5 Matroids and Submodular Functions[1]

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In 1935 H. Whitney defined the notion of matroid in:
H. Whitney (1935). On the abstract properties of linear dependence. *American J. Math.* 57, 509–533.

His main motivation was to capture the fundamental properties of linear independence by setting up an axiomatization for abstract independence. From this approach the following definition is quite natural. A *matroid* M is a pair $(S,\mathcal{I})$ consisting of a finite ground-set S and a non-empty family $\mathcal{I}$ of subsets of S satisfying the following axioms.

(I1) Each subset of every member of $\mathcal{I}$ belongs to $\mathcal{I}$,
(I2) for every subset X of S, all maximal subsets of X belonging to $\mathcal{I}$ have the same cardinality $r(X)$, called the rank of X.

(A member I of $\mathcal{F}$ is called maximal in X if $I \subseteq X$ and there is no $J \in \mathcal{I}$ with $I \subset J \subseteq X$.) The members of $\mathcal{F}$ are called *independent* sets (while all other subsets of S are *dependent.*)

Since the rank-functions of two distinct matroids are distinct, by investigating the properties of r one may obtain information about the structure of the matroid. A rank-

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function r is clearly non-negative, integer-valued and monotonously increasing in the sense that $r(X) \geq r(Y)$ whenever $X \supset Y$. It is also subcardinal, that is, $r(X) \leq |X|$ for every $X \subseteq S$. Moreover, it is not difficult to prove that r is *submodular,* that is,

$$r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y) \text{ for every } X, Y \subseteq S. \tag{1}$$

Conversely, Whitney proved that any set-fuction admitting these properties is the rank-function of a matroid.

Later, researchers realized that among the properties of a matroid rank-function the submodular inequality is the most important one and they started to investigate submodular set-functions which are not necessarily non-negative or monotonous or sub-cardinal. Though submodular functions were introduced earlier, supermodular functions are not less important in applications. (A set-function is called *supermodular* if the reverse inequality holds in (1), in other words, the negative of a submodular function is supermodular.)

J. Edmonds (1970). Submodular functions, matroids, and certain polyhedra. R. Guy, H. Hanani, N. Sauer, J. Schonheim (eds.). *Combinatorial Structures and their Applications*, Gordon and Breach, New York, 69–87.

This is a seminal paper including a systematic study of submodular functions. As a main contribution, Edmonds recognized that certain polyhedra, called polymatroids, can be associated with submodular functions, and then linear programming ideas may be used. He also showed that there is a one-to-one correspondence between polymatroids and polymatroid functions (a set-function having all the properties of a matroid rank-function except subcardinality).

Relying on Edmonds' ideas, the notion of polymatroids was later extended to basis polyhedra, submodular polyhedra and g-polymatroids. Another early paper on the use of submodular function is the following.

L. Lovász (1970). A generalization of König's theorem. *Acta. Math.* 21, 443–446.

This is the first place where a general framework concerning graphs and submodular functions is introduced. Later several other abstract models have been set up to bring graphs and sub- or supermodular functions together.

Most notable is the notion of submodular flows due to Edmonds and Giles (see §§3.1). Let $G = (V, E)$ be a directed graph, b a submodular function on the subsets of V, and $f : V \to \Re$, $g : V \to \Re$ two functions with $f \leq g$. A vector $x : E \to R$ is called a *submodular flow* if $f \leq x \leq g$ and $\sum(x(e) : e \text{ enters } Z)$ - $\sum(x(e) : e \text{ leaves } Z) \leq b(Z)$ for every $Z \subseteq V$. The fundamental result of Edmonds and Giles is that this system is totally dual integral (TDI).

Another successful way of abstraction has been to bring partially ordered sets and sub- or supermodular functions together. For example, lattice polyhedra, a notion due to A. Hoffman and his co-workers, belong to this category [Hoffman 1982] (see §§1.2). Another rich area where partially ordered sets and submodular functions are combined is the theory of greedoids [Korte, Lovász and Schrader 1991] (see §§1.1). A greedoid may be defined by a pair $(S, \mathcal{F})$ where $\mathcal{F}$ is a family of subsets of S, called feasible sets, satisfying the second independence axiom of matroids.

Before proceeding to a more systematic overview of the material let me mention a predecessor of this work, an annotated bibliography on submodular functions and related topics, written by E. Lawler (who passed away untimely in 1994).

E. Lawler (1985). Submodular functions and polymatroidal optimization. M. O'hEgearteigh, J.K. Lenstra, A.H.G. Rinnooy Kan (eds.). *Combinatorial Optimization: Annotated Bibliography*, John Wiley and Sons, New York, 32–39.

In this paper Lawler finishes his overview with the year 1984.

Let us take up the thread with this year. That is, we will concentrate on works which have appeared since 1984 but some important earlier papers will also be mentioned. Readers interested in recent developments of the topic are in a good position since the last dozen of years produced quite a few excellent books and survey papers.

1 Books and Surveys

1.1 Books

A. Recski (1989). *Matroid Theory and its Applications in Electric Network Theory and Static*, Springer Verlag, Berlin, and Akadémiai Kiadó, Budapest.

This is a basic textbook that can be useful for beginners and teachers of the topic, as well. Researchers will enjoy the book being a rich source of engineering applications of matroids and submodular functions. Rigidity problems and automatized recognition of line-drawings are among the applied areas discussed in Recski's book.

A more detailed discussion of these two topics can be found in the following two books, respectively.

J. Graver, B. Servatius, H. Servatius (1993). *Combinatorial Rigidity*, Graduate Studies in Mathematics 2, Amer. Math. Soc.

K. Sugihara (1986). *Machine Interpretation of Line-drawings*, MIT Press, Cambridge, Mass.

T. Ibaraki, N. Katoh (1988). *Resource Allocation Problems – Algorithmic Approaches*, The MIT Press Series in the Foundation of Computing.

A third, interesting application-oriented book that includes submodular functions. Section 8 of this book summarizes the basics of submodular functions while Section 9 is an exciting exhibition of resource allocations problems and their solutions under submodular constraints.

K. Murota (1987). *System analysis by graphs and matroids. Structural solvability and controllability*, Algorithms and Combinatorics 3, Springer-Verlag, Berlin.

One more recommendable application-guided monograph concerning matroids, graphs and linear algebra along with their applications in engineering.

More theoretically oriented readers are recommended the following works:

K. Trümper (1992). *Matroid Decomposition*, Academic Press, San Diego.

This book covers a great number of the deep results concerning matroid minors, decompositions, and representations, including classical theorems of Tutte and of Seymour, as well as the fundamental research achievements due to the author of the book.

J.G. Oxley (1992). *Matroid Theory,* Oxford Science Publications, Oxford.
This is another highly recommendable thorough treatment concentrating on minors, decompositions and representations.

L. Lovász, A. Recski (eds.) (1985). *Matroid Theory,* North-Holland, Amsterdam.
N. White (ed.) (1986). *Theory of Matroids,* Cambridge University Press, Cambridge
N. White (ed.) (1992). *Matroid Applications,* Cambridge University Press, Cambridge
These books are collections of papers on matroids. Some of them will also be mentioned in the sequel.

M. Grötschel, L. Lovász, A. Schrijver (1988). *Geometric Algorithms and Combinatorial Optimization,* Springer Verlag, Berlin.
This monograph describes in detail general-purpose algorithms, such as the ellipsoid methods and the basis reduction algorithm, and shows several applications of these methods concerning matroids and submodular functions. Perhaps the most important of these applications is a polynomial time algorithm to minimize a submodular function. Algorithms concerning submodular flows are also exhibited.

Among the many possible generalizations of matroids we refer here to two: greedoids and oriented matroids. The books concerning these topics are:
B. Korte, L. Lovász, R. Schrader (1991). *Greedoids,* Springer-Verlag, Berlin.
A. Björner, M. Las Vergnas, B. Sturmfels, N. White, G. Ziegler (1993). *Oriented Matroids,* Cambridge University Press, Cambridge.
A. Bachem and W. Kern (1992). *Linear Programming Duality: An Introduction to Oriented Matroids,* Springer, Berlin.

Multimatroids and Δ-matroids form yet another generalization of matroids. (A. Bouchet, who developed a great part of this flourishing theory, is writing a book on the topic.) Last but not at all least comes the following indispensible monograph on submodular functions:
S. Fujishige (1991). Submodular functions and optimization. *Ann. Discr. Math.* 47.
Several models concerning submodular functions and graphs along with their relationship are analyzed. The reader may also read about the algorithms which are far-reaching extensions of classical max-flow algorithms. Some non-linear optimization problems concerning submodular functions are also discussed.

G.L. Nemhauser, L.A. Wolsey (1988). *Integer and Cominatorial Optimization,* Wiley, New York.
This book includes a chapter on the optimization aspects of matroids and submodular functions.

R. Graham, M. Grötschel, L. Lovász (eds.) (1995). *Handbook of Combinatorics,* Elsevier Science, Amsterdam.
This monumental handbook covers the whole body of combinatorics and includes

several survey papers concerning our topics (see the beginning of next subsection).

1.2 Survey papers

D.J.A. Welsh (1995). Matroid theory: fundamental concepts. R. Graham, M. Grötschel, L. Lovász (eds.). *Handbook of Combinatorics,* Elsevier Science, Amsterdam, 481–526.
This paper provides a concise introduction starting from the basic concepts and ending with an outline of the most recent developments.

P.D. Seymour (1995). Matroid minors. R. Graham, M. Grötschel, L. Lovász (eds.). *Handbook of Combinatorics,* Elsevier Science, Amsterdam, 527-550.
This paper will be unavoidable for those studying or investigating matroid minors and decompositions. Likewise, the following article will serve a similar role in the area of matroid optimization.
R.E. Bixby, W.H. Cunningham (1995). Matroid optimization and algorithms. R. Graham, M. Grötschel, L. Lovász (eds.). *Handbook of Combinatorics,* Elsevier Science, Amsterdam, 551–609.

U. Faigle (1987). Matroids in combinatorial optimization. N. White (ed.). *Combinatorial Geometries,* Encyclopedia of Mathematics and its Applications, 29, Cambridge University Press, Cambridge, 161–210.
One more survey paper from the same area.

A. Bouchet (1995). Covering and Δ-covering. E. Balas, J. Clausen (eds.). *Integer Programming and Combinatorial Optimization,* Lecture Notes in Computer Science, 920 Springer, Berlin 228–244.
A good introduction to Δ-matroids.

A. Frank, É. Tardos (1988). Generalized polymatroids and submodular flows. *Math. Program. (B)* 42, 489–563.
As far as submodular functions are concerned, one may obtain some insight of submodular flows, polymatroids and their relationship from this paper.

A. Schrijver (1984). Total dual integrality from directed graphs, crossing families and sub- and supermodular functions. W. R. Pulleyblank (ed.). *Progress in Combinatorial Optimization,* Academic Press, New York, 315–361.
An excellent cross-section for a reader interested in the relationship between the several models and frameworks concerning submodular functions and graphs.

The following two survey papers concentrate on the applications of submodular functions in graph theory.
A. Frank (1993). Submodular functions in graph theory. *Discr. Math.* 111, 231–243.
A. Frank (1993). Applications of submodular functions. K. Walker (ed.). *Surveys in Combinatorics,* London Math. Soc. Lecture Note Series 187, Cambridge University Press, Cambridge, 85–136.

A. Frank (1990). Packing paths, circuits, and cuts – a survey. B. Korte, L. Lovász, H-J. Prömel, A. Schrijver (eds.). *Paths, Flows and VLSI-Layouts*, Springer Verlag, Berlin, 47–100.
A. Frank (1994). Connectivity augmentation problems in network design. J.R. Birge, K.G. Murty (eds.). *Mathematical Programming: State of the Art*, The University of Michigan, 34–63.

Submodular functions turned out to be a basic proof technique in edge-disjoint paths problems as well as in connectivity augmentation problems. These areas are surveyed, respectively, in these papers.

I promised to mention only few works appeared before 1984, but there are two fundamental works which certainly cannot be left out from any overview.
A. Hoffman (1982). Ordered sets and linear programming. I. Rival (ed.). *Ordered Sets*, D. Reidel Publishing Comp., 619–654.

This is a very exciting survey on *lattice polyhedra* and other models introduced and analyzed by Hoffman and his co-workers. In my view this area deserves more attention than it has got in the last decade. One beautiful application of lattice polyhedra is the strikingly simple derivation of fundamental theorems of Greene and of Greene and Kleitman on optimal families of chains and antichains of a partially ordered set.

L. Lovász (1983). Submodular functions and convexity. A. Bachem, M. Grötschel, B. Korte (eds.). *Mathematical Programming: the State of the Art*, Springer, Berlin, 235–257.

This is another basic survey on the close parallel of convex functions and their discrete counter-parts, submodular functions. For brand-new developments in this direction, see §3.

M. Iri (1983). Applications of matroid theory. A. Bachem, M. Grötschel, B. Korte (eds.). *Mathematical Programming: the State of the Art*, Springer, Berlin, 160–201.

Another important overview on the applicability of matroids.

After mentioning these older papers, let me finish the list of surveys with a very new one.
R.E. Burkard, B. Klinz, R. Rudolf (1996). Perspectives of Monge properties in optimization. *Discr. Appl. Math.* 70, 95–162.

A superb survey (with 130 items in its reference list),which is highly recommendable for those interested in greedy algorithms, a method strongly related to submodular functions.

2 Submodular Functions

2.1 Minimization

One of the most exciting problems of the area is minimizing a submodular function. The only existing polynomial time algorithm for this purpose uses the ellipsoid method [Grötschel, Lovász and Schrijver 1988] (see §1.1). However there are important special cases when purely combinatorial algorithms are available.

W.H. Cunningham (1985). On submodular function minimization. *Combinatorica* 5, 185–192.

This solves the problem for "small" submodular functions. The method relies on the theory of polymatroids and may be considered as a clever refinement of Edmonds' matroid-partition algorithm.

M. Queyranne (1996). A combinatorial algorithm to minimize symmetric submodular functions. *Math. Program. (B)*, to appear.

This recent result settles completely the minimization problem for symmetric submodular functions: the algorithm is strongly polynomial. The surprising thing here is that no polymatroid theory is needed at all. The method is an extension of the revolutionary algorithm of [Nagamochi and Ibaraki 1995] (see §5) to compute the edge-connectivity of an undirected graph. Queyranne's algorithm may also be specialized to compute the minimum cut of a hypergraph.

M.X. Goemans, V.S. Ramakrishnan (1995). Minimizing submodular functions over families of sets. *Combinatorica* 15, 499–541.

In some cases one is interested in the minimum of a submodular function over the members of a certain family of sets. Already the book of [Grötschel, Lovász and Schrijver 1988] (see §1.1) includes non-trivial results in this direction and an elegant extension is found in this paper.

2.2 Intersection theorem

Perhaps the most important central result of the whole theory is the matroid intersection theorem of Edmonds. Its non-weighted special case asserts that *two matroids have a k-element independent set in common if and only if there is no bi-partition* $\{X_1, X_2\}$ *of the ground-set for which* $r_1(X_1) + r_2(X_2) < k$ *where* r_1 *and* r_2 *are the rank-functions of the matroids.* This theorem has a great number of applications and serves as a root to many extensions. One of them, the polymatroid intersection theorem, was already found by Edmonds in his paper of 1970 mentioned above.

A. Frank (1984). Finding feasible vectors of Edmonds-Giles polyhedra. *J. Combin. Theory B* 36, 221–239.

This paper consider an interesting version of the polymatroid intersection theorem the discrete separation theorem. It asserts that, *given integer-valued super- and submodular functions p and b, there is an integer-valued modular function m for which* $p \leq m \leq b$. (A set function satisfying (1) with equality is called **modular**.) Related results are investigated in:
J. Kindler (1988). Sandwich theorems for set functions. *J. Math. Analysis Appl.* 133 529–544.

F.D.J. Dunstan, A.W. Ingleton, D.J.A. Welsh (1972). Supermatroids. D.J.A Welsh, D.R. Woodall (eds.). *Combinatorics*, The Institute of Mathematics and its Applications, London, 72–122.

A difficult extension of Edmonds' matroid intersection theorem concerns distributive supermatroids which have been introduced in the following paper. A distributive supermatroid is a family $\mathcal{F}$ of ideals of a partially ordered set satisfying the following axioms. (1) $\emptyset \in \mathcal{F}$, (2) if $X \subset Y \in \mathcal{F}$ and X is an ideal, then $X \in \mathcal{F}$, (3) if $X \subset Y \in \mathcal{F}$ and $|X| < |Y|$, then there is an element $x \in X - Y$ such that $Y \cup \{x\} \in \mathcal{F}$.

É. Tardos (1990). An intersection theorem for supermatroids. *J. Combin Theory B* 50, 150–159.

In this paper an intersection theorem is described concerning two distributive supermatroids defined on the same partially ordered set.

A. Schrijver (1985). Supermodular colorings. L. Lovász, A. Recski (eds.). *Matroid Theory*, North-Holland, Amsterdam.

This paper present another type of intersection theorem concerning common supermodular colourings of two supermodular functions.

É. Tardos (1985). Generalized matroids and supermodular colorings. L. Lovász, A. Recski (eds.). *Matroid Theory*, North-Holland, Amsterdam.

Tardos discovered a connection between supermodular colourings and generalized polymatroid intersection and provided this way an alternative proof of Schrijver's theorem.

2.3 Greedy algorithms

Any introductory course on matroid theory certainly discusses the greedy algorithm. For a given weight-function w on the ground-set of the matroid, the greedy algorithm consists of choosing step by step a not-yet-chosen element of largest positive weight in such a way that the set of chosen elements form an independent set of the matroid. The greedy algorithm theorem asserts that the final independent set is of maximum weight. It is useful to realize that the independence axioms of matroids may be interpreted as requiring that for every $0-1$ weight-function w the greedy algorithm computes correctly a maximum weight independent set. The basic algorithm has been extended in several directions and there is a vast literature of work on greedy algorithms related to matroids and extensions. For a survey, see the paper of [Burkard, Klinz, Rudolf 1996] (see §1.2). An early interesting extension of the matroid greedy algorithm is the following.

D. Kornblum (1978). *Greedy algorithms for some optimization problems on a lattice polyhedron*, Ph. D. Thesis, Graduate Center of the City University of New York.

This thesis was written under the guidance of A. Hoffman who has several papers on greedy algorithms. See, for example:
A. Hoffman (1985). On greedy algorithms that succeed. I. Anderson (ed.). *Surveys in Combinatorics*, London Math. Soc. Lecture Notes Series, 103, Cambridge University Press, Cambridge, 97–112.

In these papers the original matroid greedy algorithm is generalized.

There are other greedy-type algorithms, primarily those concerning the so-called Monge-property, which have not been known until recently to have any connection to submodularity. The following enlightening paper found a bridge between the two large classes of greedy algorithms.

M. Queyranne, F. Spieksma, F. Tardella (1993). A general class of greedily solvable linear programs. G. Rinaldi, L. Wolsey (eds.). *Proc. of the 3rd IPCO Conf.*, Erice, 385–399.

This work contains a common generalization of Edmonds' polymatroid greedy algorithm and the greedy algorithm of Hoffman for transportation problems when the cost function satisfies the Monge property. The algorithm has been further generalized in the following paper.

U. Faigle, W. Kern (1996). Submodular linear programs on forests, *Math. Program.* 72, 195–206.

Actually this is a two-phase greedy algorithm in the sense that first the primal linear program is solved in a greedy way and then the dual is solved greedily. A similar type of two-phase greedy approach is described for another model in the next paper.

A. Frank (1996). *Increasing the rooted connectivity of a digraph by one*, submitted to *Mathematical Programming Ser. B.*

This algorithm may be considered as a common generalization of Fulkerson's minimum cost arborescence algorithm and Kornblum's algorithm for (supermodular) lattice polyhedra.

3 Frameworks for Sub- and Supermodular Functions

Sub- and supermodular functions are often considered in connection with other structures like directed or undirected graphs, partially ordered sets. Several models have been developed to incorporate the various phenomena appearing in special cases.

3.1 Submodular flows

J. Edmonds, R. Giles (1977). A min-max relation for submodular functions on graphs. *Ann. Discr. Math.* 1, 185–204.

The model of submodular flows is perhaps the most convenient and flexible among the several equivalent models (such as, independent flows, polymatroidal flows, kernel systems, etc).

Successful efforts have been made to carry over the known techniques of the classical network flows to submodular flows. There follows a selection of papers of this type their titles already indicates the flow technique they have extended to submodular flows.

W. Cunningham, A. Frank (1985). A primal-dual algorithm for submodular flows *Math. Oper. Res.* 10, 251–261.

U. Zimmermann (1985). Augmenting circuit methods for submodular flow problems. L. Lovász, A. Recski (eds.). *Matroid Theory*, North-Holland, Amsterdam.
S. Fujishige (1987). An out-of-kilter method for submodular flows. *Discr. Appl. Math.* 17, 3–16.
W. Cui, S. Fujishige (1988). A primal algorithm for the submodular flow problem with minimum-mean cycle selection. *J. Oper. Res. Soc. Japan* 31, 431–441.
U. Zimmermann (1992). Negative circuits for flows and submodular flows. *Discr. Appl. Math.* 36, 179–189.
S.T. McCormick, T.R. Ervolina (1993). Canceling most helpful total submodular cuts for submodular flow. G. Rinaldi, L. Wolsey (eds.). *Proc. of the 3rd IPCO Conf.*, Erice. CIACO, 343–353.
H.N. Gabow (1993). A framework for cost-scaling algorithms for submodular flow problems. *Proc. 34th Annual IEEE Symp. Found. Comput. Sci.*, 449–458.
S. Fujishige, H. Röck, U. Zimmermann (1989). A strongly polynomial algorithm for minimum cost submodular flow problems. *Math. Oper. Res.* 14, 60–69.

H.N. Gabow (1995). Centroids, representations, and submodular flows. *J. Algorithms* 18, 586–628.

This paper presents a general technique to speed up several of the above algorithms.

3.2 Delta-matroids and submodular functions in two variables

A. Bouchet (1987). Greedy algorithm and symmetric matroids. *Math. Program.* 38 147–159,
R. Chandrasekaran, S.N. Kabadi (1988). Pseudomatroids. *Discr. Math.* 71, 206–217.

Various possible applications led researchers to investigate structures that may be associated with sub- or supermodular functions in two variables. A Δ-matroid is a pair $D = (S, \mathcal{F})$ where S is a finite ground-set and $\mathcal{F}$ is a family of subsets of S, called feasible sets, satisfying the following symmetric exchange axiom: for $F_1, F_2 \in \mathcal{F}$ and for $x \in F_1 \Delta F_2$, there is an element $y \in F_1 \Delta F_2$ with $F_1 \Delta \{x, y\} \in \mathcal{F}$. (Here Δ denotes the symmetric difference.) This was introduced independently, under different names, in these two papers.

By now the name Δ-matroid seems to become the generally accepted one. It can be shown that a Δ-matroid is a matroid (given by its bases) if and only if all the feasible sets have the same cardinality. Dress and Havel introduced a slightly weaker notion, called *metroid.*

A. Bouchet, A. Dress, T. Havel (1992). Δ-matroids and metroids. *Adv. Math.* 91, 136–142.

It was pointed out that metroids are exactly those Δ-matroids for which the empty set is feasible.

One important feature of Δ-matroids is that the greedy algorithm computes correctly a maximum weight feasible set.

M. Nakamura (1988). A characterization of those polytopes in which the greedy algorithm works (abstract). *Proc. 13th Int. Symp. on Math. Program.*, Tokyo.
S.N. Kabadi, R. Chandrasekaran (1990). On totally dual integral systems. *Discr. Appl. Math.* 26, 87–104.
L. Qi (1988). Directed submodularity, ditroids, and directed submodular flows. *Math. Program.* 42, 579–599.

The convex hull P of characteristic vectors of feasible sets has been described independently in these three papers. Namely, $P = \{x : x(A) - x(B) \leq f(A,B),\ A, B \subseteq S, A \cap B = \emptyset\}$ where $f(A,B) := \max(|F \cap A| - |F \cap B| : F \in \mathcal{F})$. Function f can be shown to be bisubmodular, that is, $f(A,B) + f(A',B') \geq f(A \cap A', B \cap B') + f((A \cup A') - (B \cup B'), (B \cup B') - (A \cup A'))$. Actually, the linear system $x(A) - x(B) \leq f(A,B),\ A, B \subseteq S, A \cap B = \emptyset\}$ is totally dual integral whenever f is a bisubmodular function.

A. Bouchet, W. Cunningham (1995). Delta-matroids, jump systems, and bisubmodular polyhedra. *SIAM J. Discr. Math.* 8, 17–32.

This paper presents more structural results on bisubmodular functions.

A. Frank, T. Jordán (1995). Minimal edge-coverings of pairs of sets. *J. Combin. Theory B*, 65, 73–110.

Another type of two-variable supermodular set-function is considered.

A non-negative integer-valued function p defined on the pairs of disjoint subsets of S is said to be *crossing bi-supermodular* if $p(X,Y) + p(X',Y') \leq p(X \cap X', Y \cup Y') + p(X \cup X', Y \cap Y')$ holds whenever $p(X,Y), p(X',Y') > 0,\ X \cap X', Y \cap Y' \neq \emptyset$. The problem is to find an integer-valued function $x \geq 0$ defined on the ordered pairs (u,v) $(u, v \in S)$ so that $\sum(x(u,v) : u \in X, v \in Y) \geq p(X,Y)$ holds for every $X, Y \subseteq S$ and so that $(\sum x(u,v) : u, v \in S)$ is as small as possible. The main result is a general min-max theorem that implies a characterization on the minimum number of new edges to be added to a given directed graph to make it k-node-connected or k-edge-connected. Other special cases are an extension of a deep theorem of E. Győri on intervals, W. Mader's theorem on splitting off edges in directed graphs, and J. Edmonds' theorem on matroid partitions.

3.3 Valuated matroids

Valuated matroids were introduced by Dress and Wenzel. The idea is that the greedy algorithm works not only for linear cost-functions but some more general ones, as well, provided the cost function satisfies an exchange-type property.

A. Dress, W. Wenzel (1992). Valuated matroids. *Adv. Math.* 93, 214–250.

By a valuation w on a matroid Dress and Wenzel mean a function $w : \mathcal{B} \rightarrow \Re$ on the family of bases $\mathcal{B}$ so that for any two bases B, B' and element $u \in B - B'$, there is an element $v \in B' - B$ for which $w(B) + w(B') \leq w(B - u + v) + w(B' - v + v)$. A matroid with a valuation is called a *valuated matroid.* Dress and Wenzel proved that a suitable version of the greedy algorithm works for valuated matroids.

K. Murota (1996). Valuated matroid intersection I: optimality criteria. *SIAM J. Discr. Math.* 9, 545–561.

K. Murota (1996). Valuated matroid intersection II: algorithms. *SIAM J. Discr. Math.* 9, 561-576.

Murota showed that not only the greedy algorithm but the matroid intersection algorithm (and theory) as well extends nicely to valuated matroids. Moreover, Murota made an important discovery. He realized that valuated matroids may be viewed as a tool combining ideas from matroid theory and non-linear programming.

K. Murota (1996). Convexity and Steinitz's exchange property. *Proc. 5th Int. Conf. on Integer Program. and Comb. Optim.*, Vancouver.

3.4 Matroid parity

Perhaps the most difficult optimization problem concerning matroids is the matroid parity problem.

L. Lovász, M. Plummer (1986). *Matching Theory*, North-Holland, Amsterdam.

This book summarizes the early fundamental results on the subject.

H. Gabow, M. Stallmann (1986). An augmenting path algorithm from the linear matroid parity problem. *Combinatorica* 6, 123–150.

J. Orlin, J. VandeVate (1996). Solving the linear matroid parity problem as a sequence of problems. *Math. Program.* to appear.

One of the main concerns is the algorithmic side of the matroid parity problem. Lovász' original polynomial-time algorithm for the (linear) matroid parity is extremely complicated. Here are two simplified versions.

Gammoids, a special class of matroids, are defined on a node-set of a directed graph $G = (V, E)$, as follows. Let S be a specified subset of k nodes of a G and let $\mathcal{B}$ be the family of k-element subsets T of V for which there are k node-disjoint paths from S to T. Then it can be proved that $\mathcal{B}$ is the basis set of a matroid and the matroids and submatroids arising this way are called *gammoids*. (The smallest matroid that is not a gammoid is the circuit matroid of K_4.)

Po Tong, E.L. Lawler, V.V. Vazirani (1984). Solving the weighted parity problem for gammoids by reduction to graphic matching. W.R. Pulleyblank (ed.). *Progress in Combinatorial Optimization*, Academic Press, New York, 363–374.

This paper does exactly what its title indicates.

There is a very exciting probabilistic approach to solve algorithmically the matroid parity problem. The underlying idea is simple and perhaps best undersood in the special case of deciding whether a bipartite graph G has a perfect matching. Assign a matrix M to G with columns corresponding to one of the two colour-classes of G and with rows corresponding to the other colour-class. Define entry a_{ij} zero if v_i and v_j are not incident and x_{ij} if they are incident. Here the non-zero entries are considered independent indeterminates. It is not difficult to see that G has a perfect matching if and only if the determinant of M is not the zero polynomial. It may not be easy to answer this question algorithmically since the naive way to compute the determinant may lead to far too many expansion terms. There is however another natural approach: substitute random integers for each indeterminate and compute the determinant of the arising integer matrix. If this number is not zero, then the polynomial is not identically zero and hence the graph has a perfect matching. If this number is zero, then we cannot be sure that the polynomial is identically zero because we just may have hit a root of this polynomial. But intuitively the chance of such an event is clearly small and it can actually be proved that the probability of hitting a root is smaller than a fixed number $\varepsilon < 1$. Therefore if we repeat the same procedure several times and every time the resulting determinant is zero, then we can be sure, with arbitrarily high probability, that the graph has no perfect matching.

L. Lovász, (1979). On determinants, matchings and random algorithms. L. Budach (ed.). *Fundamentals of Comput. Th.*, Akademia Verlag, Berlin.

Of course, there are very efficient deterministic algorithm for bipartite matching but the nice thing is that the probabilistic algorithm above can be extended to matroid parity problems as was shown in this paper.

P.M. Camerini, G. Galbiati, F. Maffioli (1992). Random pseudo-polinomial algorithms for exact matroid problems. *J. Algorithms* 13, 258–273.

This paper extends the idea to related topics. Among others, it describes a polynomial time random algorithm to determine whether two matroids with a red and blue coloured common ground-set have a common basis with exactly k red elements. No polynomial-time deterministic algorithm is known even for the special case of this problem when one wants to determine if a red-blue edge-coloured bipartite graph has a perfect matching with exactly k red edges.

J.H. VandeVate (1992). Fractional matroid matchings. *J. Comb. Theory B* 5,5 133-145.

J.H. VandeVate (1992). Structural properties of matroid matchings. *Discr. Appl. Math.* 39, 69-85,

Some interesting structural results concerning the matroid parity problem can be found in these works.

M.L. Furst, J.L. Gross, L.A. McGeogh (1988). Finding a minimum-genus graph embedding. *J. ACM* 35, 523–534.

Matroid parity finds a nice application in graph embedding problems.

L. Nebesky (1983). A Note on Upper Embeddable Graphs. *Czechslovak Math. Journal,* 33, 37–40.

This remarkable paper is concerned with the co-graphic matroid parity problem and finds a very neat characterization for graphs having a spanning tree T for which each component arising from G by deleting the edges in T has an odd number of edges.

4 Minor, Decompositions, and Representations

Let A be a matrix with entries from an arbitrary field F. The set S of columns of A forms a subset of a vector space over F and linear independence defines a matroid on

S. A fundamental question is to decide whether a given matroid can be represented in such a form. Very often the answer relies on the notion of minors. For example, a classic result of Tutte asserts that a matroid M is binary (i.e., M can be represented over GF(2)) if and only if M does not include the uniform matroid $U_{4,2}$ as a minor. As I mentioned earlier, excellent recent books and survey papers are available concerning this area therefore here I list only a few papers to provide a flavour of the type of theorems. For example, Tutte characterized matroids which are representable over every field (the so-called regular matroids) by proving that regular matroids are those not containing $U_{4,2}$, the Fano matroid, and the dual Fano matroid.

A.M.H. Gerards (1989). A short proof of Tutte's characterization of totally unimodular matrices. *Linear Algebra and its Appl.* 114, 217-222.

The original proof is very difficult but this paper turns Tutte's result accessible for everyone familiar with the basics of matroid theory.

J. Kahn, P.D. Seymour (1988). On forbidden minors of GF(3), *Proc. American Math. Soc.* 102, 437-440.

This includes a simple proof of a characterization for ternary matroids, a result proved first by R. Reid (unpublished). Structural descriptions are often related to connectivity properties.

R.E. Bixby (1974). l-marices and a characterization of binary matroids. *Discr. Math.* 8, 139–145,

This work shows that a non-binary matroid not only includes a $U_{4,2}$-minor, but every element belongs to such a minor provided that the matroid is connected (i.e. every two elements are contained in a circuit).

P.D. Seymour (1985). On minors of 3-connected matroids. *European J. Combinatorics* 6, 375–382.

It is shown that if M is 3-connected and not binary, then even every pair of elements belong to a $U_{4,2}$-minor. As far as decomposition of matroids are concerned we mention two far reaching results. Tutte's above-mentioned characterization of regular matroids provides a certificate (namely the occurrence of certain minors) to show that a certain matroid is not regular.

P.D. Seymour (1980). Decomposition of regular matroids. *J. Comb. Theory B* 28, 305–359.

This paper provides a certificate to show that a matroid is regular. The certificate is a way to build up the matroid from graphic and cographic matroids and from a specific regular matroid on ten elements, where the building operations are the so called 1-, 2- and 3-sums.

F.T. Cheng, K. Truemper (1986). A decomposition of the matroids with the max-flow min-cut property. *Discr. Appl. Math.* 15, 329–364.

Another important decomposition theorem desribes how to construct matroids with the max-flow min-cut property.

P.D. Seymour (1977). The matroids with the max-flow min-cut property. *J. Comb. Theory B* 23, 189–222.

These matroids are characterized in terms of forbidden minors.

5 Applications to Networks, Scheduling, Game Theory, and Allocation Problems

A. Frank (1992). On a theorem of Mader. *Ann. Discr. Math.* 101, 49–57.

One of the main application area of submodular and related functions concerns connectivity of graphs. For example, skew-supermodular functions are used in this paper to provide a relatively simple proof of a deep theorem of Mader on splitting of edges without decreasing the local edge-connectivity. (A set-function p is called skew-supermodular if the following inequality holds for every pair of sets X, Y: $p(X) + p(Y) \leq \max(p(X \cap Y)p(X \cup Y),\ p(X - Y) + p(Y - X))$.

D.P. Williamson, M.X. Goemans, M. Mihail, V.V. Vazirani (1995). A primal dual algorithm for generalized Steiner network problems. *Combinatorica* 15, 435-454.

Here another class of submodular-type functions is used in a fundamental new technique of the primal-dual approximation methods.

A. Frank, E. Tardos (1989). An application of submodular flows. *Linear Algebra and its Appl.* 114/115, 329–348.

In this paper the minimum cost rooted connectivity augmentation problem is reduced to submodular flows.

H.N. Gabow (1991). A matroid approach to finding edge-connectivity and packing arborescences. *Proc. 23rd ACM Symp. Theory of Comput.*, 112–122.
H.N. Gabow, K.S. Manu (1995). Packing algorithms for arborescences (and spanning trees) in capacitated graphs. E. Balas, J. Clausen (eds.). *Integer Programming and Combinatorial Optimization*, Lecture Notes in Computer Science, 920 Springer, Berlin 388–402.
H.N. Gabow (1991). Applications of a poset representation to edge connectivity and graph rigidity. *Proc. 32nd Annual IEEE Symp. Found. Comput. Sci.*, 812–820.
H.N. Gabow (1994). Efficient splitting off algorithms for graphs. *Proc. 26th ACM Symp. Theory of Comput.*, Montreal, 696–706.

These important works of H. Gabow concern algorithmic aspects of connectivity-related problems.

L.A. Wolsey (1989). Submodularity and valid inequalities in capacitated fixed charged networks. *Oper. Res. Lett.* 8, 119–124.

Another role of submodular functions is utilized here. The observation that the flow values of certain fixed charged networks are submodular is used to derive valid inequalities.

H. Nagamochi, T. Ibaraki (1995). A faster edge splitting algorithm in multigraphs and its application to the edge-connectivity augmentation problem. E. Balas, J. Clausen

(eds.). *Integer Programming and Combinatorial Optimization*, Lecture Notes in Computer Science, 920 Springer, Berlin, 403-413.
H. Nagamochi, K. Nishima, T. Ibaraki (1995). Computing all small cuts in undirected networks. D-Z.Du, X-S. Zhang (eds.). *Algorithms and Computation*, Lecture Notes in Computer Science 834, 190-198.

In connection with Queyranne's algorithm for minimizing symmetric submodular functions, I have already referred to a fundamental paper of Nagamochi and Ibaraki. These works describe interesting extensions. The submodular technique again plays an important role.

Submodular functions proved to be useful in the area of scheduling. See for example the following two papers.

F. Blanchini, M. Queyranne, F. Rinaldi, W. Ukovich (1996). A feedback strategy for periodic network flows. *Networks* 27, 25–34.
M. Queyranne, A.S. Schulz (1995). Scheduling unit jobs with compatible release dates on parallel machines with nonstationary speed. E. Balas, J. Clausen (eds.). *Integer Programming and Combinatorial Optimization*, Lecture Notes in Computer Science, 920 Springer, Berlin, 307-320.

Another large and interesting topic of applications of submodular functions is the area of resource allocations. Here I mention some of the basic results.

N. Katoh, T. Ibaraki, H. Mine (1985). An algorithm for the equipollent resource allocation problem. *Math. Oper. Res.* 10, 44–53.
A. Federgruen, H. Gronevelt(1986). Optimal flows in networks with multiple sources and sinks, with applications to oil and gas lease investment programs. *Oper. Res.* 34, 218–225.
A. Federgruen, H. Gronevelt (1986). The greedy procedure for resource allocation problems: Necessary and sufficient conditions for optimality. *Oper. Res.* 34, 909–918.
S. Fujishige, N. Katoh, T. Ichimori (1988). The fair resource allocation problem. *Math. Oper. Res.* 13, 164–173.

Finally, let me draw attention to the applicability of supermodular functions in game theory. This connection was already discovered in
L.S. Shapley (1971). Cores of convex games. *Int. J. Game Theory* 1, 11–26.

T. Ichiishi (1981). Super-modularity: Applications to convex games and to the greedy algorithm for LP. *J. Economic Theory* 25, 283–286.

This paper analyzes the relationship of greedy algorithms and convex games.

H. Kanekom, M. Fushimi (1986). A polymatroid associated with convex games. *Discr. Appl. Math.* 14, 33–45.

This work makes use of the theory of principal partition of polymatroids.

M. Iri (1979). A review of recent work in Japan on principal partitions of matroids and their applications. *Annals of the New York Academy of Sciences* 319, 306–319.

Principal partitions form a basic tool for describing the fine structure of submodular functions.