Connectivity Augmentation Problems in Network Design

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1 Introduction

In network design it is a fundamental problem to construct graphs or subgraphs of a graph of minimum cost satisfying certain connectivity specifications.

Shortest paths between two specified nodes, or minimum cost spanning trees may be viewed as (well-known) special cases of this problem. Very often a starting graph is already available and the goal is to augment the graph. For example, at least how many new edges must be added to a digraph to make it strongly connected?

Having such a broad class of problems (already a special case, the well-known Steiner-tree problem, has a vast literature), it is of no surprise that a large number of connectivity augmentation problems are NP-complete. But there are interesting special cases, as well, for which polynomial time algorithms are available. Investigations and results of this problem may be categorized into three main directions. One is concerned with heuristics, often based on deep theoretical background such as the polyhedral method, that work well in practice. An excellent survey paper by M. Grötschel, C. L. Monma and M. Stoer [30] summarizes this type of results. Another line is to develop approximation algorithms whose running time is polynomial and the output is provably not much worse than the optimum. In a recent Ph.D. thesis, D. P. Williamson [57] provides a rich class of problems of this type along with approximation algorithms for their solution.

The purpose of the present paper is to survey connectivity augmentation problems for which an algorithm is available to find the exact optimum in (strongly) polynomial time. Though polynomially solvable problems are often too restricted to be used directly in practical applications, they may serve well as building blocks in a more complex procedure. In many cases, such as the k-edge-connectivity augmentation problem on digraphs, the proposed algorithm is purely combinatorial and strongly polynomial.

Such algorithms have the feature (as opposed to the ellipsoid method, say) that they can be run in practice and used for large graphs. In some other cases, such as the k-node-connectivity augmentation problem on digraphs, a theoretical background has been developed which, at least, ensures a polynomial-time algorithm via the ellipsoid method. But such methods might never be used in practice. Their existence

should be considered merely as a challenge to design purely combinatorial solution algorithms. Also, these algorithms are important from a theoretical point of view in order to explore the borderline (if there is any) between NP-completeness and polynomial solvability.

We will use two basic measurements for connectivity. Given a directed or undirected graph G, $\lambda(x,y) = \lambda(x,y;G)$ (respectively, $\kappa(x,y) = \kappa(x,y;G)$) denotes the maximum number of edge-disjoint (openly disjoint) paths from x to y. K. Menger's classical theorem asserts that in an undirected graph if x and y are not adjacent, then $\kappa(x,y)$ is equal to the minimum number of nodes whose deletion separates x and y. Other versions of Menger's theorem provide min-max formulae for κ and λ in directed and in undirected graphs. For example, in a digraph $\lambda(x,y)$ is equal to the minimum number of edges leaving an $x\bar{y}$-subset of nodes. (A set X is called an **$x\bar{y}$-set** if $x \in X, y \notin X$.) This is an equivalent formulation of the max-flow min-cut theorem. ($\lambda(x,y)$ may be considered as the maximum flow value from x to y provided that the capacity of each edge is 1.)

An undirected graph G is called **k-edge-connected** if every cut has at least k edges. When $k=1$ we simply say G is connected. A digraph D is called **k-edge-connected** if every (non-empty, proper) subset of nodes has at least k exiting edges. When $k=1$ we call D **strongly connected**. By Menger's theorem a digraph or an undirected graph is k-edge-connected if and only if there are k edge-disjoint paths from every node to every other one.

The general form of the augmentation problems we investigate is as follows. Given a directed or undirected graph $G=(V,E)$ and a non-negative integer function $r(x,y)$ on the set of ordered pairs of nodes, serving as a **demand function**, add a minimum number of new edges to G (or, more generally, a minimum cost set of new edges, if a cost-function is given on the set of possible new edges) so that

$$\lambda(x,y;G^+) \geq r(x,y) \tag{1.1a}$$

or

$$\kappa(x,y;G^+) \geq r(x,y) \tag{1.1b}$$

holds for every pair of nodes x,y of the resulting graph (digraph) G^+. Accordingly, we may speak about **edge-connectivity augmentation** problem or **node-connectivity augmentation** problem.

Beside these minimization forms, we will consider degree-constrained augmentation problems, as well, where a lower and upper bound is given at every node v for the number of new edges incident to v.

A natural relaxation of the augmentation problem is the **max flow version.** Suppose that $g(u,v)$ is a non-negative capacity function on the pairs of nodes u,v ($u,v \in V$) and let $r(u,v)$ be a demand function. The problem is to increase the existing capacities so that in the resulting network the maximum flow value between u and v is at least $r(u,v)$ for each pair $\{u,v\}$ of nodes and such that the sum of capacity increments is minimum.

If $g(u,v)$ and $r(u,v)$ are integer-valued and the capacity increments are also required to be integer-valued, then the edge-connectivity augmentation problem is equivalent to the max-flow version. Namely, the latter problem can be formulated as a max-flow version by letting $g(u,v)=1$ when (u,v) is an edge of G and $g(u,v)=0$ otherwise. Conversely, if g is integer-valued, we can define a graph having $g(u,v)$ parallel edges between each pair of nodes u and v and then a solution to the edge-connectivity problem yields a solution to the integer-valued max-flow problem.

This equivalence, however, does not mean algorithmic equivalence. We are going to exhibit strongly polynomial time algorithms for the more difficult max-flow augmentation problem. (A polynomial time algorithm is called **strongly polynomial** if it uses only basic operations, such as comparing, adding, subtracting, multiplying, and dividing numbers, and the number of these operations is independent of the numbers occurring in the input.)

We will also investigate the question of when the fractional augmentation allows better solution than the integer-valued one. For example, let V be a set of n nodes, $g \equiv 0$ and $r \equiv 1$. If only integers are allowed for the increments, then the value of the best solution is $n-1$: take any tree on V and increase the capacity of its edges by one. If we may use fractional increments, then the value of the best solution is $n/2$: take any circuit of n nodes, increase the capacity of its edges by 1/2.

On the other hand, we will see problems (concerning mainly undirected graphs) when the optimum of the integer-valued solution is at most one half bigger than that of the fractional solution, and problems (especially when G is directed) when there is an optimal solution to the max-flow augmentation problem that is integer-valued.

Given two elements s,t and a subset X of a ground-set U, we say that X is an $s\bar{t}$**-set** if $s \in X, t \notin X$. X **separates** s **from** t (or x and t) if $|X \cap \{s,t\}| = 1$. A family $\{X_1,\dots,X_t\}$ of pairwise disjoint, non-empty subsets of U is called a **sub-partition**.

Let $G=(U,E)$ an undirected graph. $d_G(X,Y)$ denotes the number of undirected edges between $X-Y$ and $Y-X$. $\bar{d}_G(X,Y) := d_G(X,U-Y)(= d_G(U-X,Y))$. $d_G(X)$ stands for $d_G(X,U-X)$. Observe that $\bar{d}_G(X,Y) = \bar{d}_G(U-X,U-Y)$. When it does not cause ambiguity, we leave out the subscript.

For a directed graph $D=(U,A)$ $\varrho_D(X)$ denotes the number of edges entering X, $\delta_D(X) := \varrho_D(U-X)$ and $\beta_D(X) := \min(\varrho_D(X), \delta_D(X))$. Note that $\beta_D(X) = \beta_D(U-X)$. $d_D(X,Y)$ denotes the number of edges with one end in $X-Y$ and one end in $Y-X$. $\bar{d}_D(X,Y) := d_D(X,U-Y)(= d_D(U-X,Y))$. An **arborescence** F is a directed tree in which every node but one has in-degree 1 and the exceptional node, called the **root**, is of in-degree 0. (Equivalently, there is a directed path from the root to every other node of F.)

Let $M=(U, A\cup E)$ be a mixed graph composed as the union of a directed gaph $D=(U,A)$ and an undirected graph $G=(U,E)$. Let $\varrho_M(X) := \varrho_D(X) + d_G(X)$, $\delta_M(X) := \delta_D(X) + d_G(X)$, and $\beta_M(X) := \min(\varrho_M(X), \delta_M(X))$. We say that a node v of a M **is di-Eulerian** if $\varrho_D(v) = \delta_D(v)$. M is called **di-Eulerian** if every node of M is di-Eulerian.

Splitting off a pair of edges $e = us, f = st$ means that we replace e and f by a new edge ut. The resulting mixed graph will be denoted by M^{ef}. This operation is defined only if both e and f are undirected (respectively, directed) and then the newly added edge ut is considered undirected (directed). Accordingly, we speak of undirected or directed splittings.

For a function $m: V \to R$ we use the notation $m(X) := \sum(m(v) : v \in X)$. For a number x, let $x^+ := \max(x,0)$.

2 Subgraphs versus Supergraphs

To clarify a simple link between optimal subgraphs and optimal supergraphs, we start with a specific problem. We are given a digraph $D=(V,A)$ with two specified nodes s and t. One of the simplest connectivity property one may consider in D is

$$\lambda(s,t) \geq 1, \tag{2.1}$$

that is, there is a path from s to t. It is well-known that (2.1) holds if and only if every $s\bar{t}$-set has an exiting edge.

When (2.1) holds, one may consider the shortest path problem, a starting point of combinatorial optimization, that consists of finding a path from s to t of minimum cost with respect to a given cost function c on E. For non-negative c, this may be considered as a SUBGRAPH problem: given a digraph, find a minimum cost subgraph satisfying (2.1). Dijkstra's classical algorithm for finding a shortest paths is of $O(n^2)$ complexity.

If (2.1) does not hold, then a natural task is to augment optimally D so as to satisfy (2.1). Augmentation may be considered as a SUPERGRAPH problem: given a digraph D and a digraph $H=(V,F)$ of possible new edges which is endowed with a cost function c, in $D+H$ construct a minimum cost supergraph of D satisfying (2.1).

This augmentation problem may be solved by a shortest path computation in the digraph $D+H$ where the cost of the edges of H is determined by c and the cost of the original edges is defined to be 0.

This easy principle may be applied to properties other than (2.1). If one is able to solve the minimum cost subgraph problem, one can solve the corresponding supergraph (that is, the augmentation) problem, as well. Below we list some other connectivity properties when the subgraph problem is efficiently solvable and, therefore, so is the augmentation problem. But we already hasten to emphasize that the focus of this paper will be on polynomially solvable augmentation problems where the corresponding subgraph problem is NP-complete.

Properties in a digraph for which the minimum cost subgraph problem (and the minimum cost augmentation problem, as well) is solvable in strongly polynomial time are:

$$\lambda(s,t) \geq k, \tag{2.2}$$

$$\kappa(s,t) \geq k, \tag{2.3}$$

$$\lambda(s,x) \geq k \text{ for every } x \in V, \text{ and} \tag{2.4}$$

$$\kappa(s,x) \geq k \text{ for every } x \in V. \tag{2.5}$$

The minimum cost subgraph problem with respect to (2.2) is equivalent to finding min-cost flow of value k [52]. By an easy elementary construction, observed already in [12], (2.3) goes back to (2.2).

Finding a minimum cost subgraph satisfying (2.4) is trickier. First, we may assume that no edge of D enters s. Since the cost function is supposed to be non-negative, it is enough to consider digraphs satisfying (2.4) which are minimal with respect to edge-deletion. The main observation is that such digraphs are precisely those in which (a) the in-degree of every node $v \neq s$ is precisely k and (b) the underlining undirected graph is the union of k disjoint spanning trees. (The equivalence may be proved by Edmonds' [7] arborescence theorem) By this equivalent formulation the problem is to find a minimum cost common basis of two matroids M_1 and M_2 defined on the edge-set of D. Here M_1 is a partition matroid in which a set is independent if it contains at most k edges entering any node $v \neq s$. M_2 is defined to be the sum of k copies of the circuit matroid of the underlining undirected graph (that is, a subset of edges of D is independent in M_2 if it is the union of k forests).

Since there are strongly polynomial algorithms for the weighted matroid intersection problem [9, 15] the minimum cost subgraph problem with respect to (2.4) is also solvable. By exploiting the particular structure of the two matroids in question, H. Gabow [27] developed a more efficient algorithm.

Note that the special case of (2.4) when $k = 1$ is tantamount to finding a minimum cost arborescence of root s, for which D.R. Fulkerson [25] described a particularly elegant algorithm.

No elementary reduction of Property (2.5) to (2.4) is known. A solution to the subgraph minimization problem with respect to (2.5) was described in [22]. It used a tricky reduction to submodular flows [10], a common generalization of network flows and matroid intersection. Here we do not repeat the reduction but provide a min-max theorem concerning the corresponding augmentation problem, which is deducible from the theory of submodular flows but was not explicitly stated in [22].

Let us call a digraph satisfying (2.5) **k-out-connected (from s)**. Let $D = (V, E)$ be a digraph with a specified node s and assume our task is to augment D to obtain a k-out-connected digraph. Let $H = (V, F)$ denote the digraph of possible new edges and $c : F \to R_+$ a cost function. In order to have a solution at all we assume that the union graph $D + H$ is k-out-connected.

Let $\mathcal{F}$ denote the family of pairs (A, B) of two non-empty disjoint subsets of nodes so that $s \in A$. For a pair (A, B) let $\delta(A, B) := \delta_D(A, B)$ denote the number of edges of D from A to B. By a version of Menger's theorem a digraph is k-out-connected from s if and only if $|V - (A \cup B)| + \delta(A, B) \geq k$. Define the **deficiency** $p_{def}(A, B)$ of a pair (A, B) by $k - (|V - (A \cup B)| + \delta(A, B))$ when this number is positive and by zero otherwise. Clearly, adding a subset X of edges of H to D yields a k-out-connected digraph if and only if there are at least $p(A, B)$ edges in X going from A to B for every pair $(A, B) \in \mathcal{F}$.

Theorem 2.1 *The minimum cost of edges of H whose addition to D results in a digraph which is k-out-connected from s is equal to* max$(\sum y(A,B)p(A,B))$ *where $y \geq 0$ is such that, for every edge $xy \in F$, $\sum(y(A,B) : x \in A, y \in B) \leq c(xy)$. Moreover, if c is integer-valued, y may be chosen integer-valued.*

Actually, this theorem asserts that a certain linear program is totally dual integral. It turns out that the theorem can be stated in a more abstract form. Let $H = (V, F)$ be a directed graph endowed with a cost function $c : F \to R$ and a capacity function $g : F \to Z_+$. Let p be a non-negative integer-valued function on the pairs (A, B) of disjoint subsets of V. We say that p is **intersecting bi-supermodular** if

$$p(X,Y) + p(X',Y') \leq p(X \cap X', Y \cup Y') + p(X \cup X', Y \cap Y') \tag{2.6}$$

holds whenever $p(X,Y), p(X',Y') > 0, Y \cap Y' \neq \emptyset$.

For a vector x defined on the edge-set F let $\delta_x(A,B) := \sum(x(uv) : uv \in F, u \in A, v \in B)$.

Theorem 2.2 *Let $g : F \to Z_+$ be an integer-valued capacity function so that $\delta_g(A,B) \geq p(A,B)$ for every pair (A,B). Then the linear program*

$$\min(cx : 0 \leq x \leq g, \delta_x(A,B) \geq p(A,B) \text{ for every disjoint } A, B) \tag{2.7}$$

is totally dual integral. In particular, (2.7) has an integer-valued optimum and if in addition c is integer-valued, the dual linear program also has an integer-valued optimum.

(This theorem may be proved by using the standard uncrossing technique as was done in [13, 16] for the special special case when $p(A,B)$ may be positive only on complementary pairs (i.e. $A \cup B = V$.)

Note that the role of the two variables of p is not symmetric. It becomes symmetric if (2.6) is required only when $p(X,Y), p(X',Y') > 0, X \cap X', Y \cap Y' \neq \emptyset$. In this case p is called **crossing bi-supermodular.** Theorem 2.2 is no more true for crossing bi-supermodular functions. But in Section 4 we will prove that a min-max theorem does hold when H is a complete directed graph (i.e. each of the possible $n(n-1)$ edges belong to H) and $c \equiv 1$. Such a result will allow us to solve the node-connectivity augmentation problem in directed graph when arbitrary edges may be added.

3 Edge-Connectivity Augmentation of Digraphs

In the previous section we have seen that the minimum cost subgraph problem, and therefore the minimum cost augmentation problem, is tractable for properties (2.1) and (2.2).

The next natural property to be investigated is strong connectivity. The minimum cost subgraph problem reads: find a minimum cost strongly connected spanning subgraph of a given digraph. This is NP-complete even if the cost function is identically 1 since if one is able to find a strongly connected subgraph of minimum cardinality, then one is able to decide if a digraph contains a strongly connected subgraph of cardinality n (the number of nodes) and this latter property is equivalent for a digraph to have a Hamiltonian circuit.

The corresponding augmentation problem asks, given a digraph $D = (V, E)$ and another digraph $H = (V, F)$ (endowed with a non-negative cost-function c), for the minimum cost of edges of H whose addition to D creates a strongly connected digraph.

A similar argument shows that the augmentation problem is also NP-complete, even for $c \equiv 1$, if no restriction is made for H. One interesting restriction is when there is a path in D from v to u for each edge uv of H. (For example, when H arises from D by re-orienting each edge.) For the cardinality case, a theorem of Lucchesi and Younger [43] asserts in the present context that *the minimum cardinality of new edges of H to be added to D to provide a strongly connected digraph is equal to the maximum number of H-independent source-sets of D.* (A proper non-empty subset X of V is called a **source-set** if no edge of D enters X and a family of source-sets is **H-independent** if no edge of H enters more than one of them. For later purposes we define **sink-sets** as the complement of source-sets). The theory of submodular functions (established in [10] extends this theorem to the weighted case. In [14] a strongly polynomial time algorithm was developed to find the minimum in question.

The augmentation problem for strong connectivity was solved by K.P. Eswaran and R.E. Tarjan [11] in the case when any possible new edge is allowed to be added and $c \equiv 1$. In a digraph the sink-sets are closed under taking intersection and union. Hence the minimal sink-sets (with respect to containment) are pairwise disjoint. Let p_1 denote their number. Similarly, the minimal source-sets are pairwise disjoint. Let p_2 denote their number. Since in a strongly connected digraph there are no source-sets and sink-sets, at least $\max(p_1, p_2)$ new edges must be added. The next theorem asserts that this bound is achievable. Note that it is not difficult to calculate p_1 (or p_2) since p_1 is the number of sink-nodes (nodes with no leaving edges) of the digraph arising from D by contracting each strong component into one node.

Theorem 3.1 (K.P. Eswaran and R.E. Tarjan [11]) *Given a directed graph $D = (V, E)$ the minimum number of new edges whose addition to D creates a strongly connected digraph is* $\max(p_1, p_2)$.

The proof of Eswaran and Tarjan is constructive and gives rise to a linear-time algorithm.

As we mentioned before, the minimum cost version of the problem is NP-complete. However, the minimum node-cost augmentation problem is solvable as will be shown in a more general context.

In order to generalize the cardinality case of the strong connectivity augmentation problem, suppose that a subset T of nodes is specified in a digraph $D = (V, E)$ and our purpose is to add a minimum number of new edges so that every element of T be reachable from every other element of T. It is not difficult to see [18, 19] that this problem is NP-complete. We will show, however, (even in a more general context) that this problem is solvable in polynomial time if the new edges are required to have both end-nodes in T.

Let us turn to this general case when we require k-edge-connectivity for the augmented digraph, that is the demand function $r(u, v) \equiv k$. The directed k-edge-connectivity augmentation problem was solved by D.R. Fulkerson and L.S. Shapley [26] when the starting digraph $D = (V, \emptyset)$ has no edges at all, by Y. Kajitani and S. Ueno (1986) when the starting digraph is a directed tree and by Frank [18] for an arbitrary starting digraph. The major idea that led to the solution to the minimization problem came from the recognition that degree-prescribed augmentation problems serve as useful intermediate problems.

Let $D = (V, E)$ be a digraph and $m_o, m_i : V \to Z_+$ two integer-valued functions so that $m_o(V) = m_i(V)$.

Theorem 3.2 *A directed graph $D = (V, E)$ can be made k-edge-connected by adding a set F of new edges satisfying*

$$\varrho_F(v) = m_i(v) \text{ and } \delta_F(v) = m_o(v) \tag{3.1}$$

for every node $v \in V$ if and only if both

$$\varrho(X) + m_i(X) \geq k \text{ and } \delta(X) + m_o(X) \geq k \tag{3.2}$$

hold for every $X \subseteq V$.

Note that F may contain parallel edges or even loops. It is an important open problem to find characterizations for the existence of an F without loops and parallel edges. To get rid of the loops is at least easy (see, Corollary 3.6).

A crucial observation is that Theorem 3.2 is nothing but a re-formulation of W. Mader's directed splitting off theorem:

Theorem 3.3 (Mader [46]) *Let $D = (V + s, A)$ be a directed graph for which $\varrho(s) = \delta(s)$ and $(*)$ $\lambda(x, y) \geq k$ for every $x, y \in V$. Then the edges entering and leaving s can be partitioned into $\varrho(s)$ pairs so that splitting off all these pairs leaves a k-edge-connected digraph.*

To derive Theorem 3.2, extend D by a new node s and for each $v \in V$ adjoin $m_i(v)$ (respectively, $m_o(v)$) parallel edges from s to v (from v to s). Now by (3.2) the hypotheses of Theorem 3.3 are satisfied and hence we can split off γ pairs of edges to obtain a k-edge-connected digraph. The resulting set of γ new edges (connecting original nodes) satisfies the requirement.

Mader stated his theorem in the form that there is a pair of edges, entering and leaving s, which is splittable in the sense that their splitting off does not destroy $(*)$. Since $\varrho(s) = \delta(s)$, by repeated applications of this one gets Theorem 3.3. An example (in which $\varrho(s) = 1, \delta(s) = 2$) shows that the existence of a splittable pair is not necessarily true without the assumption $\varrho(s) = \delta(s)$. As a slight generalization of Mader's theorem I can prove that there is a splittable pair if $\varrho(s) \le \delta(s) < 2\varrho(s)$ but I do not know any application of this result.

In [19, 18] the following characterization was derived for the minimization problem.

Theorem 3.4 *A directed graph $D = (V, E)$ can be made k-edge-connected by adding at most γ new edges if and only if*

$$\sum_i (k - \varrho(X_i)) \le \gamma \text{ and } \sum_i (k - \delta(X_i)) \le \gamma \qquad (3.3)$$

holds for every sub-partition $\{X_1, \dots, X_t\}$ of V.

This was proved with the help of Theorem 3.2. (The proof method gives rise to a polynomial time algorithm which is actually strongly polynomial even in the capacitated case). Here we prove an extension of Theorem 3.2 and, using that, derive an extension of Theorem 3.4.

Since (3.3) is a necessary condition for the fractional augmentation, as well, we can conclude that the integer-valued optimum is the same as the fractional optimum.

Let us turn to the generalization of Theorem 3.2 and 3.4.

Theorem 3.5 *Let T be a ground-set, p a non-negative, integer-valued function defined on subsets of T for which $p(\emptyset) = p(T) = 0$ and $p(X) + p(Y) \le p(X \cap Y) + p(X \cup Y)$ holds whenever $p(X) > 0, p(Y) > 0, X \cap Y \ne \emptyset, T - (X \cup Y) \ne \emptyset$. Let m_i, m_o be two modular non-negative integer-valued functions on T for which $m_i(T) = m_o(T) = \gamma$. There exists a digraph $H = (T, F)$ for which*

$$\varrho_H(X) \ge p(X) \text{ for every } X \subseteq T \qquad (3.4)$$

and

$$\varrho_H(v) = m_i(v) \text{ for every } v \in T \qquad (3.5a)$$

$$\delta_H(v) = m_o(v) \text{ for every } v \in T \qquad (3.5b)$$

if and only if

$$m_i(X) \ge p(X) \text{ for every } X \subseteq T \qquad (3.6a)$$

and

$$m_o(T - X) \ge p(X) \text{ for every } X \subseteq T \qquad (3.6b)$$

Proof The necessity of (3.6) is straightforward. To see the sufficiency let $m := m_i + m_o$ and call a set X **in-tight** (resp., **out-tight**) if (3.6a) (resp., (3.6b)) is satisfied with equality. We need 4 easy lemmas.

Lemma 1 *If X, Y are two disjoint out-tight sets, then $m(T - (X \cup Y)) = 0$.*

Proof We have $m_i(X) \ge p(X) = m_o(T - X) = \gamma - m_o(X)$ and $m_i(Y) \ge p(Y) = m_o(T - Y) = \gamma - m_o(Y)$ from which $m_i(X) + m_i(Y) \ge 2\gamma - m_o(X) - m_o(Y)$. Therefore $m(X) + m(Y) \ge 2\gamma = m(T)$ and hence $m(T - (X \cup Y)) = 0$, as required. ■

Lemma 2 *If X, Y are two in-tight sets for which $T = X \cup Y$, then $m(X \cap Y) = 0$.*

Proof We have $m_o(T - X) + m_o(T - Y) \ge p(X) + p(Y) = m_i(X) + m_i(Y) = 2\gamma - m_i(T - X) - m_i(T - Y)$. Therefore $m(T - X) + m(T - Y) \ge 2\gamma$ and hence $m(X \cap Y) = 0$, as required. ■

Lemma 3 *If X is out-tight, Y is in-tight and the supermodular inequality holds for $p(X)$ and $p(Y)$ (for example, if $X \subseteq Y$ or $Y \subseteq X$ or X, Y are crossing), then $m(Y - X) = 0$.*

Proof From $m_o(T - X) = p(X)$ and $m_i(Y) = p(Y)$ we have $m_o(T - X) + m_i(Y) = p(X) + p(Y) \le p(X \cap Y) + p(X \cup Y) \le m_o(T - (X \cup Y)) + m_i(X \cap Y) = m_o(T - X) - m_o(Y - X) + m_i(Y) - m(Y - X)$ and hence $0 \le m(Y - X) \le 0$, as required. ■

Lemma 4 *The intersection and the union of two crossing in-tight (respectively, out-tight) sets X, Y are in-tight (resp., out-tight).*

Proof We prove the lemma only when X, Y are in-tight. Then $m_i(X) + m_i(Y) = p(X) + p(Y) \le p(X \cap Y) + p(X \cup Y) \le m_i(X \cap Y) + m_i(X \cup Y) = m_i(X) + m_i(Y)$ from which equality holds everywhere and the lemma follows. ■

To prove the theorem let t be a node for which $m_i(t)$ is positive. If there is no in-tight set containing t, define $Z_i := \emptyset$. By Lemma 2 if there are two in-tight sets containing t, then their union Z is not T, and then, by Lemma 4, Z is in-tight. Therefore the union Z_i of all in-tight sets containing t is also in-tight. If there is no out-tight set in $T - t$, define $Z_o = T$. By Lemma 1 if there are two out-tight sets in $T - t$, then their intersection is non-empty, and then, by Lemma 4, the intersection Z_o of all out-tight sets in $T - t$ is out-tight.

It follows from Lemma 3 that the supermodular inequality does not hold for $p(Z_o)$ and $p(Z_i)$. Therefore $Z_o \cap Z_i = \emptyset$ or $Z_o \cup Z_i = T$. We claim that $m_o(Z_o - Z_i) > 0$. For otherwise, if $Z_o \cup Z_i = T$, then $0 = m_o(Z_o - Z_i) = m_o(T - Z_i) \ge p(Z_i) = m_i(Z_i) \ge 0$. Hence $Z_i = \emptyset$ and $m_o(T) = 0$, a contradiction. If $m_o(Z_o - Z_i) = 0$ and $Z_o \cap Z_i = \emptyset$, then $m_i(Z_o) \ge p(Z_o) = m_o(T - Z_o) = \gamma$ and hence $m_i(T - Z_o) = 0$. But then $Z_o = T, Z_i = \emptyset$ and $m_o(T) = 0$, a contradiction.

Choose an element s in $Z_o - Z_i$ for which $m_o(s) > 0$. Define $p'(X) := p(X) - 1$ if X is a $t\bar{s}$-set, $p(X) > 0$ and define $p'(X) := p(X)$ otherwise. Clearly p' satisfies the hypothesis of the theorem. Let $m'_o(s) = m_o(s) - 1, m'_o(v) = m_o(v)$ if $v \ne s$. Let $m'_i(t) = m_i(t) - 1, m'_i(v) = m_i(v)$ if $v \ne t$. Because there is no in-tight ts-set, (3.6a) holds with respect to p', m'_i. Similarly, there is no out-tight set in $T - \{s, t\}$ and therefore (3.6b) holds with respect to p', m'_o.

By induction, there is a digraph $H' = (T, F')$ satisfying the requirements of the theorem with respect to p', m'_i, m'_o. But then $H = (T, F' + st)$ satisfies the requirements with respect to p, m_i, m_o. ■

Remark The proof of the theorem gives rise to a (strongly) polynomial time algorithm to find the desired digraph H provided that the following oracles are available for p. For any pair of nodes x, y and vector $m : V \to Z_+$, minimize $m(X) - p(X)$ over the sets X (A) containing both x and y, (B) neither x nor y.

Corollary 3.6 *In Theorem 3.5 H may be chosen loopless if and only if (3.6) holds and*

$$m(v) := m_i(v) + m_o(v) \leq \gamma \text{ for every } v \in T. \qquad (3.7)$$

Proof If there is a loopless H satisfying (3.5), then every edge entering v leaves $T - v$ and hence $m_i(v) \leq m_o(T-v)$, that is, $m_i(v) + m_o(v) \leq m_o(T-v) + m_o(v) = \gamma$ and the necessity of (3.7) follows.

To see the sufficiency, let us start with a solution $H = (T, F)$ provided by Theorem 3.5 and assume that H has a minimum number of loops. If this minimum is zero, we are done. Suppose that at a node v there is a loop e in H. If there is an edge $f = xy$ of H with end-nodes different from v then we can replace e and f by xv and vy. The revised digraph clearly satisfies (3.4) and (3.5) and has one less loop than H, a contradiction. Therefore v is one of two end-nodes of each edge of H. But then v violates (3.7). ■

Theorem 3.7 *Let p be the same as in Theorem 3.5. There exists a digraph $H = (T, F)$ satisfying (3.4) so that H has at most γ edges if and only if*

$$\sum p(X_i) \leq \gamma \qquad (3.7a)$$

and

$$\sum p(T - X_i) \leq \gamma \qquad (3.7b)$$

holds for every sub-partition $\{X_1, \ldots, X_t\}$ of T.

Proof The necessity of (3.7) is straightforward. We prove the sufficiency with the following idea. Determine first two functions m_i and m_o satisfying (3.6) and apply then Theorem 3.5. To this end let m_i and m_o be integer-valued function on T satisfying (3.6) (but not necessarily $m_o(T) = m_i(T)$) and assume that m_i and m_o are minimal with respect to this (3.6a) and (3.6b), respectively. (That is, (3.6a), say, is destroyed if we reduce m_i by one on any element v where $m_i(v) > 0$.)

Claim *$m_i(T) \leq \gamma$ and $m_o(T) \leq \gamma$.*

Proof By symmetry we may assume that $m_i(T) \geq m_o(T)$. Increase m_o so that $m_o(T) = m_i(T)$ (this way we may loose the minimality of m_o but it does not matter). Since m_i is minimal, every element $v \in T$ for which $m_i(v) > 0$ belongs to an in-tight set. Let $\mathcal{F} := \{X_1, \ldots, X_t\}$ be a family of in-tight sets so that each v with positive $m_i(v)$ belongs to a member of $\mathcal{F}$ and $|\mathcal{F}|$ is minimum. There are no two crossing members of $\mathcal{F}$ since, by Lemma 4, their union is in-tight, contradicting the minimality of $\mathcal{F}$. If $\mathcal{F}$ have two members X, Y for which $T = X \cup Y$, then by Lemma 2, $m(X \cap Y) = 0$. Applying (3.7b) to $\{T - X, T - Y\}$ we have $m_i(T) = m_i(X) + m_i(Y) = p(X) + p(Y) \geq \gamma$. Finally, if $\mathcal{F}$ consists of disjoint subsets, then by (3.7a) we get $m(T) = \sum_j m_i(X_j) = \sum_j p(X_j) \geq \gamma$. Now the theorem directly follows from Theorem 3.5. ■

Theorem 3.7 implies the following generalization of Theorem 3.4. Let $D = (V, E)$ be a directed graph and T a subset of nodes. We say that D is **k-edge-connected in T** if $\lambda(u, v) \geq k$ for every pair of nodes $u, v \in T$.

Theorem 3.8 *Given a digraph D and a subset T of nodes, it is possible to make D k-edge-connected in T by adding at most γ new edges connecting elements of T if and only if*

$$\sum_i (k - \varrho(X_i)) \leq \gamma \text{ and } \sum_i (k - \delta(X_i)) \leq \gamma \qquad (3.8)$$

holds for every family $\mathcal{F} = \{X_1, \ldots, X_t\}$ of subsets V for which $\emptyset \subset X_i \cap T \subset T$ and $\mathcal{F}|T$ is a sub-partition of T.

Proof For every subset X of T define $p(X) := \max((k - \varrho(X \cup Z))^+ : Z \subseteq V - T)$. This p satisfies the hypothesis of Theorem 3.5 and (3.7) transforms to (3.8) and hence Theorem 3.7 implies Theorem 3.8. ■

More can be said if D is di-Eulerian outside T, that is, $\varrho(v) = \delta(v)$ for every $v \in V - T$.

Corollary 3.9 *Suppose that D is di-Eulerian outside T. It is possible to make D k-edge-connected in T by adding at most γ new edges connecting elements of T if and only if (3.8) holds for every sub-partition $\mathcal{F} = \{X_1, \ldots, X_t\}$ of subsets V for which $\emptyset \subset X_i \cap T \subset T$.*

Proof If the conditions of Theorem 3.8 are satisfied, we are done. Suppose indirectly that there is a family $\mathcal{F} = \{X_1, \ldots, X_t\}$ for which $\mathcal{F}|T$ is a sub-partition and $\mathcal{F}$ violates (3.8). We may assume that $\sum |X_i|$ is minimum. Since a sub-partition of V satisfies (3.8), there are two members X, Y of $\mathcal{F}$ whose intersection is non-empty. By the hypothesis every node in $X \cap Y$ is di-Eulerian, therefore $\varrho(X) + \varrho(Y) \geq \varrho(X - Y) + \varrho(Y - X)$. Replacing X and Y by $X - Y$ and $Y - X$ we obtain a family $\mathcal{F}'$ which also violates (3.8), contradicting the minimal choice of $\mathcal{F}$. ■

Since the condition in Corollary 3.9 is necessary even if new edges are allowed to have end-nodes outside T, it also follows that the minimum number of new edges whose addition makes a digraph k-edge-connected in T does not depend on whether we may only add edges with end-nodes in T or arbitrary new edges are allowed, provided that D is di-Eulerian outside T.

The following generalization of Corollary 3.9 is due to [1]. Let $D = (V, E)$ be a digraph and let $T(D) := \{v \in V : \varrho_D(v) \neq \delta_D(v)\}$ be the set of non-di-Eulerian nodes. Let k be a positive integer and $r(x, y)$, $(x, y \in V)$ a non-negative integer-valued demand function satisfying

$$r(x,y) = r(y,x) \leq k \text{ for every } x, y \in V \text{ and} \tag{3.9a}$$

$$r(x,y) = k \text{ for every } x, y \in T(D). \tag{3.9b}$$

Let $R(\emptyset) = R(V) = 0$ and for $X \subseteq V$ let $R(X) := \max(r(x,y) : X$ separates x and $y)$. Let us define $q_i(X) := R(X) - \varrho_D(X)$, $q_o(X) := R(X) - \delta_D(X)$.

Theorem 3.10 (Bang-Jensen, Frank and Jackson[1]) *Given a digraph $D = (V,E)$, positive integers k, γ, and a demand function $r(x,y)$ satisfying (3.9), D can be extended to D^+ by adding γ new directed edges so that*

$$\lambda(x,y;D^+) \geq r(x,y) \text{ for every } x, y \in V \tag{3.10}$$

if and only if both

$$\sum_j q_i(X_j) \leq \gamma \tag{3.11a}$$

and

$$\sum_j q_o(X_j) \leq \gamma \tag{3.11b}$$

hold for every sub-partition $\{X_1, \dots, X_t\}$ of V.

Corollary 3.11 *Given an Eulerian digraph $D = (V,E)$, and a symmetric demand function $r(x,y)$, D can be extended to an Eulerian digraph D^+ by adding γ new edges so that (3.10) holds if and only if (3.11) is satisfied.*

Our next problem is to find a k-edge-connected augmentation of minimum cardinality if upper and lower bounds are imposed both on the in-degrees and on the out-degrees of the digraph of newly added edges. Let $f_i \leq g_i$ and $f_o \leq g_o$ be four non-negative integer-valued functions on V (infinite values are allowed for g_i and g_o). The following two results appeared in [18, 19].

Theorem 3.12 *Given a directed graph $D = (V,E)$ and a positive integer k, D can be made k-edge-connected by adding a set F of precisely γ new edges so that both*

$$f_i(v) \leq \varrho_F(v) \leq g_i(v) \tag{3.12a}$$

and

$$f_o(v) \leq \delta_F(v) \leq g_o(v) \tag{3.12b}$$

hold for every node v of D if and only if both

$$k - \varrho(X) \leq g_i(X) \tag{3.13a}$$

and

$$k - \delta(X) \leq g_o(X) \tag{3.13b}$$

hold for every subset $\emptyset \subset X \subset V$ and both

$$\sum_j (k - \varrho(X_j) : j = 1, \dots, t) + f_i(X_0) \leq \gamma \tag{3.14a}$$

and

$$\sum_j (k - \delta(X_j) : j = 1, \dots, t) + f_o(X_0) \leq \gamma \tag{3.14b}$$

hold for every partition $\{X_0, X_1, X_2, \dots, X_t\}$ of V where only X_0 may be empty.

One may be interested in degree-constrained augmentations when there is no requirement on the number of new edges.

Theorem 3.13 *Given a directed graph $D = (V,E)$ and a positive integer k, D can be made k-edge-connected by adding a set F of new edges satisfying (3.12) if and only if (3.13) holds and and*

$$\sum_j (k - \varrho(X_j) : j = 1, \dots, t) + f_i(X_0) \leq \alpha \tag{3.15a}$$

and

$$\sum_j (k - \delta(X_j) : j = 1, \dots, t) + f_o(X_0) \leq \alpha \tag{3.15b}$$

hold for every partition $\{X_0, X_1, X_2, \dots, X_t\}$ of V where only X_0 may be empty and $\alpha := \min(g_o(V), g_i(V))$.

We close this section by another generalization of Theorem 3.8. Let $D = (V,A)$ be a digraph with two specified non-empty subsets S, T of nodes (which may or may not be disjoint). We say that D is **k-edge-connected from S to T** if there are k edge-disjoint paths from every node of S to every node of T. (When $S = T$ we are back at k-edge-connectivity.) We say that a family of subsets of nodes is **(S,T)-independent** if it contains at most one $t\bar{s}$-set for every pair $s \in S, t \in T$.

Theorem 3.14 *A digraph $D = (V,E)$ can be made k-edge-connected from S to T by adding at most γ new edges with tails in S and heads in T if and only if*

$$\sum_j (k - \varrho(X_j)) \leq \gamma$$

holds for every choice of (S,T)-independent family of subsets $X_j \subseteq V$ where $T \cap X_j \neq \emptyset, S - X_j \neq \emptyset$ for each X_j.

In all other theorems in this section (except the Lucchesi-Younger theorem) sub-partitions played the main role in the characterization in question. In Theorem 3.14 the characterization is more complicated. In fact, its proof goes along a line completely different from the approach applied for proving the previous theorems. The theorem is a consequence of a general result of [20] on crossing bi-supermodular functions, which among others, gives rise to a solution to the node-connectivity augmentation problem of directed graphs. This is the topic of the next section.

4 Node-Connectivity Augmentation of Digraphs

Given a directed graph $D = (V, E)$, how many new edges have to be added to D to make it k-node-connected, in short, k-connected. Recall that a digraph is called k-connected if it remains strongly connected after deleting at most $k-1$ nodes. That is, k-connectivity is defined only if $k \leq n-1$. If $k = n-1$, then in a k-connected digraph xy is an edge for every ordered pair $\{x, y\}$ of nodes. This case is uninteresting so we will assume that $k \leq n-2$. Also, when $k = 1$, edge-connectivity and node-connectivity coincide (:strong connectivity) so we will assume that $k \geq 2$.

In Section 2 we indicated that a related augmentation problem, when the goal is to reach k-connectivity from a specified node, could be solved [23], including the minimum cost version. The general minimum cost k-connectivity augmentation problem is NP-complete so we concentrate only on the minimum cardinality case. Masuzawa, Hagihara and Tokura (1987) solved it when the starting digraph is an arborescence (a directed tree so that every node is reachable from a source-node). Their result easily extends to branchings:

Theorem 4.1 (Masuzawa et al. 1987) *The minimum number of edges whose addition makes a branching $D = (V, E)$ k-connected is $(\sum(k - \delta(v))^+ : v \in V)$, that is, the sum of out-deficiencies of the nodes.*

For more general digraphs stronger lower bounds are required. One natural idea is to mimic the approach applied successfully in Theorem 3.4. For a subset X of nodes (with $|V - X| \geq k+1$) let $I(X)$ (respectively, $O(X)$) denote the set of nodes in $V - X$ from which there is an edge to X (into which there is an edge from X). In a k-connected digraph the cardinality of $I(X)$ and $O(X)$ must be at least k. Therefore, if the digraph is not k-connected, we may call the quantity $Q_I(X) := (k - |I(X)|)^+$ the **in-deficiency** and $Q_O(X) := (k - |O(X)|)^+$ the **out-deficiency** of set X. Clearly, if there is a family of disjoint sets (each having cardinality at most $|V| - 1 - k$)), then the sum of the in-deficiencies and the sum of out-deficiencies are both lower bounds for the necessary number of new edges. Theorem 3.4 asserted that the maximum of the analogous lower bounds in the edge-connectivity augmentation provides the correct minimum for the number of new edges. Unfortunately, this is not the case for node-connectivity even if the starting digraph is $k-1$ connected. An example in [36] shows that the minimum of the required new edges may be $k-1$ larger than the maximum sum of out- or in-deficiencies of a sub-partition. (On the other hand, in a recent paper[21] we can derive from the general min-max theorem below that this gap actually can never get bigger than $k-1$.)

Theorem 2.1 however suggests that, instead of single sets, it might be helpful to consider pairs of disjoint sets. Let us call an ordered pair (A, B) of non-empty disjoint subsets of nodes **one-way** if there is no edge in D from A to B. The deficiency $p_{def}(A, B)$ of a one-way pair is defined by $(k - (|V - (A \cup B)|))^+$. Clearly in a k-connected augmentation of D at least that many edges from A to B must be added to D. Finally, call two pairs $(A_i, B_i)(i = 1, 2)$ **independent** if at least one of $A_1 \cap A_2$ and $B_1 \cap B_2$ is empty.

Theorem 4.2 (Frank and Jordán [20]) *A digraph $D = (V, E)$ can be made k node-connected by adding at most γ new edges if and only if*

$$\sum(p_{def}(X, Y) : (X, Y) \in \mathcal{F}) \leq \gamma \tag{4.1}$$

holds for every family $\mathcal{F}$ of pairwise independent one-way pairs.

Since this is a characterization for a general starting digraph, one may expect that Theorem 4.1, where the starting digraph is a branching, can be derived from it. So far we were not able to do that. The following conjecture, if true, is a generalization of Theorem 4.2.

Conjecture *If D is a simple acyclic digraph, then the minimum number of new edges whose addition makes D k-connected is equal to the maximum of the sum of out-deficiencies and the sum of in-deficiencies of nodes.*

Note that M. Bussieck [3] pointed out that the the corresponding statement for edge-connectivity easily follows from Theorem 3.4.

Actually, Theorem 4.2 is a special case of a more general result. Let V be a ground-set and S, T two (not-necessarily disjoint) subsets of V. Let A denote the set of all directed edges st for which $s \in S, t \in T$.

Let $\mathcal{A}$ denote the set of all ordered pairs (X, Y) with $X \subseteq S, Y \subseteq T$. We call X and Y the **tail** and the **head** of the pair, respectively. A directed edge xy **covers** a pair $(X, Y) \in \mathcal{A}$ if $x \in X, y \in Y$. We say that a sub-family $\mathcal{F}$ of $\mathcal{A}$ **is independent** if every edge of A covers at most one member of $\mathcal{F}$. This is equivalent to requiring that there are no two members $(X_i, Y_i)(i = 1, 2)$ of $\mathcal{F}$ for which $X_1 \cap X_2 \neq \emptyset$ and $Y_1 \cap Y_2 \neq \emptyset$.

Let p be a non-negative, integer-valued function defined on $\mathcal{A}$ for which $p(X, \emptyset) = p(\emptyset, Y) = 0$. We say that p is **crossing bi-supermodular** if

$$p(X, Y) + p(X', Y') \leq p(X \cap X', Y \cup Y') + p(X \cup X', Y \cap Y') \tag{4.2}$$

holds whenever $p(X, Y), p(X', Y') > 0, X \cap X', Y \cap Y' \neq \emptyset$.

For a non-negative function x defined on A, define $\delta_x(A, B) := \sum(x(s, t) : s \in S, t \in T)$. We say that x **covers** p or that x is a **covering** of p if $\delta_x \geq p$. The main result n [20, 21] is:

Theorem 4.3 *For an integer-valued crossing bi-supermodular function p the following min-max equality holds. $\tau_p := \min(z(A) : z$ an integer-valued covering of $p) = \nu_p := \max(p(\mathcal{F}) : \mathcal{F} \subseteq \mathcal{A}, \mathcal{F}$ independent).*

Theorem 4.2 as well as Theorems 3.4 and 3.7 are special cases of this result and Theorem 3.5 can also easily be derived from it. (In [20, 21] a difficult min-max theorem on intervals of E. Győri was also shown to be a consequence but this has nothing to do with connectivity augmentation). Having so many corollaries, it is indeed surprising that the proof of Theorem 4.3 is short and is rather standard (demonstrating nicely that finding the right notions and formulation of results might subsume sophisticated proofs.) This proof is, however, not constructive! Though the theorem may be used to develop a polynomial time algorithm for finding a minimum k-connected augmentation of a digraph, the algorithm is based on the ellipsoid method. Designing a combinatorial algorithm for this task is one of the most challenging algorithmical problems of the area. We do not know such an algorithm even if D is $(k-1)$-connected, that is, the goal is to increase the connectivity of D only by 1 (For $k=1$, Eswaran and Tarjan have such an algorithm. In a recent paper we developed an algorithm for $k=2$ [21]).

In the same paper we were able to show that in Theorem 4.2, if D is $(k-1)$-connected, then the optimal digraph of new edges may be chosen to consist of disjoint directed paths and circuits. This implies

Theorem 4.4 *Let D be a k-connected digraph for which the in-degree and out-degree of every node is k. Then it is possible to add disjoint directed circuits to D so that the resulting digraph D^+ is $(k+1)$-connected.*

Note that it is not always possible to increase the connectivity of a digraph by adding a directed Hamiltonian circuit. (Take a digraph arising from $K_{3,3}$ by replacing each edge by two oppositely directed edges.)

5 Edge-Connectivity Augmentation with Undirected Edges

The purpose of the present chapter is to review results concerning edge-connectivity augmentation when only undirected edges are allowed to be added. Typically the starting graph $G=(V,E)$ is also undirected but many results extend to mixed starting graphs as well.

Let $r(x,y)$ be a symmetric, non-negative, integer-valued function defined on the pairs of nodes. Add a minimum number of new edges to G so that $\lambda(x,y;G^+)\geq r(x,y)$ holds for every pair of nodes x,y of the augmented graph G^+.

The first results concerned the special case when the starting graph has no edges. For the fractional version of this case R.E. Gomory and T.C. Hu [28] provided an elegant solution and proved that the optimal (fractional) augmentation can be realized by half-integers. W. Chou and H. Frank [5] solved the integer-valued version. In another paper Frank and Chou[5] solved the restricted problem when no parallel edges are allowed to be used. J. Edmonds [6] proved that if there exists a simple graph with a specified degree sequence and each degree is at least k, then there is a k-edge-connected simple graph with the given degree sequence.

The first papers concerning general starting graphs appeared in 1976. K. Eswaran and R. E. Tarjan and J. Plesnik solved the 2-edge-connectivity augmentation problem.

Eswaran and Tarjan also provided a linear time algorithm while Plesnik's paper is the first where the idea of splitting off technique appears. In a tiny note at the end of his paper, Plesnik remarks that the 2-edge-connectivity augmentation theorem also follows from a (then recent) theorem of Lovász on splitting off edges. It turned out that this approach has far reaching consequences. The general k-edge-connectivity augmentation problem was solved by T. Watanabe and A. Nakamura[54]. In their solution there is no restriction on the number of copies a new edge may be added. It is an important open problem to find algorithms that does not add parallel edges. Very recently this task was solved for the special cases $k\leq 5$ by Taoka, Takafuji and Watanabe[51].

The fundamental min-max theorem of Watanabe and Nakamura is as follows.

Theorem 5.1 *The minimum number of edges whose addition makes an undirected graph $G=(V,E)$ k-edge-connected ($k\geq 2$) is equal to*

$$\max\lceil(\sum_i(k-d(X_i)))/2\rceil \tag{5.1}$$

where the maximum is taken over all sub-partitions $\{X_1,\dots,X_t\}$ of V.

The proof of Watanabe and Nakamura is based on the recognition that the augmentation problems for different k's are strongly related. They prove various exciting structural properties of edge-connectivity augmentations. Below we cite two of them. These are not only the basis of their augmentation algorithm but serve as a framework for subsequent improved algorithms as well [47, 27, 2].

Let us first study how an optimal sub-partition for (5.1) may be found. It is an easy observation that for any fixed integer l the relation "$\lambda(x,y)\geq l$" on the node-set of a graph $G=(V,E)$ is an equivalence relationship. An equivalence class may be called an **edge-connectivity component** (in short, **ec-component**) or an **l-component.** (That is, an l-component is a maximal subset of nodes for which $\lambda(x,y)\geq l$.) From the definition it is straightforward that the family $\mathcal{F}_{ec}$ of all ec-components is a laminar family (and hence it has at most $2n$ members). For $l=0$ the node-set V is an ec-component and for $l=|E|+1$ each single node is a ec-component.

Call a subset $X\subseteq V$ **extreme** if $d(X')>d(X)$ holds for every proper, non-empty subset X' of X. If we choose an optimal sub-partition in (5.1) so that the union of its members has minimum cardinality, then the sub-partition consists of extreme sets. In other words it suffices to restrict (5.1) on sub-partitions consisting of extreme sets. The following lemma is basic to explore the structure of extreme sets.

Lemma 5.2 *Each extreme set is an edge-connectivity component.*

Proof Let C be an extreme set and let $l:=\min(\lambda(x,y):x,y\in C)$. Since V and each singleton is an ec-component, we may assume that $1<|C|<|V|$. We have to show that $\lambda(x,y)<l$ for any pair $x\in C, y\in V-C$. This is clearly the case if $d(C)<l$ so suppose that $d(C)\geq l$. By Menger's theorem there is a set M for which $M\cap C$ and $C-M$ are non-empty and $d(M)=l$. By taking the complement if necessary, we may

assume that $x \in M$. If $y \notin M$, then we have $d(C)+d(M) \geq d(C \cap M)+d(C \cup M) > d(C)+d(C \cup M)$ and therefore $\lambda(x,y) \leq d(C \cup M) < d(M) = l$. If $y \in M$, then we have $d(C)+d(M) \geq d(C-M)+d(M-C) > d(C)+d(M-C)$ and therefore $\lambda(x,y) \leq d(M-C) < d(M) = l$. ■

By this lemma the family $\mathcal{F}^*$ of extreme sets is a sub-family of $\mathcal{F}_{ec}$ and therefore it is a laminar family. Since the ec-components of G can be computed with the help of a Gomory-Hu tree, $\mathcal{F}^*$ can also be computed. The nice thing is that the family of extreme sets includes all information which is required to determine the optimum in (5.1) for **any** k.

This may be done recursively. For each extreme set X let us define the **recursive k-deficiency** $R_k(X)$ as follows. For singletons let $R_k(v) := (k-d(v))^+$. If for all maximal extreme subsets X' of an extreme set X (which form, incidentally, a sub-partition of X) $R_k(X')$ has already been determined, then define $R_k(X) := \max((k-d(X))^+, \sum(R_k(X') : X'$ is a maximal extreme subset of $X))$. Parallel to this we may store a sub-partition $\mathcal{R}_k(X)$ of X. It consists of the single set X if the maximum in the definition of $R_k(X)$ is attained on the first term. If the maximum is attained on the second term, then let $\mathcal{R}_k(X) := \cup(\mathcal{R}_k(X') : X'$ is a maximal extreme subset of $X)$. From the definition, it is clear that, among the sub-partitions $\mathcal{R}$ of V consisting of extreme sets, $\mathcal{R}_k(V)$ maximizes the sum $\sum(k-d(X) : X \in \mathcal{R})$ and therefore $\mathcal{R}_k$ is an optimal solution to (5.1).

Not only the best sub-partitions for different values of k may be encoded into a single laminar family (of the extreme sets) but Watanabe and Nakamura proved that the optimal edge-connectivity augmentations for increasing k may be chosen as a sequence of ever increasing supergraphs:

Theorem 5.3 *Suppose that the edge-connectivity of the starting graph G is l. There is a sequence $G_l := G, G_{l+1}, G_{l+2}, \ldots$ of graphs so that for each $i \geq l$, G_{i+1} is a supergraph of G_i and G_i is an i-edge-connected augmentation of G using a minimum number of new edges.*

Watanabe and Nakamura described how to compute this sequence in polynomial time. Gusfield, Naor and Martel [47] and Gabow [27] found improvements for the complexity. One apparent disadvantage of this approach is that the resulting algorithm is not strongly polynomial if the target edge-connectivity k is very big. This is clearly so since the approach uses one-by-one augmentations. A. Benczúr [2] however devised a clever grouping technique to make the algorithm of Watanabe and Nakamura strongly polynomial.

The first strongly polynomial algorithm [18, 19] for the k-edge-connectivity augmentation problem followed a different line. One of its basic ideas, the use of the splitting off technique, was suggested by Plesnik [48] when $k=2$ and by Cai and Sun [4] for arbitrary $k \geq 2$. Using splitting off is equivalent to using degree-prescribed augmentation problems.

Theorem 5.4 *Let $G=(V,E)$ be an undirected graph and m a modular non-negative integer-valued function on V. G can be made k-edge-connected ($k \geq 2$) by adding a*

›t F of new edges so that

$$d_F(v) = m(v) \tag{5.2}$$

›lds for every node v if and only if $m(V)$ is even and

$$d(X)+m(X) \geq k \tag{5.3}$$

›lds for every non-empty proper subset X of V.

›is theorem is an immediate consequence of Lovász's theorem on splitting off:

›heorem 5.5 *(Lovász [25, 42]) Suppose that in a graph $G' = (V+s, E')$ $d'(s) > 0$ even and*

$$\lambda(x,y) \geq k \tag{5.4}$$

›lds for every pair of nodes $x,y \in V$. Then the edges incident to s can be paired into $(s)/2$ pairs so that after splitting off these pairs the resulting digraph on node-set is k-edge-connected.

› derive Theorem 5.4, add a new node s to D and $m(v)$ parallel edges between v ›d s for every $v \in V$. Then (5.3) is equivalent to (5.4) and the edge-set F arising in ›eorem 5.5 from the splitting off operations satisfies the requirements in Theorem 4.

An equivalent form of Theorem 5.1 of Watanabe and Nakamura is:

›heorem 5.1' *An undirected graph $G=(V,E)$ can be made k edge-connected by ›dding at most γ new edges if and only if*

$$\sum_i (k - d(X_i)) \leq 2\gamma \tag{5.5}$$

olds for every sub-partition $\{X_1, \ldots, X_t\}$ of V.

›roof Let m be an integer-valued function for which (5.3) holds and m is minimal ›ith respect to this property. Call a set X tight if it satisfies (5.3) with equality. We ›aim that if X and Y are intersecting tight sets, then both $X-Y$ and $Y-X$ are tight ›nd $m(X \cap Y) = 0$. Indeed, $k-m(X)+k-m(Y) = d(X)+d(Y) \geq d(X-Y)+(Y-X) \geq k-m(X-Y)+k-m(Y-X) = k-m(X)+k-m(Y)+2m(X \cap Y)$, ›om which the claim follows. ■

By the minimality of m there is a family $\mathcal{F}$ of tight sets so that each v for which (v) is positive belongs to a member of $\mathcal{F}$. Choose $\mathcal{F}$ so that $\sum(|X| : X \in \mathcal{F})$ is ›inimum. This choice and the claim shows that $\mathcal{F}$ is laminar. Therefore the maximal ›embers of $\mathcal{F}$ form a sub-partition $\{X_1, \ldots, X_t\}$ of V covering all elements v with $(v) > 0$.

It follows from (5.5) that $m(V) = \sum_i m(X_i) = \sum_i (k-d(X_i)) \leq 2\gamma$. By increasing › if necessary, we may assume that $m(V) = 2\gamma$. Finally, by applying Theorem 5.4 › this m, Theorem 5.1' follows. ■

Let us turn to the general augmentation problem when we are given an arbitrary starting graph $G = (V, E)$, an arbitrary (symmetric, non-negative, integer-valued) demand function $r(x, y)$ and the goal is to determine the minimum number of new edges to be added to G so as to obtain a graph G^+ for which

$$\lambda(x, y; G^+) \geq r(x, y) \tag{5.6}$$

holds for every pair of nodes x, y in the augmented graph G^+. We may call an augmentation satisfying (5.6) **feasible.** If F is the set of new edges in a feasible augmentation, a vector in Z^V defined by $(d_F(v) : v \in V)$ is called an **augmentation vector.**

Recall that the corresponding augmentation problem for directed graphs is NP-complete already for very special demand functions (e.g., if $r(x, y) = 1$ when both x and y belong to a subset T of nodes and 0 otherwise.) In this light it is especially surprising that the undirected augmentation problem is tractable for any starting graph and for any demand function [18, 19].

In order to formulate the general augmentation result, let us define a set-function R on the subsets of V so that $R(\emptyset) := R(V) := 0$ and

$$R(X) := \max(r(x, y) : x \in X, y \in V - X). \tag{5.7}$$

By Menger's theorem $\lambda(x, y) \geq r(x, y)$ holds in a graph for every pair of nodes x, y if and only if $d(X) \geq R(X)$ for every subset $X \subseteq V$. That is, $R(X)$ serves as a lower bound for the number of edges in a cut $[X, V - X]$ and $q(X) := (R(X) - d(X))^+$ may be considered as the deficiency of X. Now

$$\max(\lceil (\sum_i q(X_i))/2 \rceil \ : \{X_1, X_2, \ldots, X_t\} \text{ a sub-partition of } V) \tag{5.8}$$

is a lower bound for the minimum number $\gamma(G, r)$ of new edges. Theorem 5.1 asserts that this lower bound is achievable if $r \equiv k (\geq 2)$.

On the way to generalize Theorem 5.1 we have to prepare, however, to overcome a little anomaly indicated already by the fact that Theorem 5.1 is not true for $k = 1$ (take a starting graph on four nodes with no edges) while the augmentation problem when $k = 1$ is trivial. This distinction must be handled in the general case, as well. To this end let $C (\neq V)$ be the node-set of a component of G and call C a **marginal component** (with respect to the demand function r) if $q(C) \leq 1$ and $q(X) = 0$ for every proper subset of C.

The solution in [18] to find a minimal feasible augmentation of G consists of two parts. In the first part the marginal components are eliminated while the second one consists of proving (algorithmically) that $\gamma(G, r)$ is equal to the maximum in (5.8) when there are no marginal components. (This is, by the way, the case if G is connected).

Let C be a marginal component, $G_1 := G - C$ and let r_1 denote the demand function restricted on the node set of G_1. It is proved in [18] that $\gamma(G, r) = \gamma(G_1, r_1) + q(C)$. It is also shown how an optimal feasible augmentation of G_1 can be extended to an optimal feasible augmentation of G by adding $q(C)$ (which is 0 or 1) edge. This way we can eliminate the marginal components one by one.

Theorem 5.6 *If G has no marginal components, there is a feasible augmentation of G using at most γ new edges if and only if*

$$\sum_i q(X_i) \leq 2\gamma \tag{5.9}$$

holds for every sub-partition $\{X_1, X_2, \ldots, X_t\}$ of V. Or, equivalently, the minimum number of new edges $\gamma(G, r) = \max(\lceil (\sum_i q(X_i))/2 \rceil \ : \{X_1, X_2, \ldots, X_t\}$ a sub-partition of V).

Corollary 5.6' *Let $G = (V, E)$ be an undirected graph, $r(u, v)$ an integer-valued demand-function such that G has no marginal components, and g an integer-valued capacity function on E. There is an optimal solution to the undirected max-flow augmentation problem which is half integral. Furthermore, an optimal integer-valued solution is either optimal among the real-valued augmentations or its total increment is one half bigger than that of a (real-valued) optimal solution.*

The key to the proof of Theorem 5.6 is the following deep splitting off theorem of W. Mader.

Theorem 5.7 (Mader [44]) *Let $G' = (V + s, E')$ be a (connected) undirected graph in which $0 < d_G(s) \neq 3$ and there is no cut-edge incident with s. Then there exists a pair of edges $e = su, f = st$ so that*

$$\lambda(x, y; G) = \lambda(x, y; G^{ef})$$

holds for every $x, y \in V$.

(For a relatively simple proof, using submodularity, see [18, 19]) In Section 3 it was pointed out that Mader's directed splitting off theorem is equivalent to Theorem 3.2 on the existence of a k-edge-connectivity augmentation of a digraph satisfying prescriptions of the in-degree and out-degree. Analogously, Theorem 5.7 is equivalent to:

Theorem 5.7' *Let $m : V \to Z_+$ be an integer-valued function so that $m(V)$ is even and $m(C) \geq 2$ for each component C of $G = (V, E)$. There is a set F of new edges so that $G^+ = (V, E + F)$ is a feasible augmentation of G and $d_F(v) = m(v)$ for every node v (that is, m is an augmentation vector) if and only if*

$$m(X) \geq R(X) - d_G(X) \tag{5.10}$$

for every $X \subseteq V$.

The material of the closing part of this section is taken from a recent work of [1]. They proved an extension of Mader's theorem when the graph is a mixed one but all edges incident to s are undirected. This was used to derive a generalization of Theorem 5.6.

Let $N = (V, E + A)$ be a mixed graph composed from an undirected graph $G = (V, E)$ and a directed graph $D = (V, A)$ in which $T(D) := \{v \in V : \varrho_D(v) \neq \delta_D(v)\}$ is the set of non-di-Eulerian nodes. Let $k \geq 2$ be an integer and $r(x, y)$ $(x, y \in V)$ a non-negative, integer-valued demand function satisfying

$$r(x,y) = r(y,x) \leq k \text{ for every } x, y \in V \text{ and} \tag{5.11a}$$

$$r(x,y) \equiv k \text{ for every } x, y \in T(D). \tag{5.11b}$$

Let $R(\emptyset) = R(V) = 0$ and for $X \subseteq V$ let $R(X) := \max(r(x,y) : X$ separates x and $y)$. We say that a component C of N is **marginal** (with respect to r) if $r(u,v) \leq \lambda(u,v;N)$ for every $u, v \in C$ and $r(u,v) \leq \lambda(u,v;N) + 1$ for every u, v separated by C. Let $\beta_N(X) := \min(\varrho_D(X) + d_G(X), \delta_D(X) + d_G(X))$.

Theorem 5.8 (Bang-Jensen, Frank and Jackson [1]) *Given a mixed graph N, integers $k \geq 2$, $\gamma \geq 0$, and a demand function $r(x,y)$ satisfying (5.11) so that there is no marginal components, N can be extended to a mixed graph N^+ by adding γ new undirected edges so that*

$$\lambda(x, y; N^+) \geq r(x, y) \text{ for every } x, y \in V \tag{5.12}$$

if and only if

$$\sum (R(X_i) - \beta_N(X_i)) \leq 2\gamma \tag{5.13}$$

holds for every sub-partition $\{X_1, \ldots, X_t\}$ of V.

Before Theorem 5.6 we indicated how to eliminate marginal components when the starting graph is undirected. A similar reduction works for mixed undirected graphs as well. When N is an undirected graph, Theorem 5.8 specializes to Theorem 5.6. When $r \equiv k$ for an integer $k \geq 2$ Theorem 5.8 specializes to:

Corollary 5.9 *Let $N = (V, A \cup E)$ be a mixed graph and $k \geq 2, \gamma \geq 1$ integers. N can be made k-edge connected by adding γ new undirected edges if and only if*

$$\sum (k - \beta_N(X_i)) \leq 2\gamma$$

holds for every sub-partition $\{X_1, .., X_t\}$ of V.

This corollary is not true for $k = 1$. (Let N be a digraph with 4 nodes and 3 edges so that the heads of the edges are distinct but their tails coincide) However, the following can be proved.

Theorem 5.10 *A mixed graph N with connected underlying graph can be made strongly connected by adding γ new undirected edges if and only if $(*)$ for any family $\mathcal{F}$ of $\gamma + 1$ disjoint subsets of nodes contains (not necessarily distinct) members X, Y for which $\varrho_N(Y) > 0$ and $\delta_N(Y) > 0$.*

(A mixed graph is **strongly connected** if every node is reachable from every other node along a path not using oppositely oriented edges.)

6 Node-Connectivity Augmentation of Undirected Graphs

In this section we want to make an undirected graph $G = (V, E)$ k-connected by adding a minimum number of new edges. In previous sections we pointed out that the k-edge-connectivity augmentation problem is tractable for both directed and undirected graphs. It turned out that undirected edge-connectivity augmentation behaves better than the directed one in the sense that even the general-demand augmentation is solvable in the undirected case. Section 4 described a nice min-max theorem for the directed k-connectivity augmentation. From these data one hopes that the undirected k-connectivity augmentation problem also has a solution. Unfortunately, at present, it is not known if the problem is **NP**-complete or is perhaps in co-**NP**$\cap$**NP** or even in **P**.

For general k, F. Harary [31] found the solution when the starting graph has n nodes but no edges. Wang and Kleitman [53] determined a necessary and sufficient condition for the existence of a k-connected graph with specified degree sequence.

For general starting graphs, solutions are known only for small k. When $k = 1$ the problem is obvious. Plesnik [48] and Eswaran and Tarjan [11] proved a min-max formula for $k = 2$ and the latter paper described a linear-time algorithm, as well, to construct the optimal augmentation. For a subset X of nodes let $N(X)$ denote the set of nodes in $V - X$ which have a neighbor in X. In a k-connected graph $|N(X)| \geq k$ whenever $|X| \leq |V| - k - 1$. Therefore, in a k connected augmentation of G at least $q_\kappa(X) := (k - N(X))^+$ new edges must be in the cut $[X, V - X]$. Hence

$$\max(\lceil (\sum_i q_\kappa(X_i))/2 \rceil \ : \{X_i\} \text{ a sub-partition of } V) \tag{6.1}$$

is a lower bound for the minimum number of new edges. Theorem 5.1 asserted that in the edge-connectivity augmentation an analogous lower bound is achievable, except the (otherwise trivial) case of $k = 1$. The difficulty of the node-connectivity augmentation arises from the fact that this kind of trouble may occur for any big k. For example, let G be a star, that is, G is a simple graph on n nodes in which every edge is incident to a node s. The bound of (6.1) when $k = 2$ is $\lceil (n-1)/2 \rceil$. But to make G 2-connected one needs $n - 2$ edges. From this example we may extract another lower bound for $k = 2$. Let $c(X)$ denote the number of components of the graph arising from G by deleting the node-set X. Then at least $c(v) - 1$ new edges have to be added to G to make it 2-connected,

Theorem 6.1 (Eswaran and Tarjan [11], Plesnik [48]) *An undirected graph G can be made 2-connected by adding at most γ new edges if and only if*

$$\sum_i (2 - |N(X_i)|) \leq 2\gamma \text{ for every sub-partition } \{X_i\} \text{ of } V \tag{6.2}$$

where $(|X_i| \leq |V| - 3)$ and

$$c(v) \leq \gamma + 1 \tag{6.3}$$

for every node $v \in V$.

For the 3-connectivity augmentation problem an analogous theorem holds:

Theorem 6.2 (Watanabe and Nakamura [54, 55]) *An undirected graph G can be made 3-connected by adding at most γ new edges if and only if*

$$\sum_i (3 - |N(X_i)|) \leq 2\gamma \text{ for every sub-partition } \{X_i\} \text{ of } V \tag{6.2'}$$

where $(|X_i| \leq |V| - 4)$ *and*

$$c(X) \leq \gamma + 1 \tag{6.3'}$$

for every 2-element subset X of V.

Watanabe and Nakamura also developed a polynomial-time algorithm to find the optimal augmentation. A linear time algorithm (and another proof) was given by Hsu and Ramachandran [33]. Jordán [37] solved the degree-constrained 3-connectivity augmentation problem when the starting graph is 2-connected. In particular, he proved that G can be made optimally 3-connected by adding disjoint paths.

More generally, if the starting graph G is $k-1$ connected, then both $\max\lceil(\sum_i q_\kappa(X_i))/2\rceil$ and $\max(c(X)-1)$ are lower bounds for the number of new edges where $\{X_i\}$ is a sub-partition of V consisting of subsets of at most $|V| - k - 1$ elements and X is a cut-set of precisely $k-1$ elements. Theorems 6.1 and 6.2 are equivalent to saying that the maximum of these two lower bounds are achievable when $k = 2, 3$. But not so if $k = 4$ as is shown by $K_{3,3}$. Here the optimum is 4 while the first maximum is 3 and the second is 2. For this case (that is, to find a minimal 4-connected augmentation of a 3-connected graph) a polynomial algorithm was developed by T. Hsu [3]. He also showed that the maximum of the two lower bounds is achievable except for some well-described small graphs. A related nice result of Jordán [37] asserts that a 3-regular 3-connected graph on at least 8 nodes can be made 4-connected by adding a perfect matching.

For higher k the complete bipartite graph $K_{k-1,k-1}$ shows that the gap between the maximum and the minimum may be as big as $k-3$. T. Jordán [35] proved that this is the largest possible gap for any $(k-1)$-connected starting graph. Relying on this theorem, he developed a polynomial time algorithm to make an arbitrary $(k-1)$-connected graph k-connected and the solution provided by the algorithm uses at most $(k-3)^+$ more edges than the optimum. To compute this optimum exactly seems to be out of reach at the time of writing this paper.

There is a fractional version of the connectivity augmentation problem. We say that a weighting on the edge-set of a graph G is **connected** if the total weight of every cut is at least 1. We say that a weighting is **k-connected,** if leaving out any set of at most $k-1$ nodes leaves a graph with connected weighting. Clearly if all weights are 1, then this definition gives back the original definition of k-connectivity.

The fractional node-connectivity augmentation problem is the following. Let $G = (V, E)$ be a simple undirected graph. Define $w_G(x, y) = 1$ if xy is an edge of G and $w_G(x, y) = 0$ otherwise. The goal is to determine a non-negative weighting w on the edge set of the complete graph on V so that $w + w_G$ is k-connected and $1w$ is minimum.

By an elementary construction this problem can be reduced to the directed node-connectivity augmentation problem. As a consequence of Theorem 4.2 one may derive that the optimal fractional augmentation may be chosen half-integral. For example, if the starting graph is $k_{k-1,k-1}$, then an optimum fractional augmentation is 1/2 on the edges of a circuit in both parts, that is, its total value is $k-1$. Compare this with the integer optimum which is $2(k-2)$.

7 Polyhedral and Algorithmic Aspects

In Sections 3 and 5 we saw that the cardinality version of the k-edge-connectivity augmentation problem is tractable for both directed and undirected graphs, while the minimum cost version is **NP**-complete. But there is a restricted class of cost functions, the node-induced costs, when even the minimum cost augmentation problem is solvable. The idea is based on the fact that the degree sequences belonging to possible graphs of new edges in feasible augmentations of the starting graph span a g-polymatroid. G-polymatroids generalize polymatroids which are generalization of matroid polyhedra. The matroid greedy algorithm may be extended to g-polymatroids. This feature makes it possible to solve minimum node-cost augmentation problems.

Another important property of g-polymatroids is that their intersection with a box is again a g-polymatroid. This is why the degree-constrained augmentation problems can be handled (see, for example Theorem 3.12).

Here we illustrate some details of this polymatroidal approach concerning the undirected case. The basic framework is similar for directed augmentation. Details may be found in [18, 19].

Let V be a finite ground-set and $b : 2^V \to Z \cup \{\infty\}$ an integer-valued set-function which is zero on the empty set. We call b **fully (intersecting) submodular** if

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y) \tag{7.1}$$

holds for every (intersecting) $X, Y \subseteq V$. A set function p is called **supermodular** if $-p$ is submodular. We say that p is **skew supermodular** if for every $X, Y \subseteq V$ at least one of the following inequalities holds:

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y),$$

$$p(X) + p(Y) \leq p(X - Y) + p(Y - X).$$

Note that intersecting supermodular functions are skew supermodular.

We say that a pair (p, b) of set-functions is a **strong pair** if p (resp. b) is fully supermodular (submodular) and they are **compliant**, that is,

$$b(X) - p(Y) \geq b(X - Y) - p(Y - X) \tag{7.2}$$

holds for every $X, Y \subseteq S$. If p and b are intersecting super- and submodular functions and (7.2) holds for intersecting X, Y, then (p, b) is called a **weak pair.**

Given a strong pair (p, b), the polyhedron

$$Q(p,b) := \{x \in R^S : p(A) \leq x(A) \leq b(A) \text{ for every } A \subseteq S\}.$$

is called a **generalized polymatroid** (in short, **g-polymatroid.**) For technical reasons the empty set is also considered as a g-polymatroid. Properties of g-polymatroids were extensively studied in [22]. It can be proved that $Q := Q(p,b)$ is non-empty and that Q uniquely determines p and q, that is, different strong pairs define different g-polymatroids. On the other-hand (p', b') is a g-polymatroid for any weak pair. Let f and g be two integer-valued functions on V with $f \leq g$ and let $B := \{x : f \leq x \leq g\}$ a box.

Theorem 7.1 *For a weak pair (p', b') the intersection $M := Q(p', b') \cap B$ is an integral g-polymatroid. M is non-empty if and only if*

$$g(Z_0) + \sum_{i \geq 1} b'(Z_i) \geq p'(\cup_{i \geq 0} Z_i) \text{ and}$$

$$f(Z_0) + \sum_{i \geq 1} p'(Z_i) \leq b'(\cup_{i \geq 0} Z_i)$$

holds for every sub-partition $\{Z_i\}$ of V where only part Z_0 may be empty.

This theorem is in the background of connectivity augmentation results (such as Theorem 3.12) when upper and lower bounds are imposed on the degrees of the augmented graph.

Contra-polymatroids form a special class of g-polymatroids. Given a fully supermodular, monotone increasing function p,

$$C(p) := \{x \in R^S : x \geq 0, x(A) \geq p(A) \text{ for every } A \subseteq S\}$$

is called a **contra-polymatroid.** Such a p is uniquely determined by the polyhedron but weaker functions may also define contra-polymatroids.

Theorem 7.2 *Let p^* be a skew supermodular function. Then $C(p^*)$ is a contra-polymatroid whose unique (monotone, fully supermodular) defining function p is given by $p(X) := \max(\sum p^*(X_i))$ where the maximum is taken over all sub-partitions $\{X_i\}$ of X.*

This theorem (proved in [18, 19]) made it possible for minimum node-cost and degree-constrained augmentation problems to become tractable. The link between augmentations and contra-polymatroids is revealed by the observation that the function $q(X) := R(X) - d_G(X)$ is skew supermodular. (For the notation, see Theorem 5.6.)

Recall the notion of an augmentation vector. Combining Theorem 7.12 and Theorem 5.6 together we obtain:

Corollary 7.3 *For a connected graph G, an integer vector z is an augmentation vector if and only if $z(V)$ is even and $z \in C(q)$.*

Suppose we are given a non-negative cost-function c on V. For each possible new edge xy define a cost by $c(x) + c(y)$. We call such a function a node-induced cost function. With a slight modification of the greedy algorithm one can find an integer-valued element z of $C(q)$ with $z(V)$ even which minimizes cz. Therefore, for node-induced cost-functions, the minimum cost edge-connectivity augmentation problem can be solved in polynomial time.

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