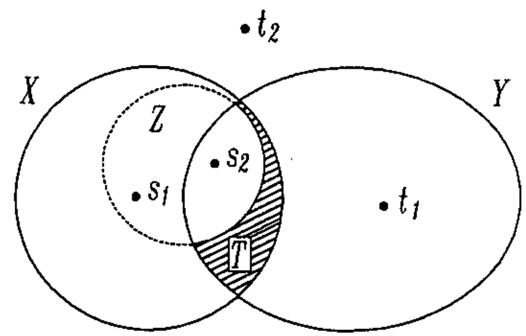


Fig. 7.1

Case 2. Precisely one member, say s_1 and s_2, of both terminal pairs has odd degree. If both c_1 and c_2 are even (resp. odd), define H to consist of c_1 edges between s_1 and t_1, $c_{12} - c_1$ edges between s_2 and t_2 and just one edge between s_1 and s_2 (resp. t_1 and t_2). If one of c_1 and c_2 is odd, say c_1, then let H consist of c_1 edges between s_1 and t_1, $c_{12} - c_1 + 1$ edges between s_2 and t_2 and one edge between t_1 and t_2. With some care one can check that in each case $G + H$ is Eulerian and the cut criterion holds. Therefore Theorem 3.8 implies the existence of the desired c_{12} paths.

If c_{12} is odd, then one member, say s_1 and s_2, of both terminal pairs has odd degree. Furthermore any set X for which $\nabla(X)$ separates both terminal pairs and $d(X) = c_{12}$ contains exactly one of s_1 and s_2. Therefore if we add a new edge $e = s_1s_2$ to G, then the new c'_{12} is one bigger that c_{12}. So it is even. For even c'_{12} we have proved already the existence of c'_{12} edge-disjoint paths in $G + e$ between the two terminal pairs. If we take back the newly added edge e, we still have the c_{12} paths in G, as required. □

Let us call a cut C *critical* if it has a minimum number of edges from G among all the cuts that separate the same terminal pairs as C does. Recall the definition of separating and parallel sets (before Theorem 4.4)

Theorem 7.2 (Lomonosov 1983). *Suppose that $k = 2$ and $G + \{s_1t_1, s_2t_2\}$ is planar. Then M is either $c_{12} - 1$ or c_{12}. $M = c_{12} - 1$ if and only if there is an odd cut $\nabla(T)$ which does not separate either of the two terminal pairs and which can be covered by three separating critical cuts $\nabla(X), \nabla(Y), \nabla(Z)$. Moreover X, Y, Z, T can be chosen in such a way that $Z \subset X$, X and Y are non-parallel and $T = (X \cap Y) - Z$. (See Figure 7.1)*

Proof. The original proof of Lomonosov consists of a direct construction and is rather complicated. Here we derive the result from Theorem 4.4 of Seymour. We can assume that $c_1 > 0$, $c_2 > 0$ and $c_{12} > \max(c_1, c_2)$ since otherwise the situtation is trivial.

Let H consist of $c_1 - 1$ parallel edges between s_1 and t_1 and $c_{12} - c_1$ parallel edges between s_2 and t_2, then the cut criterion holds and there is no tight separating cut. By Theorem 4.4 we see that $M \geq c_{12} - 1$.

The necessity of the condition in the second half of the theorem is straightforward.

Case 1. $c_1 + c_2 = c_{12}$.

Let H consist of c_i edges between s_i and t_i $(i = 1, 2)$. We are done if there is a solution to the feasibility problem concerning G and H. Suppose there is none. Then we have

Claim. *There are tight non-parallel separating sets X, Y and tight sets $Z_1 \subseteq X - Y$, $Z_2 \subseteq X \cap Y$ such that $d_{G+H}(X \cap Y)$ is odd and Z_1 separates one of the two terminal pairs while Z_2 separates the other.*

Proof. By Theorem 4.4 there are tight non-parallel separating sets X and Y so that $d_{G+H}(X \cap Y)$ is odd (and then automatically $d_{G+H}(X - Y)$, $d_{G+H}(Y - X)$ and $d_{G+H}(V - (X \cup Y))$ are all odd). Assume that $s_1 \in X - Y$, $t_1 \in Y - X$, $s_2 \in V - (X \cup Y)$ and $t_2 \in X \cap Y$.

Let Z_1 be a minimal tight set separating s_1 and t_1 such that $s_2, t_2 \notin Z_1$. Then $d_G(Z_1) = c_1$ and Z_1 contains one of s_1 and t_1, say s_1. Since $d_H(X, Z_1) = 0$, by Lemma 3.1 $Z_1 \cap Y$ is tight. Therefore $Z_1 \subseteq X$ by the minimal choice of Z_1. Since $d_H(Z_1, V - Y) = 0$, by Lemma 3.1 $Z_1 - Y$ is tight so we obtain that $Z_1 \subseteq X - Y$.

It can be seen analogously that there is a tight cut $\nabla(Z_2)$ separating s_2 and t_2 for which s_1, $t_1 \notin Z_2$ and Z_2 is either in $X \cap Y$ or in $V - (X \cup Y)$. We can assume that $Z_2 \subseteq X \cap Y$ for otherwise we can work with $V - Y$ on place of Y. □

Let $Z = Z_1 \cup Z_2$ and $T = X \cap Y - Z$. Since $c_{12} \leq d_G(Z) \leq d_G(Z_1) + d_G(Z_2) = c_1 + c_2 = c_{12}$ we have $d_G(Z) = c_{12}$. Furthermore $d_G(T) = d_{G+H}(T) = d_{G+H}(X \cap Y) - d_{G+H}(Z_2) + 2d_{G+H}(Z_2, T)$. Since $d_{G+H}(X \cap Y)$ is odd and $d_{G+H}(Z_2)$ is even, $d_G(T)$ is odd and therefore the theorem is proved for Case 1.

Case 2. $c_1 + c_2 > c_{12}$.

Let H' consist of c_1 edges between s_1 and t_1 and $c_{12} - c_1$ parallel edges between s_2 and t_2. If there is a solution to the feasibility problem concerning $G + H'$, then we are done. Otherwise, since the cut criterion holds, there are separating non-parallel tight sets X' and Y' such that $s_1 \in X' - Y'$, $s_2 \notin X' \cup Y'$ and $d_{G+H'}(X' \cap Y')$ is odd. Suppose that X' and Y' are minimal such sets. (To avoid confusion we will call these sets H'–tight.)

Next, let H'' consist of $c_1 - 1$ edges between s_1 and t_1 and $c_{12} - c_1 + 1$ parallel edges between s_2 and t_2. ($c_1 > 0$ since otherwise we are at Case 1). Again the cut criterion holds, so if the feasibility problem has no solution then (by Theorem 4.4) there are separating non-parallel H''-tight set X'', Y'' such that $s_1 \in X'' - Y''$, $s_2 \notin X'' \cap Y''$ and $d_{G+H}(X'' \cap Y'')$ is odd. Suppose that X'' and Y'' are minimal such sets.

It is not possible that $X' = X''$ and $Y' = Y''$ since $d_{G+H'}(X' \cap Y')$ is odd, therefore $d_{G+H''}(X' \cap Y')$ is even and $d_{G+H''}(X'' \cap Y'')$ is odd. Assume that $X' \neq Y'$ By symmetry we can assume that $X' \nsubseteq X''$, that is $Z := X' \cap X'' \subset X'$. Let $T = X' \cap Y' - Z$. Since $d_G(X'') = c_{12}$, X'' is H'-tight. By Lemma 3.1 Z is H'–tight and separating. Since Z is a proper subset of X' and X' was chosen minimal, $d_{G+H'}(Z \cap Y')$ must be even. But then $d_G(T) = d_{G+H'}(T) = d_{G+H'}(X' \cap Y') - d_{G+H'}(Z \cap Y') + 2d_{G+H}(Z \cap Y', T)$ from which we conclude that $d_G(T)$ is odd and sets X', Y', Z, T satisfy the requirements in the theorem. □

There are cases when the maximization form is easier to handle:

Theorem 7.3a (Lovász 1976b, Cherkasskij 1977). *If the demand edges form a complete graph induced by A $(A \subseteq V)$ and $d_G(v)$ is even for $v \in V - A$, then $m = M$.*

Proof. For $a \in A$ let c_a denote the maximum number of egde-disjoint paths connecting a and $A - a$. By Menger's theorem this is the minimum cardinality of a cut separating a and $A - a$. Let us denote $\frac{1}{2}c(A) = \Sigma(c_a : a \in A)$. Obviously $c(A) \geq m \geq M$. We are going to show, by induction on the number of edges, that $c(A) = M$.

We can assume that G is connected. If $A = V$ there is nothing to prove. Call a set $X \subset C$ *critical* with respect to $a \in A$ if $X \cap A = \{a\}, |X| \geq 2$ and $d(X) = c_a$.

Case 1. There is a set X critical with respect to a certain $a \in A$. Contract the elements of X into one node a'. In the contracted graph there are $c(A)$ edge-disjoint A'–paths where $A' = A - a + a'$. In X there are $d(X)$ edge disjoint paths from a to the edges in $\nabla(X)$. Pasting together the two sets of paths we obtain $c(A)$ edge-disjoint A–paths in G.

Case 2. There are no critical sets. Choose any two edges e, f which are incident to a node v not in A. Because there are no critical sets, splitting off e and g does not reduce $c(A)$ and we are done by induction. □

Generalizing this result to non-Eulerian graphs W. Mader (1978b) found the following (much more difficult) characterization for M.

Theorem 7.3. *Let $G = (V, E)$ be a graph and A a specified subset of nodes. The maximum number of edge-disjoint paths connecting distinct elements of A is* $\min 1/2(\Sigma(d(V_i) - q(\cup V_i)))$ *where the minimum is taken over all collections of disjoint subsets $V_1, V_2, \ldots, V_{|A|}$ for which $|V_i \cap A| = 1$. (Here $d(X)$ denotes the edges leaving X and $q(X)$ denotes the number of components C of $G - X$ for which $d(C)$ is odd.)*

To formulate a node-disjoint version of Theorem 7.3 suppose that A is independent.

Theorem 7.4 (Mader 1978c). *The maximum number of openly node-disjoint paths connecting distinct members of A is equal to* $\min(|V_0| + \Sigma(\lfloor 1/2b(V_i, V_0)\rfloor : i = 1, 2, \ldots, k)$ *where the minimum is taken over all collections of disjoint subsets $V_0, V_1, \ldots, V_k$ of $V - A$ $(k \geq 0)$ (where only V_0 can be empty) such that*

$G - V_0 - \cup(E(V_i) : i = 1, \ldots, k)$ *contains no path connecting distinct nodes of A. In the formula* $b(V_i, V_0)$ *denotes the number of nodes of* V_i *which have a neighbour outside* V_0.

In the next figure the solid points belong to A. There are two openly disjoint A-paths and the only family where the minimum is attained is shown in the picture.

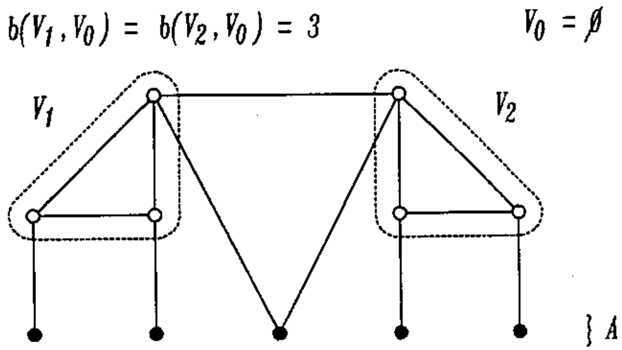


Fig. 7.2

This result can be regarded as a common generalization of Menger's theorem and the Berge-Tutte theorem. An immediate corollary of Theorem 7.4 is a result of T. Gallai (1961):

Corollary 7.5. *The maximum number of disjoint paths having end nodes in T is* $\min_{K \subseteq V}(|K| + \Sigma\lfloor 1/2\,|C \cap T|\rfloor)$ *where the sum is taken over the components C of* $G - K$.

Let us turn back to edge-disjoint paths. A common generalization of Theorems 7.1 and 7.2 has been found recently:

Theorem 7.6 (Karzanov 1985b). *Let* $H = (T, F)$ *denote the demand graph. If the maximal independent sets of H can be partitioned into two classes such that both classes consist of disjoint sets (which is equivalent to saying that the complement of H is the line graph of a bipartite graph) and, in addition, if* $d_G(v)$ *is even for* $v \in V - T$, *then* $m = M$.

As far as the maximization problem is concerned for digraphs we mention the following counter-part of Theorem 7.3a.

Theorem 7.7 (Frank 1989). *In an Eulerian digraph* $D = (V, A)$ *the maximum number of arc-disjoint paths connecting distinct nodes of a specified subset A of V is equal to the minimum of* $\Sigma\varrho(V_i)$ *over all families of disjoint subsets* $V_1, V_2, \ldots, V_{|A|}$ *of V for which* $|V_i \cap A| = 1$ $(i = 1, 2, \ldots, |A|)$.

The proof goes along the same line as that of Theorem 7.3a.

Professor Karzanov kindly informed me that this theorem was proved much earlier by Lomonosov. Karzanov presented Lomonosov's proof in: Combinatorial Methods for Flow Problems (Inst. for System Studies, Moscow 1979, issue 3), 6-69, in Russian.

8. T-joins and T-cuts

Let $G = (V, E)$ be an undirected graph. Let T be a subset of nodes with even cardinality. A cut $\nabla(K)$ is called a *T-cut* if $|K \cap T|$ is odd. A *T-join* F is a set of edges that has an odd number of edges incident to a node v if and only if v is in T. G has a T-join if and only if every component of G has an even number of nodes from T (easy exercise). Obviously any T-join and T-cut has an odd number of edges in common, in particular at least one. Therefore the maximum number of disjoint T-cuts cannot exceed the minimum cardinality of a T-join. The complete graph on 4 nodes, when $T = V$, shows that we do not have equality in general. However,

Theorem 8.1 (Seymour 1981). *In bipartite graphs the maximum number of disjoint T-cuts is equal to the minimum cardinality of a T-join.*

This theorem implies that (Lovász 1976b) in an arbitrary graph G one half of the maximum number of half-disjoint T-cuts is equal to the minimum cardinality of a T-join. (Here half-disjoint means that each edge can be in at most 2 T-cuts.) Indeed, subdivide each edge of G by a new node and apply Theorem 8.1. A weaker version of this result, stating that the minimum cardinality of a T-join is equal to the maximum of a fractional packing of T–cuts, was proved algorithmically in (Edmonds and Johnson 1973).

Because minimum T-joins and T-cuts packings have a great number of applications we say some words about the algorithmic aspects.

A weighted generalization of Theorem 8.1 is the following.

Theorem 8.1w. *Let* $d : E \to \mathbb{Z}_+$ *be a non-negative integer-valued function with the even-circuit property, that is, the d-weight of every circuit of G is even. Then the minimum weight of a T-join is equal to* $\max(\sum(y(A) : A \subseteq V, |A \cap T| \text{ odd}),$ *y non-negative and integer-valued,* $d(uv) \geq \sum(y(B) : |B \cap T| \text{ odd}, |B \cap \{u, v, \}| = 1$ *for every* $uv \in E))$.

If we choose each weight to be 1, we are back at Theorem 8.1. On the other hand this weighted version can easily be derived from Theorem 8.1. The problem is to find algorithmically the minimum and maximum in question.

Let $m(uv)$ denote the minimum d–weight of a path in G between u and v, $(v, u \in V)$. Obviously $m(uv) \leq d(uv)$, m satisfies the triangle inequality, and m has the even-circuit property if d has.

In order to construct a minimum weight T-join Edmonds and Johnson (1973) associated the T-join problem with the following minimum weight perfect matching problem. For each pair $u, v \in T$ compute $m(uv)$ between u and v. Construct then a minimum m-weight perfect matching M in the complete graph on T. Finally, look at the union F of the minimum weight paths in G that connect the pairs of nodes determined by M. It is easy to see that these $|M|$ paths are pairwise edge-disjoint and F is a minimum weight T-join. Because the mimimum weight perfect matching problem is solvable in strongly polynomial time (as was shown first by Edmonds (1965)), so is the minimum weight T-join problem.

To construct the optimal packing of T-cuts several algorithms have been devised (Korach 1982, Karzanov 1986, Barahona 1987).

Here we exhibit the newest algorithm that seems to be conceptionally the simplest. Its basic idea, due to A. Sebő (1988), is that not only the minimum weight T-join problem reduces so handily to a minimum weight perfect matching problem, but also its dual reduces relatively easily to the dual of the associated matching problem.

To obtain an integral optimal solution to the dual matching problem we need the following observation of Barahona and Cunningham (1988). Let C_T be the complete graph on a set T of even cardinality and let m be a non-negative integer-valued weighting on the edges of C_T with the even-circuit property.

Barahona and Cunningham proved the following.

Lemma 8.2. *If m satisfies the even-circuit property, then the minimum weight of a perfect matching in C_T is equal to $\max(\sum(y(A) : A \subseteq T, |A| \text{ odd}), y(A) \geq 0$ for $|A| \geq 3$, y integer-valued and $m(uv) \geq \sum(y(B) : |B| \text{ odd}, |B \cap \{u,v\}| = 1$ for every edge $uv))$.*

This follows easily from Theorem 8.1w of Seymour. The main point in Barahona and Cunningham's paper is the observation that a natural modification of Edmonds' algorithm provides not only the minimum weight perfect matching but also the integer-valued dual y occuring in the lemma. In other words, the minimum and the maximum in the lemma can be computed in strongly polynomial time.

The modification consists of two parts. First, the minimum weight of a perfect matching is considered rather than the maximum. Second, unlike Edmonds' algorithm, where an alternating forest is grown, Cunningham and Barahona's algorithm grows only one alternating tree at a time. This ensures that the current dual variables are automatically integer-valued if the even-circuit property holds.

It is obvious from the algorithm that the family $\mathscr{F} := \{A : y(A) > 0\}$ is laminar. Our second observation is the following.

Lemma 8.3. *If, in addition to the assumptions in Lemma 8.2, m satisfies the triangle inequality, then y can be chosen non-negative.*

Proof. Let us start with an optimal y occuring in Lemma 8.2. If this is non-negative, we are done, so suppose that $y(z) < 0$ for some $z \in T$. For any set $A \in \mathscr{F}$ containing z increase $y(T - A)$ by $y(A)$ and then revise $y(A)$ to be 0. This way we get another optimal solution such that no member of $\mathscr{F}$ contains z. (Such a change keeps $\mathscr{F}$ laminar).

Denote $p(uv) := \sum(y(A) : |A \cap \{u,v\}| = 1)$. The dual constraint in Lemma 8.2 requires that $p(uv) \leq m(uv)$ for every $u, v \in T$. There must be an edge uz incident to z for which $p(uz) = m(uz)$ since otherwise by increasing $y(z)$ by 1 we would get a better y.

Let A be a maximal member of $\mathscr{F}$ containing u. For $v \notin A$ we have

$$(*) \qquad p(uv) = p(uz) + p(vz) - 2y(z)$$

Let $\Delta := \min(-y(z), y(A))$ and revise y by increasing $y(z)$ by Δ and decreasing $y(A)$ by Δ.

We claim that the revised dual solution is feasible. To see this all we have to show is that $m(vz) - p(vz) \geq \Delta$ for every $v \notin A$. Actually, this inequality turns out to be strict. Indeed, using (*) and the triangle inequality we get $m(vz) \geq m(uv) - m(uz) = m(uv) - p(uz) \geq p(uv) - p(uz) = p(vz) - 2y(z) > p(vz) + \Delta$, as required.

Therefore we have another optimal dual solution. Repeat this procedure as long as there is a point z in T with $y(z)$ negative. We claim that after at most $2|T|$ iterations y becomes non-negative. Indeed, at every iteration the number of points v with negative $y(v)$ plus $|\mathscr{F}|$ reduces and this sum is at most $2|T|$. □

Let $\mathscr{F} \subseteq 2^V$ be a laminar family and $y : \mathscr{F} \to \mathbb{Z}_+$ a function. We call the pair $(y, \mathscr{F})$ a *weighted laminar family* on S as follows. Let $\mathscr{F}_S := \{X = F \cap S :$ for some $F \in \mathscr{F}\}$ and let $y_S(X) := \sum(y(F) : F \in \mathscr{F}, X = S \cap F)$ for $X \in \mathscr{F}_S$. We will say that $(y, \mathscr{F})$ is an *extension of* $(y_S, \mathscr{F}_S)$ *on* V.

Let m be a metrics on V. We say that a w–laminar family $(y, \mathscr{F})$ on S is *feasible* if $m(uv) \geq \sum(y(F) : |F \cap \{u,v\}| = 1, F \in \mathscr{F})$ holds for every $u, v \in S$.

Lemma 8.4 (Sebő 1988). *Every feasible w–laminar family $(y, \mathscr{F})$ on S can be extended (in polynomial time) to a w–laminar family on V.*

Obviously, if we apply the lemma to the w-laminar family $(y, \mathscr{F})$ on T obtained in Lemmas 8.2 and 8.3, we obtain an optimal solution to the T-packing problem in Theorem 8.1w.

Originally, the lemma was proved, using a different method, by A. Sebő (1988). The present proof has a slight advantage that it provides a conceptionally simpler algorithm.

Proof. We are going to prove only that $(y, \mathscr{F})$ can be extended on a set $S + t$ ($t \in V - S$) because then, element by element, we can extend $(y, \mathscr{F})$ on V. So suppose that $V = S + t$.

It is well known that a laminar family $\mathscr{F}$ can be represented with the help of an arborescence $D = (V', A)$ (with $V' \cap V = \emptyset$) and a mapping from V to V' as follows. There is a one to one correspondence between the edges of D and the members of $\mathscr{F}$ with the property that for every edge e of D the corresponding member of $\mathscr{F}$ consists precisely of those elements of S whose map is reachable in D from the head of E. We denote the map of an element $u \in V$ by u'.

For an edge f of D let $y(f) := y(F)$ where F is the member of $\mathcal{F}$ corresponding to f. For $u', v' \in V'$ let $y(u'v')$ denote the y-length of the unique (undirected) path in D between u' and v'. In this representation the feasibility of $(y, \mathcal{F})$ means that $m(uv) \geq y(u'v')$ for every $u, v \in S$ and the lemma follows from the following

Claim. *Either there is a node t' of D for which*

$$(*) \qquad m(tu) \geq y(t'u') \text{ for every } u \in S$$

or there is an edge $e = s'z'$ of D and an integer $0 < h < y(s'z')$ so that subdividing e by t' and defining $y(s't') := h$, $y(t'z') := y(s'z') - h$ $(*)$ holds true.

Proof. Let r' denote the root of D. If $m(tu) \geq y(r'u')$ holds for every $u \in S$, then $t' := r'$ will do. So suppose that $y(r'u') - m(tu) > 0$ for some $u \in S$ and let $u_0 \in S$ be an element for which $M = y(r'u_0') - m(tu_0)$ is maximum.

Let $s'z'$ be an edge of the path from r' to u_0' in D for which $y(r's') \leq M < y(r'z')$. If $y(r's') = M$, then choose $t' := s'$. If $y(r's') < M$, then subdivide $s'z'$ by a new node t' and choose $h := M - y(r's')$. With this choice we have $y(t'u_0') = m(tu_0)$ and we claim that $(*)$ is satisfied. To see this let D' denote the subdivided arborescence and let $u \in S$ be arbitrary.

Case 1. The path in D' from r' to u' contains t'. Then $y(r'u_0') = y(r't') + y(t'u_0')$. By the maximal choice of u_0 we have $y(r'u_0') - m(tu_0) \geq y(r'u') - m(tu)$. Therefore $m(tu) \geq y(r'u') + m(tu_0) - y(r'u_0') = y(r'u') - y(r't') = y(t'u')$.

Case 2. The path in D' from r' to u' does not contain t'. Then $y(u'u_0') = y(t'u_0') + y(t'u')$. We have $m(tu) \geq m(uu_0) - m(tu_0) = m(uu_0) - y(t'u_0') \geq y(u'u_0') - y(t'u_0') = y(t'u')$. □

This one element extension can be carried out in $O(n)$ steps, therefore the complete extension needs no more than $O(n^2)$ steps.

There is a version of Theorem 8.1 that ensures a maximum packing of T-cuts with a special structure. For a subset X of nodes let $q_T(X)$ denote the number of T-odd components in $G - X$ (a component is *T-odd* if it contains an odd number of nodes in T).

Theorem 8.5 (Frank, Sebő and Tardos 1984). *In a bipartite graph $G = (V_1, V_2; E)$ the minimum cardinality of a T-join is equal to* $\max \Sigma q_T(X_i)$ *taken over all partitions $\{X_i\}$ of V_1.*

This result immediately implies Theorem 8.1: for an optimal partition $\{X_i\}$ take the T-cuts defined by the T-odd-components of $G - X_i$ $(i = 1, 2 \ldots)$.

Before deriving these results let us mention an easy but useful lemma.

Lemma 8.6 (Mei-Gu Guan). *A T-join F is of minimum cardinality if and only if no circuit of G uses more edges from F than from $E - F$ (or in other words, there is no circuit of negative total weight in G where the edges of F have weight -1 the other edges have weight 1).*

We call a circuit of negative total weight a *negative circuit.*

To prove the non-trivial direction of Theorem 8.1 one starts with a minimum T-join F and wants to find $|F|$ disjoint T-cuts each of which containing one element of F. Obviously F is a forest since the edge-set of a circuit could be left out of F without changing the parity of the degrees. Here comes an observation: a cut that contains exactly one element of F is automatically a T-cut. Therefore Theorem 8.1 follows from the following.

Theorem 8.1$'$. *In a ± 1 edge-weighted bipartite graph there is no negative circuit if and only there are edge-disjoint cuts such that each contains one negative edge and each negative edge is contained in one cut.*

(This is exactly Theorem 3.6$'$ except that the wording is different.)

Actually Theorem 8.1$'$ follows from Theorem 8.1, as well. Indeed, the "if" part is easy. To see the "only if" part let F denote the set of edges of weight -1. Let T consist of those nodes that have an odd number of edges incident to F. Then F is a T-join and, by the lemma, F is a minimum T-join. By Theorem 8.1 there are $|F|$ disjoint T-cuts each of which necessarily contains exactly one element of F.

Using the same idea, Theorem 8.1w transforms into:

Theorem 8.1$'$w. *Let $G = (V, F^+ \cup F^-)$ be a graph and $d : F^+ \cup F^- \to \mathbb{Z}$ an integer-valued weight-function for which $d(e) \geq 0$ if $e \in F^+$ and $d(e) < 0$ if $e \in F^-$ and d satisfies the even-circuit property. There is no negative-circuit in G if and only if there is an integer-valued vector $y : 2^V \to \mathbb{Z}_+$ so that $y(A) > 0$ implies that $d_{F^-}(A) = 1$, that $\sum(y(A) : |A \cap \{u, v\}| = 1) \leq d(u, v)$ for every $uv \in F^+$ and $\sum(y(A) : |A \cap \{u, v\}| = 1) = d(uv)$ for every $uv \in F^-$.*

Note that the algorithm given after Theorem 8.1w can be used to construct either a negative circuit or a packing y. Namely, define $T = \{v \in V, d_{F^-}(v) \text{ odd}\}$ and find a minimum weight T-join F with respect to the weight function d' $|d(e)|$. If $d'(F) < d'(F^-)$, then the symmetric difference $F^- \oplus F$ contains a circuit of negative d-weight. If $d'(F) = d'(F^-)$, then the vector y in Theorem 8.1w will do for Theorem 8.1$'$w.

For later purpose we phrase here a fractional version of Theorem 8.1$'$.

Theorem 8.1$''$. *Let $\overline{G} = (V, \overline{E})$ be a graph where $\overline{E}$ is partitioned into two sets E and F. Let $y : E \to R$ be a rational vector for which $y(e) \geq 0$ if $e \in E$ and $y(e) \leq 0$ if $e \in F$. There is an assignment of non-negative variables $z(B)$ to cuts B containing exactly one edge from F for which $y(e) \geq \Sigma(z(B) : e \in B)$ for every $e \in E$ and $-y(e) \leq \Sigma(z(B) : e \in B)$ for every $e \in F$ if and only if there is no circuit of G with negative y-weight.*

If G is planar, we can take the dual graph and then Theorem 8.1 transforms into Theorem 3.6.

We can reformulate also Theorem 8.5, as follows.

Theorem 8.5′. *In a ±1 edge-weighted bipartite graph $G = (V_1, V_2; E)$ there is no negative circuit in G if and only if there is a partition $\{X_1, X_2, \ldots, X_k\}$ of V_1 such that no component of $G - X_i$ is entered by more than one negative edge.* $(i = 1, \ldots, k)$

Proof ("only if" part). We use induction on the number of nodes. If there are two nodes x and y in the same class V_1 with no negative path connecting them, then identify x and y into a single new node z. By induction there is a partition with the desired property. After splitting up z the same partition of V_1 satisfies the requirements.

So assume that there is a negative path between any two nodes in the same class. We are done by the following lemma of A. Sebő. See (Frank, Sebő and Tardos).

Lemma 8.7. *Let $G = (V_1, V_2; E)$ be a simple bipartite graph with at least three nodes and w a ±1-weighting on E. Suppose that there is no negative circuit but there is a negative path between every two nodes in the same class V_i. Then G is a tree and w is identically −1.*

Proof. Let P be a path of minimum weight m that has as few edges as possible. Let t be an end node of P and tx the first edge of P. Then (*) any starting segment of P has negative weight, in particular, $w(tz) = -1$. By induction the next claim implies the lemma.

Claim. *tx is the only edge of G incident to t.*

Proof. Suppose ty is another edge. $w(ty)$ must be positive since, if $y \in P$, then $P[t, y] \cup ty$ is a negative circuit, if $y \notin P$, then $P \cup ty$ is a path of weight $m - 1$.

By hypothesis there is a negative path Q between x and y. By parity, $w(Q) \leq -2$. Q passes through t since otherwise $Q + xt + ty$ would form a negative circuit. Moreover Q traverses the edge xt. For otherwise the weight of segment $Q[t, x]$ is at least 1, therefore the weight of $Q[t, y]$ is at most -3, and then $Q[t, y] + ty$ would form a negative circuit.

We also see that circuit $C = Q[t, y] + ty$ must have weight 0. Now C and P have solely node t in common. Indeed, let $z \in P \cap C - t$ be the first node of P (starting at t) distinct from t. By (*) $w(P[t, z]) < 0$. Hence the weight of both segments of C between z and t must be positive contradicting $w(C) = 0$. But now the paths P and $Q[t, y]$ together form a path of weight smaller than m, a contradiction. □

Theorem 8.5′ immediately implies the Berge-Tutte formula for the maximum cardinality of a matching. It also has the following pretty corollary.

Corollary 8.8 (Frank, Sebő and Tardos 1984). *A graph can be made Eulerian by doubling (parallelly) at most k edges if and only if $\Sigma q(V_i) \leq 2k$ for all partitions $\{V_1, \ldots, V_m\}$ of V where $q(X)$ denotes the number of components C of $G - X$ with $\nabla(C)$ odd.*

It is interesting to observe that function q also played a role in Theorem 7.3.

Exploiting further the idea introduced in Lemma 8.7, A. Sebő (1987) found the following refinement of Theorem 8.5′.

Theorem 8.9 (Sebő 1987). *Let G be a partite graph and w a ±1 weighting on the edges such that there is no negative circuit. Let s be a specified node of G and let $\lambda(v)$ denote the minimum w-weight of a path from s to v. Then for any integer i and for a component D of the subgraph induced by $V_i := \{v : \lambda(v) \leq i\}$ the cut $\nabla(D)$ contains exactly one negative edge if $s \notin D$ and no negative edge if $s \in D$.*

This theorem implies Theorem 8.1′: the set of cuts of form $\nabla(D)$, where D is a component of the subgraph induced by V_i and $s \notin D$, provides the desired packing. It is also easily seen that Theorem 8.5′ follows from Theorem 8.9.

Let F denote the set of edges e for which $w(e)$ is -1. Let T consist of those nodes of G that have an odd number of incident edges from F. For two nodes s, v let T' be the symmetric difference of T and $\{s, v\}$. It is not hard to see that $\lambda(v)$ is the difference of the minimum cardinality of a T'-join and the minimum cardinality of a T-join. Thus, by the remark after Theorem 8.1w, $\lambda(v)$ can be computed.

A striking consequence of Sebő's theorem is that, once the values $\lambda(v)$ have been computed, the packing of cuts in Theorem 8.1′ is also immediately available.

If G is planar in Theorem 8.9, a planar-dual form of the theorem can be stated. In a certain sense we obtain this way a canonized form of Theorem 3.6. Namely, suppose that $G + H$ is planar and Eulerian and $G + H$ has a fixed embedding into the plane. Assume that the cut criterion holds for G and H. For any face C of $G + H$ let $\lambda(C) := \min(|E \cap P| - |F \cap P| : P$ a dual path from the unbounded face to a $C)$. For each integer i let S_i denote the union of faces C of $G + H$ for which $\lambda(C) \leq i$. Then each non-empty S_i uniquely partitions into connected regions of the plane (connected in planar sense, that is, its boundary is a circuit of $G + H$). Call such a region an *island* if it is bounded.

Corollary 8.10 (Sebő 1989). *A circuit of $G + H$ bounding an island contains exactly one edge of H. Moreover, these circuits are edge-disjoint and each demand edge is contained in one of them.*

We conclude this section by mentioning a recent theorem by A.M.H. Gerards (1988). By an odd K_4 we mean a subdivided K_4 such that each face is odd. By a prism we mean the graph on six nodes consisting of two disjoint triangles and three disjoint edges connecting the two triangles. By an odd prism we mean a subdivided prism so that the two subdivided triangles are odd, while the two four-gons are even (see Figure 8.2).

Theorem 8.11 (Gerards 1988). *Let $G = (V, E)$ be a graph and $T \subseteq V$ a subset of nodes of even cardinality. If G contains neither odd K_4 nor odd prism, then the maximum number of disjoint T-cuts is equal to the minimum cardinality of a T-join.*

Note that both bipartite graphs and series-parallel graphs satisfy the assumptions therefore the theorem can be considered as a common generalization of two earlier results of Seymour.

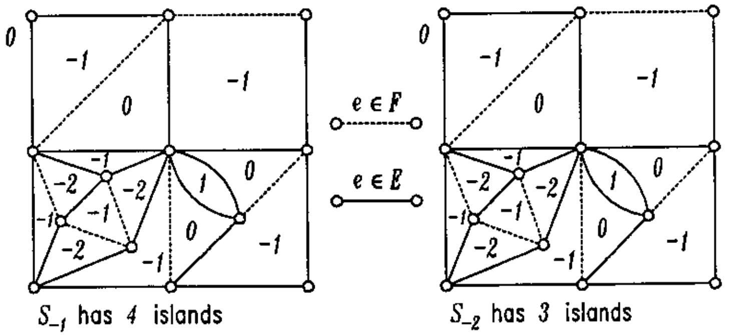
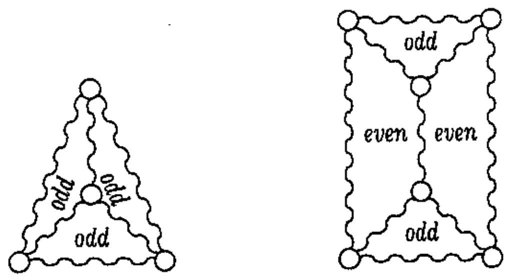


Fig. 8.1



odd K_4 odd prism

Fig. 8.2

Almost bipartite graphs (graphs with a node covering all odd circuits) also satisfy the assumption. Applying first Theorem 8.11 for planar almost bipartite graphs and then taking the planar dual one can easily derive the following extension of Theorem 3.6.

Corollary 8.12. *Suppose that $G+H$ is planar and the degree $d_{G+H}(v)$ of every node v not on the infinite face of $G+H$ is even. Then the cut criterion is necessary and sufficient for the solvability of the edge-disjoint paths problem.*

9. Packing Cuts and Circuits

There is another fundamental theorem concerning packing of cuts. Let $G=(V,E)$ be a directed graph. For a subset X of nodes, if there is no edge leaving X, the (non-empty) set of edges entering X is called a *directed cut*.

Theorem 9.1 (Lucchesi and Younger 1978). *The maximum number of disjoint directed cuts is equal to the minimum number of edges covering all the cuts.*

For a short proof, see (Lovász 1976a). (Frank 1981) includes an algorithmic proof. By planar dualization one obtains:

Theorem 9.1′. *In a planar directed graph the maximum number of edge-disjoint directed circuits is equal to the minimum number of edges covering all the directed circuits.*

An analogous min-max relation holds for minimum directed cuts. By a *minumum directed cut* we mean a directed cut of least cardinality. The following result is a special case of a theorem of Edmonds and Giles (1977).

Theorem 9.2. *In a directed graph the maximum number of disjoint minimum directed cuts is equal to the minimum number of edges covering all the minimum directed cuts.*

This result can also be dualized. For example one gets: In a planar directed graph with no oppositely directed edges the maximum number of edge-disjoint directed triangles is equal to the minimum number of edges covering all directed triangles.

Sometimes the theorem of Lucchesi and Younger can be used for undirected graphs. For example, given a planar Eulerian graph G, what is the maximum number of circuits into which the edge set of G can be partitioned? D. Younger observed (as was communicated to me by W. Pulleyblank) that if we orient the edges of G in such a way that each face is surrounded by a directed circuit (we assume that G is 2-connected), then the maximum number of edge-disjoint directed circuits in the orientation of G is the same as the maximum number of edge-disjoint circuits in the undirected graph. (This is a useful exercise).

In Section 3 we briefly indicated how to derive Theorems 3.2-3.6 from their fractional forms. In the next few paragraphs we exhibit an approach, related to packing of cuts, by which the cut criterion can be proved to be sufficient for the existence of a multiflow, at least in some special cases.

Let us recall Theorem 2.0: a multiflow problem has a solution if and only if the distance criterion holds. Therefore if we want to show that in a certain case already the cut criterion is sufficient, we have to show that the cut criterion implies the distance criterion. One way to do so is, roughly, to point out that the vector w in Theorem 2.0 can be expressed as a non-negative linear combination of cuts.

Let $G=(V,E)$ and $H=(V,F)$ be graphs and w a non-negative rational weight function on E. Let $dist_w(u,v)$ denote the minimum w-weight of a path in G connecting u and v.

Theorem 9.3. *Suppose that either*

(a) (Schrijver 1990) $G=(V,E)$ is planar, C_1 and C_2 are two specified faces of G and $H=(V,E)$ is the union of two complete graphs on $V(C_1)$ and $V(C_2)$, or

(b) $G+H$ is planar, or

(c) $H=(V,F)$ is either K_4 or C_5 (Karzanov 1985a) or a double-star (Seymour 1978).

(A) Then there exists a fractional packing of cuts, that is an assigment on non-negative variables $x(B)$ to cuts B such that for each edge $uv \in F$ $dist_w(u,v) = \Sigma(x(B) : B$ a cut, $uv \in B)$ and for each edge $uv \in E$ $w(uv) \geq \Sigma(x(B) : B$ a cut, $uv \in B)$.

(B) Moreover, if w is integer-valued such that every circuit of G has even w-weight, then x can be chosen integer-valued.

Proof of the fractional versions of Theorems 3.3, 3.6 and 3.8. We use part (A) of Theorem 9.3. The statement corresponding to cases (a),(b) and (c) will imply the fractional versions of Theorems 3.3, 3.6 and 3.8, respectively. Indeed, assume that there is a w violating the distance criterion. Let x be the variables in Theorem 9.3 assigned to the cuts. We have $\Sigma(dist_w(u,v) : uv \in F) = \Sigma(\Sigma x(B) : B$ a cut and $uv \in B) : uv \in F) = \Sigma(x(B)|F \cap B| : B$ a cut$) \leq \Sigma(x(B)|E \cap B| : B$ a cut$) = \Sigma(\Sigma x(B) : B$ a cut and $uv \in B) : uv \in E) \leq \Sigma(w(uv) : uv \in E)$, contradicting the assumption that w violates the distance criterion. Here the first inequality follows from the cut criterion. □

Notice that the above derivation works in the other direction as well, that is the fractional versions of Theorems 3.3, 3.6 and 3.8 imply part (A) of Theorem 9.3.

Remark. In this application we used only part (A) of Theorem 9.3. Part (B) should be considered interesting for its own sake. Actually, Schrijver, Karzanov and Seymour proved part B of cases (a) and (c), respectively, and observed that part (B) immediately implies part (A). (A relatively simple proof of Karzanov's theorem can be found in (Schrijver 1988a). Karzanov (1986b) gave a constructive proof of part B in case (a) that provides a strongly polynomial algorithm). The story of case (b) is different. We are going to show that part (B) of case (b) is equivalent to the following theorem of P. Seymour.

Theorem 9.4 (Seymour 1979). *Let $G' = (V', \overline{E})$ be a planar graph and $w' : \overline{E} \to \mathbb{Z}_+$ such that $w(B)$ is even for every cut B of G. There are non-negative integer variables $x(C)$ assigned to the circuits C of G' such that $w'(e) = \Sigma(x(C) : C$ a circuit and $e \in C)$ holds for every edge e if and only if $w'(e) \leq w'(B-e)$ holds for every cut B and edge $e \in B$.*

(The proof of this theorem is rather difficult.) By planar dualization we obtain

Theorem 9.4′. *Let $\overline{G} = (V, \overline{E})$ be a planar graph and $w' : \overline{E} \to \mathbb{Z}_+$ such that $w'(C)$ is even for every circuit C of $\overline{G}$. There are non-negative integer variables $x(B)$ assigned to the cuts B of $\overline{G}$ such that $w'(e) = \Sigma(x(B) : B$ a cut and $e \in B)$ holds for every edge e if and only if (*) $w'(e) \leq w'(C-e)$ holds for every cut C and edge $e \in C$.*

Proof of part (B) of Theorem 9.3b. Replace each edge e of G by a path of two edges e' and e'' (that is subdivide each edge by a new node) and let $w'(e') := \lfloor (w(e)/2) \rfloor$, $w'(e'') = \lceil (w(e)/2 \rceil$. Since $w(e) = w'(e') + w'(e'')$ this operation does not affect the w-distances of original nodes. For $uv \in F$ let $w'(uv) := dist_w(u,v)$ and let $\overline{G} = (V, \overline{E})$ be a graph where $\overline{E} = F \cup E' \cup E''$.
(Here E' and E'' denote the corresponding copies of E).

Since every circuit of G has even w-weight, every circuit of $\overline{G}$ has even w'-weight. An easy argument shows that the w'-distance of u and $v (uv \in F)$ in $\overline{G}$ is $dist_w(u,v)$. Therefore the hypotheses and (*) of Theorem 8.4′ hold and then there is an x as described in the theorem. Since every cut of $\overline{G}$ which is not a star of a new node determines a cut of G, by leaving out these stars we obtain from x the desired solution to part (B) of Theorem 9.3b. □

Proof of Theorem 9.4′ from Theorem 9.3. Let E and F be two copies of $\overline{E}$ (that is, to each edge $e \in \overline{E}$ there corresponds an edge in E and an edge in F that are parallel). Apply part B of Theorem 9.3b and let x be the integer vector provided by the theorem. Since (*) implies that $dist_{w'}(u,v) = w'(e)$ for every edge $e = uv \in \overline{E}$, x will do for Theorem 9.4′ as well. □

As far as part (A) of Theorem 9.3b is concerned, it follows from part (B) Theorem 9.3b but there is a more general result here. An equivalent reformulation of part (A) of Theorem 9.3b is the following.

Theorem 9.5′. *Let $G = (V,E)$ and $H = (V,F)$ be graphs for which $G+H$ is planar and let w and w' be two non-negative rational weight functions on E and on F, respectively. Then there exists a fractional packing of cuts, that is an assignment of non-negative variables $x(B)$ to cuts B such that for each edge $uv \in F$ $w'(f) \leq \Sigma(x(B) : B$ a cut, $f \in B)$ and for each edge $uv \in E$ $w(uv) \geq \Sigma(x(B) : B$ a cut, $uv \in B)$ if and only if $w'(f) \leq w(C-f)$ holds (equivalently, $w'(f) \leq dist_w(u,v)$) for each circuit C of $G+H$ containing exactly one edge $f = uv$ from F.*

Now the promised generalization states that if we take the planar dual form of Theorem 9.5′, then planarity can be left out from the premisses.

Theorem 9.5 (Seymour 1979). *Let $G = (V,E)$ and $H = (V,F)$ be two graphs and let w and w' be two non-negative rational weight functions on E and on F, respectively. Then there exists a fractional packing of circuits, that is an assignment of non-negative variables $x(C)$ to circuits C such that for each edge $uv \in F$ $w'(f) \leq \Sigma(x(C) : C$ a circuit, $uv \in C)$ if and only if $w'(f) \leq w(B-f)$ holds for each cut B of $G+H$ containing exactly one edge $f = uv$ from F.*

Proof. By Farkas' lemma if the desired x does not exist, then there is a vector $y : E \to \mathbb{R}$ with $y(e) \geq 0$ if $e \in E$, $y(e) \leq 0$ if $e \in F$ such that $y(C) \geq 0$ for every circuit of $G+H$ and (*) $\Sigma(y(e)w(e) : e \in E) + \Sigma(y(f)w'(f) : f \in F) < 0$.

Apply Theorem 8.3″ and let z be the vector in the theorem. We have $\Sigma(-y(f)w'(f) : f \in F) \leq \Sigma(w'(f)\Sigma(z(B) : e \in B$, B a cut containing solely f from $F) : e \in F) \leq \Sigma(w(B-f)\Sigma(z(B) : B$ a cut containing solely f from

$F) : f \in F) = \Sigma(w(e)\Sigma(z(B) : e \in B$, B a cut containing one element of $F) : e \in E) \leq \Sigma(y(e)w(e)) : e \in E)$, contradicting (*). □

(Note that the relation between Theorems 9.5 and 8.1″ is the same as the relation between the fractional versions of Theorems 3.3, 3.6 and 3.8 and part A of Theorem 9.3a, b and c, respectively.)

The problem of Theorem 9.5 can be interpreted so that one wants to find a fractional packing of circuits of $G+H$ such that certain edges satisfy an upper bound condition (edges in E) while other edges satisfy a lower bound condition (edges in F). We can impose both lower and upper bounds for every edge:

Theorem 9.6 (Seymour 1979). *Let $\overline{G} = (V, \overline{E})$ be an undirected graph endowed with two functions $f : \overline{E} \to \mathbb{R}_+$, $g : \overline{E} \to \mathbb{R}_+ \cup \{\infty\}$ for which $f \leq g$. There are non-negative variables $x(C)$ assigned to the circuits C of G for which $f(e) \leq \Sigma(x(C) : C$ a circuit and $e \in C) \leq g(e)$ holds for every edge e if and only if $\Sigma(f(e) : e$ enters $X) \leq \Sigma(g(e) : e$ leaves $X)$ holds for every subset X of nodes. Moreover, if f and g are integer-valued, x can be chosen integer-valued.*

An important difference between the directed and the undirected case is that the special case $f \equiv g$ is trivial for directed graphs while this is the crucial part in Seymour's proof of the undirected case.

Another essential difference is that for directed graphs one has the integrality result which is not so for undirected graphs. The Petersen graph shows that the integral packing of circuits does not necessarily exist: define f and g to be 2 on the edges of a specified perfect matching of the Petersen graph and 1 otherwise. In this view we should even more appreciate Theorem 9.4. (We note that even for planar graphs there is no known characterization for the existence of packing circuits if lower and upper bounds are imposed on the edges).

Let us conclude this section by presenting a generalization of Theorem 9.4. Let $G = (V, E)$ be an Eulerian graph. At every node $v \in V$ a partition $\mathcal{P}(v)$ of the edges incident to v is specified. A member of $\mathcal{P}(v)$ is called a *forbidden part* and a subset of a forbidden part is called a *forbidden set* if it has at least two elements. Let $\mathcal{P} := \cup(\mathcal{P}(v) : v \in V)$ denote the set of forbidden parts.

A circuit of G is called *good* if it includes no forbidden sets. If a cut S contains more than $|S|/2$ elements from a forbidden part P, then S is called *bad* (with respect to $\mathcal{P}$).

Theorem 9.7 (Fleischner and Frank 1988). *The edge set of a planar Eulerian graph can be partitioned into good circuits if and only if there are no bad cuts.*

This theorem immediately implies Theorem 9.4: replace each edge e by $w(e)$ parallel edges and let the forbidden parts consist of the sets of parallel edges. Another special case of the theorem is an earlier result of H. Fleischner (1980) when each forbidden part has at most two elements.

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